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Existence results for a self-adjoint coupled system of nonlinear second-order ordinary differential inclusions with nonlocal integral boundary conditions

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Abstract. A coupled system of nonlinear self-adjoint second-order ordinary differential inclusions supplemented with nonlocal non-separated coupled integral boundary conditions on an arbitrary domain is studied. The existence results for convex and non-convex valued maps involved in the given problem are proved by applying nonlinear alternative of Leray-Schauder type for multi-valued maps, and Covitz-Nadler's fixed point theorem for contractive multi-valued maps, respectively. Illustrative examples for the obtained results are presented. The paper concludes with some interesting observations.

AMS Subject Classifications: 34A60, 34B10, 34B15.

Keywords: Self-adjoint ordinary differential inclusions; coupled; nonlocal integral boundary conditions; existence; fixed point.

Contents

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1. Introduction

Inspired by the work of Bitsadze and Samarskii [6] on nonlocal elliptic boundary value problems, Il'in and Moiseev [19, 20] initiated the study of nonlocal boundary value problems for second order ordinary differential equations. Nonlocal boundary value problems constitute an important area of research as such problems find their applications in chemical engineering, thermo-elasticity, underground water flow and population dynamics, for details and examples, see [5, 30]. For a variety of interesting results on nonlocal boundary value problems, we refer the reader to the works [1–3, 8, 12–14, 16, 17, 23, 26, 28, 29] and the references cited therein. Self-adjoint differential equations are found to be of great interest in certain disciplines, for example, see [7, 11, 25, 27]. In [24], a self-adjoint coupled system of nonlinear ordinary differential equations with nonlocal multi-point boundary conditions was studied. In a recent article [4], the authors established existence results for a self-adjoint coupled system of nonlinear second-order ordinary differential equations complemented with nonlocal non-separated integral boundary conditions.

The aim of the present paper is to consider and investigate the existence of solutions for the multi-valued case of the problem discussed in [4]. In precise terms, we consider a self-adjoint coupled system of second-order ordinary differential inclusions on an arbitrary domain, subject to the nonlocal non-separated integral coupled boundary conditions given by

$$\begin{cases} \big(p(t)u'(t)\big)' \in \mu_1\, F(t,u(t),v(t)),\ t\in[a,b], \\ \big(q(t)v'(t)\big)' \in \mu_2\, G(t,u(t),v(t)),\ t\in[a,b], \\ \alpha_1 u(a)+\alpha_2 u(b)=\lambda_1\displaystyle\int_a^\eta v(s)ds,\ \ \alpha_3 u'(a)+\alpha_4 u'(b)=\lambda_2\displaystyle\int_a^\eta v'(s)ds, \\ \beta_1 v(a)+\beta_2 v(b)=\lambda_3\displaystyle\int_\xi^b u(s)ds,\ \ \beta_3 v'(a)+\beta_4 v'(b)=\lambda_4\displaystyle\int_\xi^b u'(s)ds, \end{cases} \tag{1.1}$$

where, $a<\eta<\xi<b$, $p,q\in C([a,b],\mathbb{R}^+)$, $\alpha_i,\beta_i,\lambda_i\in\mathbb{R}^+, i=1,2,3,4, \mu_j\in\mathbb{R}^+, j=1,2.$ and $F,G:[a,b]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathcal{P}(\mathbb{R})$ are given multivalued maps, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

We establish existence criteria for solutions of the problem (1.1) for convex and non-convex valued multivalued maps F and G by applying the nonlinear alternative of Leray-Schauder type for multi-valued maps in the convex case and Covitz and Nadler's fixed point theorem for contractive multi-valued maps in the non-convex case, respectively. The tools of the fixed point theory employed in our analysis are standard, however their application to the problem (1.1) is new. We emphasize that the results derived in this paper are new and enrich the literature on self-adjoint multivalued nonlocal boundary value problems.

The rest of the paper is organized as follows. We present background material about multivalued analysis in Section 2, while the main results are presented in Section 3. Numerical examples illustrating the obtained results are constructed in Section 4.

2. Preliminaries.

We begin this section by reviewing some basic definitions, lemmas, and theorems on multivalued maps from [10, 18] which are related to study of the problem (1.1).

For a normed space $(\mathcal{X},\|\cdot\|)$, we define the following:

(**i**) $P_{cl}(\mathcal{X}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{X}) : \mathcal{Y}$ is closed$\}$,

(**ii**) $P_{b}(\mathcal{X}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{X}) : \mathcal{Y}$ is bounded$\}$,

(**iii**) $P_{cp}(\mathcal{X}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{X}) : \mathcal{Y}$ is compact$\}$,

(**iv**) $P_{cp,c}(\mathcal{X}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{X}) : \mathcal{Y}$ is compact and convex$\}$.

A multi-valued map $F : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ is:

(**a**) convex (closed) valued if $F(x)$ is convex (closed) for all $x \in \mathcal{X}$.

(**b**) F is called upper semi-continuous (u.s.c.) on $\mathcal{X}$ if for each $x_0 \in \mathcal{X}$, the set $F(x_0)$ is a nonempty closed subset of $\mathcal{X}$, and if for each open set $\mathcal{N}$ of $\mathcal{X}$ containing $F(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $F(\mathcal{N}_0) \subseteq \mathcal{N}$.

(**c**) The map F is bounded on bounded sets if $F(\mathbb{B}) = \cup_{x\in\mathbb{B}} F(x)$ is bounded in $\mathcal{X}$ for all $\mathbb{B} \in \mathcal{P}_b(\mathcal{X})$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in F(x)\}\} < \infty$).

(**d**) F is said to be completely continuous if $F(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(\mathcal{X})$. F has a fixed point if there is $x \in \mathcal{X}$ such that $x \in F(x)$.

Remark 2.1. *A multivalued map $F : W \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $b \in \mathbb{R}$, the function $t \longmapsto d(b, F(t)) = \inf\{|b - c| : c \in F(t)\}$ is measurable.We define the graph of F to be the set $Gr(F) = \{(x, y) \in X \times Y, y \in F(x)\}$.The fixed point set of the multivalued operator F will be denoted by $FixF$.*

Remark 2.2. *We recall the relationship between closed graphs and upper-semicontinuity ([10]): If $F : \mathcal{X} \to \mathcal{P}_{cl}(\mathcal{X})$ is u.s.c., then $Gr(F)$ is a closed subset of $X \times Y$, i.e. for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ and $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in F(x_n)$, then $y_* \in F(x_*)$. Conversely, if F is completely continuous and has a closed graph, then it is upper semi-continuous.*

Definition 2.3. *A multivalued map $F : [a, b] \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(**i**) *$t \longmapsto F(t, u, v)$ is measurable for each $u, v \in \mathbb{R}$;*

(**ii**) *$(u, v) \longmapsto F(t, u, v)$ is upper semicontinuous for almost all $t \in [a, b]$;*

Further a Carathéodory function F is called L^1-Carathéodory if

(**iii**) *for each $\rho > 0$, there exists $\Omega_\rho \in L^1([a, b], \mathbb{R}^+)$ such that*

$$\|F(t, u, v)\| = \sup\{|x| : x \in F(t, u, v)\} \leq \Omega_\rho(t)$$

for all $\|u\|, \|v\| \leq \rho$ and for a.e. $t \in [a, b]$.

Definition 2.4. *A function $(u, v) \in \mathcal{F} \times \mathcal{F}$, where $\mathcal{F} = C^2([a, b], \mathbb{R})$ is a solution of the self-adjoint coupled system in (1.1) if it satisfies the coupled boundary conditions of (1.1) and there exist functions $\hat{f}, \hat{g} \in L^1([a, b], \mathbb{R})$ such that $\hat{f}(t) \in F(t, u(t), v(t)), \hat{g}(t) \in G(t, u(t), v(t))$ a.e on $[a, b]$.*

Let us now recall the following lemma from [4].

Lemma 2.5. *For $f_1, g_1 \in C([a,b],\mathbb{R})$ and $R \neq 0$, $E \neq 0$, the solution of the linear system*

$$\begin{cases} \big(p(t)u'(t)\big)' = \mu_1 f_1(t), \quad t \in [a,b], \\ \big(q(t)v'(t)\big)' = \mu_2 g_1(t), \quad t \in [a,b], \\ \alpha_1 u(a) + \alpha_2 u(b) = \lambda_1 \displaystyle\int_a^\eta v(s)ds, \;\; \alpha_3 u'(a) + \alpha_4 u'(b) = \lambda_2 \displaystyle\int_a^\eta v'(s)ds, \\ \beta_1 v(a) + \beta_2 v(b) = \lambda_3 \displaystyle\int_\xi^b u(s)ds, \;\; \beta_3 v'(a) + \beta_4 v'(b) = \lambda_4 \displaystyle\int_\xi^b u'(s)ds, \end{cases} \tag{2.1}$$

can be expressed by the formulas:

$$\begin{aligned} u(t) = &\int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u f_1(z)dz\Big)du + \frac{1}{R}\bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u f_1(z)dz\Big)du \\ &+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u g_1(z)dz\Big)du\,ds - \lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u g_1(z)dz\Big)du \\ &+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u f_1(z)dz\Big)du\,ds\bigg] \\ &+\frac{1}{ER}\bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\ &+E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\ &-RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b f_1(z)dz\Big) + \Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz \\ &+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz \\ &+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s g_1(z)dz\,ds\Big) \\ &+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\ &+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\ &-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b g_1(z)dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz \\ &+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz \\ &+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s f_1(z)dz\,ds\Big)\bigg], \end{aligned}$$

and

$$\begin{aligned} v(t) = &\int_a^t \Big(\frac{\mu_2}{q(u)}\int_a^u g_1(z)dz\Big)du + \frac{1}{R}\bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u f_1(z)dz\Big)du \\ &+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u g_1(z)dz\Big)du\,ds - \beta_2(\alpha_1+\alpha_2)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u g_1(z)dz\Big)du \\ &+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u f_1(z)dz\Big)du\,ds\bigg] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{ER}\Bigg[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b f_1(z)dz\Big)+\Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz\\
&+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz\\
&+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s g_1(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b g_1(z)dz\Big)+\Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz\\
&+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz\\
&+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s f_1(z)dz\,ds\Big)\Bigg].
\end{aligned}
$$

where

$$
\begin{aligned}
R &= (\alpha_1+\alpha_2)(\beta_1+\beta_2)-\lambda_1\lambda_3(\eta-a)(b-\xi),\\
E &= E_1E_4-E_2E_3,\\
E_1 &= \frac{\alpha_3}{p(a)}+\frac{\alpha_4}{p(b)},\ E_2=\int_a^\eta\frac{\lambda_2}{q(s)}ds,\ E_3=\int_\xi^b\frac{\lambda_4}{p(s)}ds,\ E_4=\frac{\beta_3}{q(a)}+\frac{\beta_4}{q(b)}.
\end{aligned}
$$

Let $(\mathcal{F},\|\cdot\|)$ denote the Banach space of all continuous real valued functions where $\mathcal{F}=\{u(t)|u(t)\in C([a,b],\mathbb{R})\}$ and $\|u\|=\sup\{|u(t)|,\quad t\in[a,b]\}$. Evidently the product space $(\mathcal{F}\times\mathcal{F},\|(u,v)\|)$ is a Banach space with the norm given by $\|(u,v)\|=\|u\|+\|v\|$ for any $(u,v)\in\mathcal{F}\times\mathcal{F}$.

Let us consider the set of selections functions F and G for each $(u,v)\in\mathcal{F}\times\mathcal{F}$ defined by

$$S_{F,(u,v)}:=\{\hat{f}\in L^1([a,b],\mathbb{R}):\hat{f}(t)\in F(t,u(t),v(t)) \text{ for a.e. } t\in[a,b]\},$$

and

$$S_{G,(u,v)}:=\{\hat{g}\in L^1([a,b],\mathbb{R}):\hat{g}(t)\in G(t,u(t),v(t)) \text{ for a.e. } t\in[a,b]\}.$$

Define the operators $\Theta_1,\Theta_2:\mathcal{F}\times\mathcal{F}\to\mathcal{P}(\mathcal{F}\times\mathcal{F})$ by

$$
\begin{aligned}
\Theta_1(u,v)=\{h_1\in\mathcal{F}\times\mathcal{F}:\ &\text{there exists }\ \hat{f}\in S_{F,(u,v)},\hat{g}\in S_{G,(u,v)} \text{ such that}\\
&h_1(u,v)(t)=\mathcal{Z}_1(t,u,v),\forall t\in[a,b]\},
\end{aligned}\tag{2.2}
$$

and

$$
\begin{aligned}
\Theta_2(u,v)=\{h_2\in\mathcal{F}\times\mathcal{F}:\ &\text{there exists }\ \hat{f}\in S_{F,(u,v)},\hat{g}\in S_{G,(u,v)} \text{ such that}\\
&h_2(u,v)(t)=\mathcal{Z}_2(t,u,v),\forall t\in[a,b]\},
\end{aligned}\tag{2.3}
$$

where

$$\begin{aligned}
\mathcal{Z}_1(u,v)(t) = & \int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)dz\Big)du \\
& +\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)dz\Big)du\,ds - \lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)dz\Big)du \\
& +\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)dz\Big)du\,ds\Big] \\
& +\frac{1}{ER}\Big[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
& +E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
& -RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_1(z)dz\Big) + \Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz \\
& +E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz \\
& +E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_1(z)dz\,ds\Big) \\
& +\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
& +E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
& -RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_1(z)dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz \\
& +E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz \\
& +E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_1(z)dz\,ds\Big)\Big],
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Z}_2(u,v)(t) = & \int_a^t \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)dz\Big)du \\
& +\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)dz\Big)du\,ds - \beta_2(\alpha_1+\alpha_2)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)dz\Big)du \\
& +\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)dz\Big)du\,ds\Big] \\
& +\frac{1}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
& +E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
& -RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_1(z)dz\Big) + \Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz \\
& +E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz \\
& +E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_1(z)dz\,ds\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
&-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_1(z)dz\Big) + \Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz \\
&+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz \\
&+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_1(z)dz\,ds\Big)\Big].
\end{aligned}
$$

Next we introduce an operator $\Theta : \mathcal{F}\times\mathcal{F} \to \mathcal{P}(\mathcal{F}\times\mathcal{F})$ as

$$
\Theta(u,v)(t) = \begin{pmatrix} \Theta_1(u,v)(t) \\ \Theta_2(u,v)(t) \end{pmatrix},
$$

where Θ_1 and Θ_2 are defined by (2.2) and (2.3) respectively.

For the sake of computational convenience, we set the notation:

$$
\mathcal{E}_1 = \mathcal{D}_1 + \mathcal{D}_3, \quad \mathcal{E}_2 = \mathcal{D}_2 + \mathcal{D}_4, \tag{2.4}
$$

where

$$
\begin{aligned}
\mathcal{D}_1 =\ &\frac{\mu_1}{|R\bar{p}|}\Big[\frac{(b-a)^2}{2}\Big(|R| + \alpha_2(\beta_1+\beta_2)\Big) + \frac{\lambda_1\lambda_2(\eta-a)\big[(b-a)^3-(\xi-a)^3\big]}{6}\Big] \\
&+\frac{1}{|RE|}\Big[\Big(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}} + \frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}} + \frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}} \\
&+\frac{E_4\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}} + \frac{RE_4(b-a)}{\bar{p}}\Big)\Big(\frac{\alpha_4\mu_1(b-a)}{|p(b)|}\Big) \\
&+\Big(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}} + \frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}} + \frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}} \\
&+\frac{E_2\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}} + \frac{RE_2(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_4\mu_1\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}\Big)\Big], \\
\mathcal{D}_2 =\ &\frac{\mu_2}{|2R\bar{q}|}\Big[\frac{\lambda_1(\beta_1+\beta_2)(\eta-a)^3}{3} + \lambda_1\beta_2(\eta-a)(b-a)^2\Big] \\
&+\frac{1}{|RE|}\Big[\Big(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}} + \frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}} + \frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}} \\
&+\frac{E_4\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}} + \frac{RE_4(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_2\mu_2(\eta-a)^2}{2\bar{q}}\Big) \\
&+\Big(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}} + \frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}} + \frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}} \\
&+\frac{E_2\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}} + \frac{RE_2(b-a)}{\bar{p}}\Big)\Big(\frac{\beta_4\mu_2(b-a)}{|q(b)|}\Big)\Big], \\
\mathcal{D}_3 =\ &\frac{\mu_1}{|R\bar{p}|}\Big[\frac{(b-a)^2}{2}\Big(\alpha_2\lambda_3(b-\xi)\Big) + \frac{\lambda_3(\alpha_1+\alpha_2)\big[(b-a)^3-(\xi-a)^3\big]}{6}\Big] \\
&+\frac{1}{RE}\Big[\Big(\frac{E_4\alpha_2\lambda_3(b-\xi)(b-a)}{\bar{p}} + \frac{E_3\lambda_1\lambda_3(b-\xi)(\eta-a)^2}{2\bar{q}} + \frac{E_3\beta_2(\alpha_1+\alpha_2)(b-a)}{\bar{q}} \\
&+\frac{E_4\lambda_3(\alpha_1+\alpha_2)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}} + \frac{RE_3(b-a)}{\bar{p}}\Big)\Big(\frac{\alpha_4\mu_1(b-a)}{|p(b)|}\Big)
\end{aligned}
$$

$$+\Big(\frac{E_2\alpha_2\lambda_3(b-\xi)(b-a)}{\bar{p}}+\frac{E_1\lambda_1\lambda_3(b-\xi)(\eta-a)^2}{2\bar{q}}+\frac{E_1\beta_2(\alpha_1+\alpha_2)(b-a)}{\bar{q}}$$
$$+\frac{E_2\lambda_3(\alpha_1+\alpha_2)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_1(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_4\mu_1\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}\Big)\Big],$$
$$\mathcal{D}_4 = \frac{\mu_2}{|R\bar{q}|}\Big[\frac{(b-a)^2}{2}\Big(|R|+\beta_2(\alpha_1+\alpha_2)\Big)+\frac{\lambda_1\lambda_3(b-\xi)(\eta-a)^3}{6}\Big]$$
$$+\frac{1}{RE}\Big[\Big(\frac{E_4\alpha_2\lambda_3(b-\xi)(b-a)}{\bar{p}}+\frac{E_3\lambda_1\lambda_3(b-\xi)(\eta-a)^2}{2\bar{q}}+\frac{E_3\beta_2(\alpha_1+\alpha_2)(b-a)}{\bar{q}}$$
$$+\frac{E_4\lambda_3(\alpha_1+\alpha_2)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_3(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_2\mu_2(\eta-a)^2}{2\bar{q}}\Big)$$
$$+\Big(\frac{E_2\alpha_2\lambda_3(b-\xi)(b-a)}{\bar{p}}+\frac{E_1\lambda_1\lambda_3(b-\xi)(\eta-a)^2}{2\bar{q}}+\frac{E_1\beta_2(\alpha_1+\alpha_2)(b-a)}{\bar{q}}$$
$$+\frac{E_2\lambda_3(\alpha_1+\alpha_2)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_1(b-a)}{\bar{p}}\Big)\Big(\frac{\beta_4\mu_2(b-a)}{|q(b)|}\Big)\Big], \tag{2.5}$$
$$\bar{p}=\inf_{z\in[a,b]}|p(z)|,\quad \bar{q}=\inf_{z\in[a,b]}|q(z)|. \tag{2.6}$$

3. The Carathéodory case

To prove our first existence result for the multivalued problem (1.1), we need the following known results.

Lemma 3.1. *([22]) Let X be a Banach space. Let $F:[a,b]\times\mathbb{R}^2\to\mathcal{P}_{cp,c}(\mathbb{R})$ be an L^1- Carathéodory multivalued map and let φ be a linear continuous mapping from $L^1([a,b],\mathbb{R})$ to $C([a,b],\mathbb{R})$. Then the operator*

$$\varphi\circ S_{F,u}: C([a,b],\mathbb{R})\to P_{cp,c}(C([a,b],\mathbb{R})),\quad u\mapsto(\varphi\circ S_{F,u})(u)=\varphi(S_{F,u})$$

is a closed graph operator in $C([a,b],\mathbb{R})\times C([a,b],\mathbb{R})$.

Lemma 3.2. *(Nonlinear alternative of Leray-Schauder type [15]). Let $\mathcal{S}$ be a Banach space, $\mathcal{S}_1$ a closed convex subset of $\mathcal{S}$, U an open subset of $\mathcal{S}_1$ and $0\in U$. Suppose that $F:\overline{U}\to\mathcal{P}_{c,cv}(\mathcal{S}_1)$ is a upper semicontinuous compact map; here $\mathcal{P}_{c,cv}(\mathcal{S}_1)$ denotes the family of nonempty, compact convex subsets of $\mathcal{S}_1$. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u\in\partial U$ and $\lambda\in(0,1)$ with $u\in\lambda F(u)$.

Now we are in a position to present our first main result.

Theorem 3.3. *Assume that*

(H_1) *$F,G:[a,b]\times\mathbb{R}^2\longrightarrow\mathcal{P}(\mathbb{R})$ are L^1-Carathéodory possessing compact and convex values;*

(H_2) *There exist continuous nondecreasing functions $\psi_1,\psi_2,\phi_1,\phi_2:[0,\infty)\longrightarrow(0,\infty)$ such that*

$$\|F(t,u,v)\|_{\mathcal{P}}:=\sup\{|\hat{f}|:\hat{f}\in F(t,u,v)\}\leqslant p_1(t)[\psi_1(\|u\|)+\phi_1(\|v\|)],$$

and

$$\|G(t,u,v)\|_{\mathcal{P}}:=\sup\{|\hat{g}|:\hat{g}\in G(t,u,v)\}\leqslant p_2(t)[\psi_2(\|u\|)+\phi_2(\|v\|)],$$

for each $(t,u,v)\in[a,b]\times\mathbb{R}^2$, where $p_1,p_2\in C([a,b],\mathbb{R}^+)$;

(H_3) *There exists a constant $N>0$ such that*

$$\frac{N}{\mathcal{E}_1\|p_1\|[\psi_1(N)+\phi_1(N)]+\mathcal{E}_2\|p_2\|[\psi_2(N)+\phi_2(N)]}>1,$$

where $\mathcal{E}_i$ $(i=1,2)$ are given in (2.4).

Then self-adjoint coupled multi-valued system (1.1) *has at least one solution on* $[a,b]$.

Proof. Consider the operators $\Theta_1,\Theta_2:\mathcal{F}\times\mathcal{F}\to\mathcal{P}(\mathcal{F}\times\mathcal{F})$ defined by (2.2) and (2.3) respectively. It follows from the assumption (H_1) that the sets $S_{F,(u,v)}$ and $S_{G,(u,v)}$ are nonempty for each $(u,v)\in\mathcal{F}\times\mathcal{F}$. Then, for $\hat{f}\in S_{F,(u,v)},\hat{g}\in S_{G,(u,v)}$ and $\forall\,(u,v)\in\mathcal{F}\times\mathcal{F}$, we have

$$
\begin{aligned}
h_1(u,v)(t) = {} & \int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du + \frac{1}{R}\bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du \\
& +\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du\,ds - \lambda_1\beta_2(\eta-a)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du \\
& +\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du\,ds\bigg] \\
& +\frac{1}{ER}\bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds \\
& +E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds \\
& -RE_4\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}(z)dz\Big) + \Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz \\
& +E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz \\
& +E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds + RE_4\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}(z)dz\,ds\Big) \\
& +\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds \\
& +E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds \\
& -RE_2\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}(z)dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz \\
& +E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz \\
& +E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds + RE_2\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}(z)dz\,ds\Big)\bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
h_2(u,v)(t) = {} & \int_a^t \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du + \frac{1}{R}\bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du \\
& +\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du\,ds - \beta_2(\alpha_1+\alpha_2)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du \\
& +\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du\,ds\bigg] \\
& +\frac{1}{ER}\bigg[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds \\
& +E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds \\
& -RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}(z)dz\Big) + \Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz
\end{aligned}
$$

$$+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)dz\Big) + \Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz$$
$$+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)dz\,ds\Big)\Big],$$

where $h_1 \in \Theta_1(u,v), h_2 \in \Theta_2(u,v)$ and hence $(h_1,h_2) \in \Theta(u,v)$.

Now, we will verify the operator Θ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. In the first step, we show that $\Theta(u,v)$ is convex valued for each $(u,v) \in \mathcal{F}\times\mathcal{F}$. Let $(h_i, \tilde{h_i}) \in (\Theta_1,\Theta_2), i=1,2$. Then there exist $\hat{f}_i \in S_{F,(u,v)}, \hat{g}_i \in S_{G,(u,v)}, i=1,2$, such that, for each $t\in[a,b]$, we have

$$h_i(t) = \int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_i(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_i(z)dz\Big)du$$
$$+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_i(z)dz\Big)du\,ds - \lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_i(z)dz\Big)du$$
$$+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_i(z)dz\Big)du\,ds\Big]$$
$$+\frac{1}{ER}\Big[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_i(z)dz\Big) + \Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz$$
$$+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz$$
$$+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_i(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_i(z)dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz$$
$$+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz$$
$$+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_i(z)dz\,ds\Big)\Big],$$

and

$$\begin{aligned}
\tilde{h_i}(t) =& \int_a^t \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_i(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_i(z)dz\Big)du \\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_i(z)dz\Big)du\,ds - \beta_2(\alpha_1+\alpha_2)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_i(z)dz\Big)du \\
&+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_i(z)dz\Big)du\,ds\Big] \\
&+\frac{1}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
&+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
&-RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_i(z)dz\Big) + \Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz \\
&+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz \\
&+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_i(z)dz\,ds\Big) \\
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
&-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_i(z)dz\Big) + \Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz \\
&+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz \\
&+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_i(z)dz\,ds\Big)\Big].
\end{aligned}$$

Let $0 \le \omega \le 1$. Then, for each $t \in [0,1]$, we have

$$\begin{aligned}
[\omega h_1 + (1-\omega)h_2](t) =& \int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u [\omega\hat{f}_1(z) + (1-\omega)\hat{f}_2(z)]dz\Big)du \\
&+\frac{1}{R}\Big[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u [\omega\hat{f}_1(z) + (1-\omega)\hat{f}_2(z)]dz\Big)du \\
&+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u [\omega\hat{g}_1(z) + (1-\omega)\hat{g}_2(z)]dz\Big)du\,ds \\
&-\lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u [\omega\hat{g}_1(z) + (1-\omega)\hat{g}_2(z)]dz\Big)du \\
&+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u [\omega\hat{f}_1(z) + (1-\omega)\hat{f}_2(z)]dz\Big)du\,ds\Big] \\
&+\frac{1}{ER}\Big[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
&+E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
&-RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b [\omega\hat{f}_1(z) + (1-\omega)\hat{f}_2(z)]dz\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&-E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&+RE_4\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)]dz\,ds\Big)\\
&+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz-E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz-E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_2\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)]dz\Big)\\
&+\Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&-E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&+RE_2\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s[\omega\hat{f}_1(z)+(1-\omega)\hat{f}_2(z)]dz\,ds\Big)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
[\omega\tilde{h_1}+(1-\omega)\tilde{h_2}](t)=&\int_a^t\Big(\frac{\mu_2}{q(u)}\int_a^u[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)]dz\Big)du\\
&+\frac{1}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u[\omega\hat{f}_1(z)+(1-\omega)\hat{f}_2(z)]dz\Big)du\\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)]dz\Big)du\,ds\\
&-\beta_2(\alpha_1+\alpha_2)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)]dz\Big)du\\
&+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u[\omega\hat{f}_1(z)+(1-\omega)\hat{f}_2(z)]dz\Big)du\,ds\Big]\\
&+\frac{1}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b[\omega\hat{f}_1(z)+(1-\omega)\hat{f}_2(z)]dz\Big)\\
&+\Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&-E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_3\int_a^t\frac{1}{p(z)}dz\Big)\\
&\times\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)]dz\,ds\Big)\\
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds
\end{aligned}
$$

$$
\begin{aligned}
&-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)]dz\Big)\\
&+\Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&-E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&+RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s[\omega\hat{f}_1(z)+(1-\omega)\hat{f}_2(z)]dz\,ds\Big)\Bigg].
\end{aligned}
$$

Since $S_{F,(u,v)}, S_{G,(u,v)}$ are convex valued as F and G are convex valued maps, therefore, $\omega h_1+(1-\omega)h_2\in\Theta_1, \omega\tilde{h_1}+(1-\omega)\tilde{h_2}\in\Theta_2$ and hence $\omega(h_1,\tilde{h_1})+(1-\omega)(h_2,\tilde{h_2})\in\Theta$.

Now, we show that Θ maps bounded sets into bounded sets in $\mathcal{F}\times\mathcal{F}$. For a positive number ν^*, let $B_{\nu^*}=\{(u,v)\in\mathcal{F}\times\mathcal{F}:\|(u,v)\|\le\nu^*\}$ be a bounded set in $\mathcal{F}\times\mathcal{F}$. Then, for each $h_i\in\Theta_i,(i=1,2),\ (u,v)\in B_{\nu^*}$, there exist $\hat{f}\in S_{F,(u,v)},\hat{g}\in S_{G,(u,v)}$ such that

$$
\begin{aligned}
h_1(u,v)(t) =& \int_a^t\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du+\frac{1}{R}\Bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du\\
&+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du\,ds-\lambda_1\beta_2(\eta-a)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du\\
&+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du\,ds\Bigg]\\
&+\frac{1}{ER}\Bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz-E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz-E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_4\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}(z)dz\Big)+\Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz\\
&+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz\\
&+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_4\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz-E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz-E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_2\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}(z)dz\Big)+\Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz\\
&+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz\\
&+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_2\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}(z)dz\,ds\Big)\Bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
h_2(u,v)(t) =& \int_a^t\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du+\frac{1}{R}\Bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du\\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du\,ds-\beta_2(\alpha_1+\alpha_2)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du
\end{aligned}
$$

$$+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du\,ds\Big]$$
$$+\frac{1}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds$$
$$+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds$$
$$-RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}(z)dz\Big)+\Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz$$
$$+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz$$
$$+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds$$
$$+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds$$
$$-RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}(z)dz\Big)+\Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz$$
$$+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz$$
$$+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}(z)dz\,ds\Big)\Big].$$

Then, for $t\in[a,b]$, we have

$$|h_1(u,v)(t)|\leq\int_a^t\Big(\frac{|\mu_1|}{|p(u)|}\int_a^u|\hat{f}(z)|dz\Big)du+\frac{1}{|R|}\Big[|\alpha_2(\beta_1+\beta_2)|\int_a^b\Big(\frac{|\mu_1|}{|p(u)|}\int_a^u|\hat{f}(z)|dz\Big)du$$
$$+|\lambda_1(\beta_1+\beta_2)|\int_a^\eta\int_a^s\Big(\frac{|\mu_2|}{|q(u)|}\int_a^u|\hat{g}(z)|dz\Big)du\,ds+|\lambda_1\beta_2(\eta-a)|\int_a^b\Big(\frac{|\mu_2|}{|q(u)|}\int_a^u|\hat{g}(z)|dz\Big)du$$
$$+|\lambda_1\lambda_3(\eta-a)|\int_\xi^b\int_a^s\Big(\frac{|\mu_1|}{|p(u)|}\int_a^u|\hat{f}(z)|dz\Big)du\,ds\Big]$$
$$+\frac{1}{|ER|}\Big[\Big(|E_4\alpha_2(\beta_1+\beta_2)|\int_a^b\frac{1}{|p(z)|}dz+|E_3\lambda_1(\beta_1+\beta_2)|\int_a^\eta\int_a^s\frac{1}{|q(z)|}dz\,ds$$
$$+|E_3\lambda_1\beta_2(\eta-a)|\int_a^b\frac{1}{|q(z)|}dz+|E_4\lambda_1\lambda_3(\eta-a)|\int_\xi^b\int_a^s\frac{1}{|p(z)|}dz\,ds$$
$$+|RE_4|\int_a^t\frac{1}{|p(z)|}dz\Big)\Big(\frac{|\alpha_4\mu_1|}{|p(b)|}\int_a^b|\hat{f}(z)|dz\Big)+\Big(|E_4\alpha_2(\beta_1+\beta_2)|\int_a^b\frac{1}{|p(z)|}dz$$
$$+|E_3\lambda_1(\beta_1+\beta_2)|\int_a^\eta\int_a^s\frac{1}{|q(z)|}dz\,ds+|E_3\lambda_1\beta_2(\eta-a)|\int_a^b\frac{1}{|q(z)|}dz$$
$$+|E_4\lambda_1\lambda_3(\eta-a)|\int_\xi^b\int_a^s\frac{1}{|p(z)|}dz\,ds+|RE_4|\int_a^t\frac{1}{|p(z)|}dz\Big)\Big(\int_a^\eta\frac{|\lambda_2\mu_2|}{|q(s)|}\int_a^s|\hat{g}(z)|dz\,ds\Big)$$
$$+\Big(|E_2\alpha_2(\beta_1+\beta_2)|\int_a^b\frac{1}{|p(z)|}dz+|E_1\lambda_1(\beta_1+\beta_2)|\int_a^\eta\int_a^s\frac{1}{|q(z)|}dz\,ds$$
$$+|E_1\lambda_1\beta_2(\eta-a)|\int_a^b\frac{1}{|q(z)|}dz+|E_2\lambda_1\lambda_3(\eta-a)|\int_\xi^b\int_a^s\frac{1}{|p(z)|}dz\,ds$$

$$
\begin{aligned}
& +|RE_2|\int_a^t \frac{1}{|p(z)|}dz\Big)\Big(\frac{|\beta_4\mu_2|}{|q(b)|}\int_a^b |\hat{g}(z)|dz\Big) + \Big(|E_2\alpha_2(\beta_1+\beta_2)|\int_a^b \frac{1}{|p(z)|}dz \\
& +|E_1\lambda_1(\beta_1+\beta_2)|\int_a^\eta\int_a^s \frac{1}{|q(z)|}dz\,ds + |E_1\lambda_1\beta_2(\eta-a)|\int_a^b \frac{1}{|q(z)|}dz \\
& +|E_2\lambda_1\lambda_3(\eta-a)|\int_\xi^b\int_a^s \frac{1}{|p(z)|}dz\,ds + |RE_2|\int_a^t \frac{1}{|p(z)|}dz\Big)\Big(\int_\xi^b \frac{|\lambda_4\mu_1|}{|p(s)|}\int_a^s |\hat{f}(z)|dz\,ds\Big)\Big] \\
\le & \Big\{\frac{\mu_1}{|R\bar{p}|}\Big[\frac{(b-a)^2}{2}\Big(|R|+\alpha_2(\beta_1+\beta_2)\Big) + \frac{\lambda_1\lambda_2(\eta-a)\big[(b-a)^3-(\xi-a)^3\big]}{6}\Big] \\
& +\frac{1}{|RE|}\Big[\Big(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}} + \frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}} + \frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}} \\
& +\frac{E_4\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}} + \frac{RE_4(b-a)}{\bar{p}}\Big)\Big(\frac{\alpha_4\mu_1(b-a)}{|p(b)|}\Big) \\
& +\Big(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}} + \frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}} + \frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}} \\
& +\frac{E_2\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}} + \frac{RE_2(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_4\mu_1\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}\Big)\Big]\Big\} \\
& \times\|p_1\|\Big[\psi_1(\nu^*)+\phi_1(\nu^*)\Big] \\
& +\Big\{\frac{\mu_2}{|2R\bar{q}|}\Big[\frac{\lambda_1(\beta_1+\beta_2)(\eta-a)^3}{3} + \lambda_1\beta_2(\eta-a)(b-a)^2\Big] \\
& +\frac{1}{|RE|}\Big[\Big(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}} + \frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}} + \frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}} \\
& +\frac{E_4\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}} + \frac{RE_4(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_2\mu_2(\eta-a)^2}{2\bar{q}}\Big) \\
& +\Big(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}} + \frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}} + \frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}} \\
& +\frac{E_2\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}} + \frac{RE_2(b-a)}{\bar{p}}\Big)\Big(\frac{\beta_4\mu_2(b-a)}{|q(b)|}\Big)\Big]\Big\} \\
& \times\|p_2\|\Big[\psi_2(\nu^*)+\phi_2(\nu^*)\Big] \\
= & \mathcal{D}_1\|p_1\|[\psi_1(\nu^*)+\phi_1(\nu^*)] + \mathcal{D}_2\|p_2\|[\psi_2(\nu^*)+\phi_2(\nu^*)].
\end{aligned}
$$

Similarly, we can obtain that

$$|h_2(u,v)(t)| \le \mathcal{D}_3\|p_1\|[\psi_1(\nu^*)+\phi_1(\nu^*)] + \mathcal{D}_4\|p_2\|[\psi_2(\nu^*)+\phi_2(\nu^*)].$$

Thus, we get

$$
\begin{aligned}
\|h_1(u,v)\| &\le \mathcal{D}_1\|p_1\|[\psi_1(\nu^*)+\phi_1(\nu^*)] + \mathcal{D}_2\|p_2\|[\psi_2(\nu^*)+\phi_2(\nu^*)], \\
\|h_2(u,v)\| &\le \mathcal{D}_3\|p_1\|[\psi_1(\nu^*)+\phi_1(\nu^*)] + \mathcal{D}_4\|p_2\|[\psi_2(\nu^*)+\phi_2(\nu^*)],
\end{aligned}
$$

where $\mathcal{D}_i, (i=1,2,3,4)$ are defined by (2.5). In consequence, we have

$$
\begin{aligned}
\|(h_1,h_2)\| &= \|h_1(u,v)\| + \|h_2(u,v)\| \\
&\le (\mathcal{D}_1+\mathcal{D}_3)\|p_1\|[\psi_1(\nu^*)+\phi_1(\nu^*)] + (\mathcal{D}_2+\mathcal{D}_4)\|p_2\|[\psi_2(\nu^*)+\phi_2(\nu^*)] \\
&= \mathcal{E}_1\|p_1\|[\psi_1(\nu^*)+\phi_1(\nu^*)] + \mathcal{E}_2\|p_2\|[\psi_2(\nu^*)+\phi_2(\nu^*)] \\
&= \ell \text{ (constant)},
\end{aligned}
$$

where $\mathcal{E}_i, i=1,2$, are defined in (2.4).

Next, we verify that $\Theta(u,v)$ is equicontinuous. Let $t_1,\ t_2 \in [a,b]$ with $t_1 < t_2$. Then, for $\hat{f} \in S_{F,(u,v)}, \hat{g} \in S_{G,(u,v)}$, we get

$$\Big|h_1(u,v)(t_2) - h_1(u,v)(t_1)\Big|$$

$$
\begin{aligned}
&= \Big| \int_a^{t_2} \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}(\tau)dz\Big) du - \int_a^{t_1} \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}(\tau)dz\Big) du \\
&\quad + \Big(\frac{E_4}{E}\Big(\int_a^{t_2} \frac{1}{p(z)}dz - \int_a^{t_1} \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)} \int_a^b \hat{f}(\tau)dz\Big)\Big) \\
&\quad + \Big(\frac{E_4}{E}\Big(\int_a^{t_2} \frac{1}{p(z)}dz - \int_a^{t_1} \frac{1}{p(z)}dz\Big)\Big(\int_a^{\eta} \frac{\lambda_2\mu_2}{q(s)} \int_a^s \hat{g}(\tau)dz\, ds\Big)\Big) \\
&\quad + \Big(\frac{E_2}{E} \int_a^{t_2} \frac{1}{p(z)}dz - \int_a^{t_1} \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)} \int_a^b \hat{g}(\tau)dz\Big)\Big) \\
&\quad + \Big(\frac{E_2}{E}\Big(\int_a^{t_2} \frac{1}{p(z)}dz - \int_a^{t_1} \frac{1}{p(z)}dz\Big)\Big(\int_{\xi}^{b} \frac{\lambda_4\mu_1}{p(s)} \int_a^s \hat{f}(\tau)dz\, ds\Big)\Big) \Big| \\
&\leq \Big[\Big(\frac{\mu_1}{|\bar{p}|}\Big)\frac{(t_2-a)^2-(t_1-a)^2}{2} + \frac{E_4}{E|\bar{p}|}\Big(\frac{\alpha_4\mu_1}{|p(b)|}\Big)(t_2-t_1)(b-a) \\
&\quad + \frac{E_2}{E|\bar{p}|}\frac{(\lambda_4\mu_1)(t_2-t_1)\big[(b-a)^2-(\xi-a)^2\big]}{2}\Big]\|p_1\|[\psi_1(\nu^*)+\phi_1(\nu^*)] \\
&\quad + \Big[\frac{E_4}{E|\bar{p}|}\frac{(\lambda_2\mu_2)(t_2-t_1)(\eta-a)^2}{2\bar{q}} + \frac{E_2}{E|\bar{p}|}\Big(\frac{\beta_4\mu_2}{|q(b)|}\Big)(t_2-t_1)(b-a)\Big] \\
&\quad \times\|p_2\|[\psi_2(\nu^*)+\phi_2(\nu^*)] \to 0 \text{ as } t_2 \to t_1 \text{ independent of } (u,v).
\end{aligned}
$$

Analogously, it can be shown that

$$|h_2(u,v)(t_2) - h_2(u,v)(t_1)| \to 0 \text{ as } t_2 \to t_1 \text{ independent of } (u,v).$$

Therefore, the operator $\Theta(u,v)$ is equicontinuous and hence we deduce that $\Theta(u,v) : \mathcal{F}\times\mathcal{F} \to \mathcal{P}(\mathcal{F}\times\mathcal{F})$ is completely continuous by the Arzelá-Ascoli Theorem.

In the next step, we show that $\Theta(u,v)$ is upper semicontinuous. Instead it will be established that $\Theta(u,v)$ has a closed graph in view of the fact that a completely continuous operator is upper semicontinuous if it has a closed graph. Let $(u_k,v_k) \longrightarrow (u_*,v_*)$ and $(h_k,\tilde{h_k}) \in \Theta(u_k,v_k)$ and $(h_k,\tilde{h_k}) \longrightarrow (h_*,\tilde{h_*})$. Then we have to show that $(h_*,\tilde{h_*}) \in \Theta(u_*,v_*)$. Associated with $(h_k,\tilde{h_k}) \in \Theta(u_k,v_k)$ and $\hat{f}_k \in S_{F,(u,v)}, \hat{g}_k \in S_{G,(u,v)}$, for each $t \in [a,b]$, we have

$$
\begin{aligned}
h_k(u_k,v_k)(t) &= \int_a^t \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_k(z)dz\Big) du + \frac{1}{R}\Big[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_k(z)dz\Big) du \\
&\quad + \lambda_1(\beta_1+\beta_2)\int_a^{\eta}\int_a^s \Big(\frac{\mu_2}{q(u)} \int_a^u \hat{g}_k(z)dz\Big) du\, ds - \lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)} \int_a^u \hat{g}_k(z)dz\Big) du \\
&\quad + \lambda_1\lambda_3(\eta-a)\int_{\xi}^b\int_a^s \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_k(z)dz\Big) du\, ds\Big] \\
&\quad + \frac{1}{ER}\Big[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^{\eta}\int_a^s \frac{1}{q(z)}dz\, ds \\
&\quad + E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_{\xi}^b\int_a^s \frac{1}{p(z)}dz\, ds \\
&\quad - RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)} \int_a^b \hat{f}_k(z)dz\Big) + \Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz \\
&\quad + E_3\lambda_1(\beta_1+\beta_2)\int_a^{\eta}\int_a^s \frac{1}{q(z)}dz\, ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz \\
&\quad + E_4\lambda_1\lambda_3(\eta-a)\int_{\xi}^b\int_a^s \frac{1}{p(z)}dz\, ds + RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^{\eta} \frac{\lambda_2\mu_2}{q(s)} \int_a^s \hat{g}_k(z)dz\, ds\Big) \\
&\quad + \Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^{\eta}\int_a^s \frac{1}{q(z)}dz\, ds
\end{aligned}
$$

$$
\begin{aligned}
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_k(z)dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz\\
&+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz\\
&+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_k(z)dz\,ds\Big)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{h_k}(u_k,v_k)(t) =& \int_a^t \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)dz\Big)du\\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)dz\Big)du\,ds - \beta_2(\alpha_1+\alpha_2)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)dz\Big)du\\
&+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)dz\Big)du\,ds\Big]\\
&+\frac{1}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_k(z)dz\Big) + \Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz\\
&+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz\\
&+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_k(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_k(z)dz\Big) + \Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz\\
&+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz\\
&+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_k(z)dz\,ds\Big)\Big].
\end{aligned}
$$

Consider the continuous linear operators $\Psi_1, \Psi_2 : L^1([a,b],\mathcal{F}\times\mathcal{F}) \longrightarrow C([a,b],\mathcal{F}\times\mathcal{F})$ given by

$$
\begin{aligned}
\Psi_1(u,v)(t) =& \int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du\\
&+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du\,ds - \lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du\\
&+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du\,ds\Big]
\end{aligned}
$$

$$+\frac{1}{ER}\Bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}(z)dz\Big) + \Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz$$
$$+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz$$
$$+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz$$
$$+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz$$
$$+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)dz\,ds\Big)\Bigg],$$

and

$$\Psi_2(u,v)(t) = \int_a^t \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du + \frac{1}{R}\Bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du$$
$$+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du\,ds - \beta_2(\alpha_1+\alpha_2)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du$$
$$+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du\,ds\Bigg]$$
$$+\frac{1}{ER}\Bigg[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}(z)dz\Big) + \Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz$$
$$+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)dz\Big) + \Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz$$

$$+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)dz\,ds\Big)\Big].$$

From Lemma 3.1, we know that $(\Psi_1,\Psi_2)\circ(S_F,S_G)$ are closed graph operators. Moreover, we have $(h_k,\tilde{h_k})\in(\Psi_1,\Psi_2)\circ(S_{F,(u_k,v_k)},S_{G,(u_k,v_k)})$ for all k. Since $(u_k,v_k)\longrightarrow(u_*,v_*)$, $(h_k,\tilde{h_k})\longrightarrow(h_*,\tilde{h_*})$, it follows that $\hat{f}_*\in S_{F,(u,v)},\hat{g}_*\in S_{G,(u,v)}$ such that

$$\begin{aligned}
h_*(u_*,v_*)(t) =& \int_a^t\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_*(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2(\beta_1+\beta_2)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_*(z)dz\Big)du\\
&+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_*(z)dz\Big)du\,ds - \lambda_1\beta_2(\eta-a)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_*(z)dz\Big)du\\
&+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_*(z)dz\Big)du\,ds\Big]\\
&+\frac{1}{ER}\Big[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&+E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_*(z)dz\Big) + \Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz\\
&+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz\\
&+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_*(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_*(z)dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz\\
&+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz\\
&+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_*(z)dz\,ds\Big)\Big],
\end{aligned}$$

and

$$\begin{aligned}
\tilde{h_*}(u_*,v_*)(t) =& \int_a^t\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_*(z)dz\Big)du + \frac{1}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_*(z)dz\Big)du\\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_*(z)dz\Big)du\,ds - \beta_2(\alpha_1+\alpha_2)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_*(z)dz\Big)du\\
&+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_*(z)dz\Big)du\,ds\Big]\\
&+\frac{1}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds
\end{aligned}$$

$$
\begin{aligned}
&+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_*(z)dz\Big)+\Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz\\
&+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz\\
&+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds + RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_*(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}_*(z)dz\Big)+\Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz\\
&+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz\\
&+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds + RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}_*(z)dz\,ds\Big)\Bigg],
\end{aligned}
$$

which lead to the conclusion that $(h_k,\tilde{h_k})\in\Theta(u_*,v_*)$.

Finally, we show that there exists an open set $U\subseteq\mathcal{F}\times\mathcal{F}\to\mathcal{P}(\mathcal{F}\times\mathcal{F})$ with $(u,v)\notin\epsilon\Theta(u,v)$ for any $\epsilon\in(0,1)$ and all $(u,v)\in\partial U$. Let $\epsilon\in(0,1)$ and $(u,v)\in\epsilon\Theta(u,v)$. Then there exist $\hat{f}\in S_{F,(u,v)}$ and $\hat{g}\in S_{G,(u,v)}$ such that, for $t\in[a,b]$, we have

$$
\begin{aligned}
u(t) =\ &\epsilon\int_a^t\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du + \frac{\epsilon}{R}\Bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du\\
&+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du\,ds - \lambda_1\beta_2(\eta-a)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du\\
&+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du\,ds\Bigg]\\
&+\frac{\epsilon}{ER}\Bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_4\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}(z)dz\Big)+\Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz\\
&+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz\\
&+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds + RE_4\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_2\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}(z)dz\Big)+\Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz
\end{aligned}
$$

$$+E_1\lambda_1(\beta_1+\beta_2)\int_a^{\eta}\int_a^{s}\frac{1}{q(z)}dz\,ds-E_1\lambda_1\beta_2(\eta-a)\int_a^{b}\frac{1}{q(z)}dz$$
$$+E_2\lambda_1\lambda_3(\eta-a)\int_{\xi}^{b}\int_a^{s}\frac{1}{p(z)}dz\,ds+RE_2\int_a^{t}\frac{1}{p(z)}dz\Big)\Big(\int_{\xi}^{b}\frac{\lambda_4\mu_1}{p(s)}\int_a^{s}\hat{f}(z)dz\,ds\Big)\Big],$$

and

$$\begin{aligned}
v(t) = {} & \epsilon\int_a^t\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du+\frac{\epsilon}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du\\
&+\lambda_1\lambda_3(b-\xi)\int_a^{\eta}\int_a^{s}\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du\,ds-\beta_2(\alpha_1+\alpha_2)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)dz\Big)du\\
&+\lambda_3(\alpha_1+\alpha_2)\int_{\xi}^{b}\int_a^{s}\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)dz\Big)du\,ds\Big]\\
&+\frac{\epsilon}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_3\lambda_1\lambda_3(b-\xi)\int_a^{\eta}\int_a^{s}\frac{1}{q(z)}dz\,ds\\
&+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_4\lambda_3(\alpha_1+\alpha_2)\int_{\xi}^{b}\int_a^{s}\frac{1}{p(z)}dz\,ds\\
&-RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}(z)dz\Big)+\Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz\\
&+E_3\lambda_1\lambda_3(b-\xi)\int_a^{\eta}\int_a^{s}\frac{1}{q(z)}dz\,ds-E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz\\
&+E_4\lambda_3(\alpha_1+\alpha_2)\int_{\xi}^{b}\int_a^{s}\frac{1}{p(z)}dz\,ds+RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^{\eta}\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_1\lambda_1\lambda_3(b-\xi)\int_a^{\eta}\int_a^{s}\frac{1}{q(z)}dz\,ds\\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_2\lambda_3(\alpha_1+\alpha_2)\int_{\xi}^{b}\int_a^{s}\frac{1}{p(z)}dz\,ds\\
&-RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}(z)dz\Big)+\Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz\\
&+E_1\lambda_1\lambda_3(b-\xi)\int_a^{\eta}\int_a^{s}\frac{1}{q(z)}dz\,ds-E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz\\
&+E_2\lambda_3(\alpha_1+\alpha_2)\int_{\xi}^{b}\int_a^{s}\frac{1}{p(z)}dz\,ds+RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_{\xi}^{b}\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}(z)dz\,ds\Big)\Big].
\end{aligned}$$

Using the arguments employed in the second step, we find that

$$\|u\|\leq\mathcal{D}_1\|p_1\|[\psi_1(\|u\|)+\phi_1(\|v\|)]+\mathcal{D}_2\|p_2\|[\psi_2(\|u\|)+\phi_2(\|v\|)],$$

and

$$\|v\|\leq\mathcal{D}_3\|p_1\|[\psi_1(\|u\|)+\phi_1(\|v\|)]+\mathcal{D}_4\|p_2\|[\psi_2(\|u\|)+\phi_2(\|v\|)].$$

Then we have

$$\begin{aligned}
\|(u,v)\|\|\|u\|+\|v\| &\leq (\mathcal{D}_1+\mathcal{D}_3)\|p_1\|[\psi_1(\|u\|)+\phi_1(\|v\|)]+(\mathcal{D}_2+\mathcal{D}_4)\|p_2\|[\psi_2(\|u\|)+\phi_2(\|v\|)]\\
&\leq \mathcal{E}_1\|p_1\|[\psi_1(\|u\|)+\phi_1(\|v\|)]+\mathcal{E}_2\|p_2\|[\psi_2(\|u\|)+\phi_2(\|v\|)],
\end{aligned}$$

where $\mathcal{E}_i, i=1,2$, are given by (2.4). Consequently, we have

$$\frac{\|(u,v)\|}{\mathcal{E}_1\|p_1\|[\psi_1(\|u\|)+\phi_1(\|v\|)]+\mathcal{E}_2\|p_2\|[\psi_2(\|u\|)+\phi_2(\|v\|)]}\leq 1.$$

According to (H_3), there exists N such that $\|(u,v)\| \neq N$. Let us set

$$U = \{(u,v) \in (\mathcal{F} \times \mathcal{F}) : \|(u,v)\| < N\}.$$

Observe that the operator $\Theta : \bar{U} \longrightarrow \mathcal{P}_{cp,cv}(\mathcal{F}) \times \mathcal{P}_{cp,cv}(\mathcal{F})$ is completely continuous and upper semicontinuous. From the choice of U, there is no $(u,v) \in \partial U$ such that $(u,v) \in \epsilon\Theta(u,v)$ for some $\epsilon \in (0,1)$. Therefore, by nonlinear alternative of Leray-Schauder type (Lemma 3.2), we deduce that Θ has a fixed point $(u,v) \in \bar{U}$ which is a solution of the problem (1.1). □

4. The Lipschitz case.

The forthcoming result is based on the fixed point theorem for contraction multivalued operators due to Covitz-Nadler [9], which is stated below.

Lemma 4.1. *(Covitz-Nadler) Let (X,d) be a complete metric space. If $G : X \to P_{cl}(X)$ is a contraction, then $FixG \neq \emptyset$.*

Remark 4.2. *Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by*

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [21]).

Theorem 4.3. *Assume that the following conditions hold:*

(H_5) *$F, G : [a,b] \times \mathbb{R}^2 \to \mathcal{P}_{cp}(\mathbb{R})$ are such that $F(\cdot,u,v), G(\cdot,u,v) : [a,b] \to \mathcal{P}_{cp}(\mathbb{R})$ are measurable for each $u,v \in \mathbb{R}$;*

(H_6) *For almost all $t \in [a,b]$ and $u,v,\bar{u},\bar{v} \in \mathbb{R}$ with $\mathcal{B}_1, \mathcal{B}_2 \in C([a,b],\mathbb{R}^+)$,*

$$H_d(F(t,u,v), F(t,\bar{u},\bar{v}) \leq \mathcal{B}_1(t)(|u-\bar{u}| + |v-\bar{v}|),\ H_d(G(t,u,v), G(t,\bar{u},\bar{v}) \leq \mathcal{B}_2(t)(|u-\bar{u}| + |v-\bar{v}|),$$

and $d(0, F(t,0,0)) \leq \mathcal{B}_1(t), d(0, G(t,0,0)) \leq \mathcal{B}_2(t)$.

Then the self-adjoint coupled multi-valued system (1.1) has at least one solution on $[a,b]$ if

$$\mathcal{E}_1\|\mathcal{B}_1\| + \mathcal{E}_2\|\mathcal{B}_2\| < 1,$$

where $\mathcal{E}_1, \mathcal{E}_2$ are given in (2.4).

Proof. Consider the operators $\Theta_1, \Theta_2 : \mathcal{F} \times \mathcal{F} \to \mathcal{P}(\mathcal{F} \times \mathcal{F})$ defined by (2.2) and (2.3) respectively.

Notice that the sets $S_{F,(u,v)}$ and $S_{G,(u,v)}$ are nonempty and consequently $\Theta \neq \emptyset$ for each $(u,v) \in \mathcal{F} \times \mathcal{F}$. Then, by the assumption (H_5), the multivalued maps $F(\cdot,(u,v))$ and $G(\cdot,(u,v))$ are measurable, and thus admit measurable selections.

Now we shall show that the operator $\Theta(u,v)$ satisfies the hypothesis of Lemma 4.1. Firstly, we verify that $\Theta(u,v) \in \mathcal{P}_{cl}(\mathcal{F}) \times \mathcal{P}_{cl}(\mathcal{F})$ for each $(u,v) \in \mathcal{F} \times \mathcal{F}$. Let $(h_k, \tilde{h}_k) \in \Theta(u_k, v_k)$ such that $(h_k, \tilde{h}_k)$ converges to $(h, \tilde{h})$ as $k \to \infty$ in $\mathcal{F} \times \mathcal{F}$. So $(h, \tilde{h}) \in \mathcal{F} \times \mathcal{F}$ and there exist $\hat{f}_k \in S_{F,(u_k,v_k)}$ and $\hat{g}_k \in S_{G,(u_k,v_k)}$ such that, for each $t \in [a,b]$, we have

$$\begin{aligned}
h_k(u_k,v_k)(t) = & \int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)dz\Big)du + \frac{1}{R}\Bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)dz\Big)du \\
& +\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)dz\Big)du\,ds - \lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)dz\Big)du \\
& +\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)dz\Big)du\,ds\Bigg] \\
& +\frac{1}{ER}\Bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
& +E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds
\end{aligned}$$

$$
\begin{aligned}
&-RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_k(z)dz\Big)+\Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz\\
&+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds-E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz\\
&+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds+RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_k(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz-E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz-E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_k(z)dz\Big)+\Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz\\
&+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds-E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz\\
&+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds+RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_k(z)dz\,ds\Big)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{h_k}(u_k,v_k)(t)=&\int_a^t\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)dz\Big)du+\frac{1}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)dz\Big)du\\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)dz\Big)du\,ds-\beta_2(\alpha_1+\alpha_2)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)dz\Big)du\\
&+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)dz\Big)du\,ds\Big]\\
&+\frac{1}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz-E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz-E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_k(z)dz\Big)+\Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz\\
&+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds-E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz\\
&+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds+RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_k(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz-E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds\\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz-E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds\\
&-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_k(z)dz\Big)+\Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz\\
&+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds-E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz\\
&+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds+RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_k(z)dz\,ds\Big)\Big].
\end{aligned}
$$

Since F and G have compact values, we pass onto a subsequences (if necessary) to get that $\hat{f}_k$ and $\hat{g}_k$ converge to $\hat{f}$ and $\hat{g}$ in

$L^1([a,b],\mathbb{R})$ respectively. Then $\hat{f} \in S_{F,(u,v)}$ and $\hat{g} \in S_{G,(u,v)}$ and for each $t \in [a,b]$, we have

$$
\begin{aligned}
& h_k(u_k,v_k)(t) \to h(u,v)(t) \\
= & \int_a^t \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}(z)dz\Big)du + \frac{1}{R}\bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du \\
& +\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du\,ds - \lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du \\
& +\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du\,ds\bigg] \\
& +\frac{1}{ER}\bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
& +E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
& -RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}(z)dz\Big) + \Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz \\
& +E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz \\
& +E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)dz\,ds\Big) \\
& +\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
& +E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
& -RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz \\
& +E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz \\
& +E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)dz\,ds\Big)\bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
& \tilde{h_k}(u_k,v_k)(t) \to \tilde{h}(u,v)(t) \\
= & \int_a^t \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du + \frac{1}{R}\bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du \\
& +\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du\,ds - \beta_2(\alpha_1+\alpha_2)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)dz\Big)du \\
& +\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)dz\Big)du\,ds\bigg] \\
& +\frac{1}{ER}\bigg[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds \\
& +E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds \\
& -RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}(z)dz\Big) + \Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz
\end{aligned}
$$

$$+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz$$
$$+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds$$
$$+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds$$
$$-RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}(z)dz\Big)+\Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz$$
$$+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz$$
$$+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}(z)dz\,ds\Big)\Bigg].$$

Therefore $(u,v)\in\Theta$ and hence $\Theta(u,v)$ is closed.

Next we show that Θ is a contraction on $\mathcal{P}_{cl}(\mathcal{F})\times\mathcal{P}_{cl}(\mathcal{F})$, that is, there exists a positive number $\gamma<1$ such that

$$H_d(\Theta(u,v),\Theta(\bar{u},\bar{v}))\le\gamma(\|u-\bar{u}\|+\|v-\bar{v}\|)\ \text{ for each }\ u,v,\bar{u},\bar{v}\in\mathcal{F}.$$

Let $(u,\bar{u}),(v,\bar{v})\in\mathcal{F}\times\mathcal{F}$, and $(h_1,\tilde{h_1})\in\Theta(u,v)$. Then there exist $\hat{f}_1(t)\in S_{F,(u,v)}$ and $\hat{g}_1(t)\in S_{G,(u,v)}$ such that, for each $t\in[a,b]$, we obtain

$$h_1(u,v)(t)=\int_a^t\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}_1(z)dz\Big)du+\frac{1}{R}\Bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}_1(z)dz\Big)du$$
$$+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_1(z)dz\Big)du\,ds-\lambda_1\beta_2(\eta-a)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_1(z)dz\Big)du$$
$$+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}_1(z)dz\Big)du\,ds\Bigg]$$
$$+\frac{1}{ER}\Bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz-E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds$$
$$+E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz-E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds$$
$$-RE_4\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}_1(z)dz\Big)+\Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz$$
$$+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz$$
$$+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_4\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}_1(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz-E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds$$
$$+E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz-E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds$$
$$-RE_2\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}_1(z)dz\Big)+\Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}dz$$
$$+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}dz$$

$$+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_2\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}_1(z)dz\,ds\Big)\Big],$$

and

$$\begin{aligned}\tilde{h}_1(u,v)(t) = &\int_a^t\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_1(z)dz\Big)du+\frac{1}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}_1(z)dz\Big)du\\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_1(z)dz\Big)du\,ds-\beta_2(\alpha_1+\alpha_2)\int_a^b\Big(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_1(z)dz\Big)du\\
&+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\Big(\frac{\mu_1}{p(u)}\int_a^u\hat{f}_1(z)dz\Big)du\,ds\Big]\\
&+\frac{1}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}_1(z)dz\Big)+\Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz\\
&+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz\\
&+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_3\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}_1(z)dz\,ds\Big)\\
&+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz-E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds\\
&+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz-E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds\\
&-RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}_1(z)dz\Big)+\Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}dz\\
&+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}dz\,ds-E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}dz\\
&+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}dz\,ds+RE_1\int_a^t\frac{1}{p(z)}dz\Big)\Big(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}_1(z)dz\,ds\Big)\Big].\end{aligned}$$

By (H_6), we have that

$$H_d(F(t,u,v),F(t,\bar{u},\bar{v}))\le\mathcal{B}_1(t)(|u(t)-\bar{u}(t)|+|v(t)-\bar{v}(t)|),$$

and

$$H_d(G(t,u,v),G(t,\bar{u},\bar{v}))\le\mathcal{B}_2(t)(|u(t)-\bar{u}(t)|+|v(t)-\bar{v}(t)|).$$

So there exist $\hat{\vartheta}_f\in F(t,u(t),v(t))$ and $\hat{\vartheta}_g\in G(t,u(t),v(t))$ such that

$$|\hat{f}_1(t)-\hat{\vartheta}_f|\le\mathcal{B}_1(t)(|u(t)-\bar{u}(t)|+|v(t)-\bar{v}(t)|),$$

$$|\hat{g}_1(t)-\hat{\vartheta}_g|\le\mathcal{B}_2(t)(|u(t)-\bar{u}(t)|+|v(t)-\bar{v}(t)|).$$

Define $W_1,W_2:[a,b]\to\mathcal{P}(\mathbb{R})$ by

$$W_1(t)=\{\hat{\vartheta}_f\in L^1([a,b),\mathbb{R}):|\hat{f}_1(t)-\hat{\vartheta}_f|\le\mathcal{B}_1(t)(|u(t)-\bar{u}(t)|+|v(t)-\bar{v}(t)|)\},$$

and

$$W_2(t)=\{\hat{\vartheta}_g\in L^1([a,b),\mathbb{R}):|\hat{g}_1(t)-\hat{\vartheta}_g|\le\mathcal{B}_2(t)(|u(t)-\bar{u}(t)|+|v(t)-\bar{v}(t)|)\}.$$

Since the multivalued operators $W_1(t) \cap F(t, u(t), v(t))$ and $W_2(t) \cap G(t, u(t), v(t))$ are measurable, there exist functions $\hat{f}_2(t), \hat{g}_2(t)$ which are measurable selections for W_1 and W_2. Thus $\hat{f}_2(t) \in F(t, u(t), v(t)), \hat{g}_2(t) \in G(t, u(t), v(t))$ and for each $t \in [a, b]$, we have

$$|\hat{f}_1(t) - \hat{f}_2(t)| \leq \mathcal{B}_1(t)(|u(t) - \bar{u}(t)| + |v(t) - \bar{v}(t)|),$$

and

$$|\hat{g}_1(t) - \hat{g}_2(t)| \leq \mathcal{B}_2(t)(|u(t) - \bar{u}(t)| + |v(t) - \bar{v}(t)|).$$

For each $t \in [a, b]$, let us define

$$\begin{aligned}
h_2(u,v)(t) = & \int_a^t \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_2(z)dz\Big)du + \frac{1}{R}\bigg[- \alpha_2(\beta_1+\beta_2) \int_a^b \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_2(z)dz\Big)du \\
& + \lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \Big(\frac{\mu_2}{q(u)} \int_a^u \hat{g}_2(z)dz\Big)du\, ds - \lambda_1\beta_2(\eta - a) \int_a^b \Big(\frac{\mu_2}{q(u)} \int_a^u \hat{g}_2(z)dz\Big)du \\
& + \lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_2(z)dz\Big)du\, ds\bigg] \\
& + \frac{1}{ER}\bigg[\Big(E_4\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}dz\, ds \\
& + E_3\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}dz\, ds \\
& - RE_4 \int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)} \int_a^b \hat{f}_2(z)dz\Big) + \Big(- E_4\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}dz \\
& + E_3\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}dz\, ds - E_3\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}dz \\
& + E_4\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}dz\, ds + RE_4 \int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)} \int_a^s \hat{g}_2(z)dz\, ds\Big) \\
& + \Big(E_2\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}dz\, ds \\
& + E_1\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}dz\, ds \\
& - RE_2 \int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)} \int_a^b \hat{g}_2(z)dz\Big) + \Big(- E_2\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}dz \\
& + E_1\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}dz\, ds - E_1\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}dz \\
& + E_2\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}dz\, ds + RE_2 \int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)} \int_a^s \hat{f}_2(z)dz\, ds\Big)\bigg],
\end{aligned}$$

and

$$\begin{aligned}
\tilde{h}_2(u,v)(t) = & \int_a^t \Big(\frac{\mu_2}{q(u)} \int_a^u \hat{g}_2(z)dz\Big)du + \frac{1}{R}\bigg[- \alpha_2\lambda_3(b-\xi) \int_a^b \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_2(z)dz\Big)du \\
& + \lambda_1\lambda_3(b-\xi) \int_a^\eta \int_a^s \Big(\frac{\mu_2}{q(u)} \int_a^u \hat{g}_2(z)dz\Big)du\, ds - \beta_2(\alpha_1+\alpha_2) \int_a^b \Big(\frac{\mu_2}{q(u)} \int_a^u \hat{g}_2(z)dz\Big)du \\
& + \lambda_3(\alpha_1+\alpha_2) \int_\xi^b \int_a^s \Big(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_2(z)dz\Big)du\, ds\bigg] \\
& + \frac{1}{ER}\bigg[\Big(E_4\alpha_2\lambda_3(b-\xi) \int_a^b \frac{1}{p(z)}dz - E_3\lambda_1\lambda_3(b-\xi) \int_a^\eta \int_a^s \frac{1}{q(z)}dz\, ds \\
& + E_3\beta_2(\alpha_1+\alpha_2) \int_a^b \frac{1}{q(z)}dz - E_4\lambda_3(\alpha_1+\alpha_2) \int_\xi^b \int_a^s \frac{1}{p(z)}dz\, ds
\end{aligned}$$

$$-RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_2(z)dz\Big)+\Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz$$
$$+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds-E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds+RE_3\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_2(z)dz\,ds\Big)$$
$$+\Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz-E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz-E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_2(z)dz\Big)+\Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}dz$$
$$+E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds-E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}dz$$
$$+E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds+RE_1\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_2(z)dz\,ds\Big)\Bigg].$$

Then

$$\big|h_1(u,v)(t)-h_2(u,v)(t)\big|$$
$$\leq \int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u \big|\hat{f}_1(z)-\hat{f}_2(z)\big|dz\Big)du$$
$$+\frac{1}{R}\Bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \big|\hat{f}_1(z)-\hat{f}_2(z)\big|dz\Big)du$$
$$+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \big|\hat{g}_1(z)-\hat{g}_2(z)\big|dz\Big)du\,ds$$
$$-\lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \big|\hat{g}_1(z)-\hat{g}_2(z)\big|dz\Big)du$$
$$+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \big|\hat{f}_1(z)-\hat{f}_2(z)\big|dz\Big)du\,ds\Bigg]$$
$$+\frac{1}{ER}\Bigg[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz-E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$
$$+E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz-E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$
$$-RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \big|\hat{f}_1(z)-\hat{f}_2(z)\big|dz\Big)+\Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz$$
$$+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds-E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz$$
$$+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds+RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \big|\hat{g}_1(z)-\hat{g}_2(z)\big|dz\,ds\Big)$$

$$+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$

$$+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$

$$-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \big|\hat{g}_1(z)-\hat{g}_2(z)\big|dz\Big) + \Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz$$

$$+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz$$

$$+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \big|\hat{f}_1(z)-\hat{f}_2(z)\big|dz\,ds\Big)\Big]$$

$$\leq \int_a^t \Big(\frac{\mu_1}{p(u)}\int_a^u \mathcal{B}_1(z)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\Big)du$$

$$+\frac{1}{R}\Big[-\alpha_2(\beta_1+\beta_2)\int_a^b \Big(\frac{\mu_1}{p(u)}\int_a^u \mathcal{B}_1(z)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\Big)du$$

$$+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \Big(\frac{\mu_2}{q(u)}\int_a^u \mathcal{B}_2(z)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\Big)du\,ds$$

$$-\lambda_1\beta_2(\eta-a)\int_a^b \Big(\frac{\mu_2}{q(u)}\int_a^u \mathcal{B}_2(z)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\Big)du$$

$$+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \Big(\frac{\mu_1}{p(u)}\int_a^u \mathcal{B}_1(z)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\Big)du\,ds\Big]$$

$$+\frac{1}{ER}\Big[\Big(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$

$$+E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds$$

$$-RE_4\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \mathcal{B}_1(z)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\Big)$$

$$+\Big(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz + E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$

$$-E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz + E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\,ds + RE_4\int_a^t \frac{1}{p(z)}dz\Big)$$

$$\times\Big(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \mathcal{B}_2(z)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\,ds\Big)$$

$$+\Big(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\,ds$$

$$
\begin{aligned}
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\, ds \\
&-RE_2\int_a^t \frac{1}{p(z)}dz\Big)\Big(\frac{\beta_4\mu_2}{q(b)}\int_a^b \mathcal{B}_2(\tau)(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)dz\Big) \\
&+\Big(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}dz + E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}dz\, ds \\
&-E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}dz + E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}dz\, ds + RE_2\int_a^t \frac{1}{p(z)}dz\Big) \\
&\times\Big(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \mathcal{B}_1(\tau)(|u(\tau)-\bar{u}(z)|+|v(z)-\bar{v}(z)|)|dz\, ds\Big)\Big] \\
\le&\ \Big\{\frac{\mu_1}{|R\bar{p}|}\Big[\frac{(b-a)^2}{2}\Big(|R|+\alpha_2(\beta_1+\beta_2)\Big)+\frac{\lambda_1\lambda_2(\eta-a)\big[(b-a)^3-(\xi-a)^3\big]}{6}\Big] \\
&+\frac{1}{|RE|}\Big[\Big(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}} \\
&+\frac{E_4\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_4(b-a)}{\bar{p}}\Big)\Big(\frac{\alpha_4\mu_1(b-a)}{|p(b)|}\Big) \\
&+\Big(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}} \\
&+\frac{E_2\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_2(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_4\mu_1\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}\Big)\Big]\Big\} \\
&\times\|\mathcal{B}_1\|\big(\|u-\bar{u}\|+\|v-\bar{v}\|\big) \\
&+\Big\{\frac{\mu_2}{|2R\bar{q}|}\Big[\frac{\lambda_1(\beta_1+\beta_2)(\eta-a)^3}{3}+\lambda_1\beta_2(\eta-a)(b-a)^2\Big] \\
&+\frac{1}{|RE|}\Big[\Big(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}} \\
&+\frac{E_4\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_4(b-a)}{\bar{p}}\Big)\Big(\frac{\lambda_2\mu_2(\eta-a)^2}{2\bar{q}}\Big) \\
&+\Big(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}} \\
&+\frac{E_2\lambda_1\lambda_3(\eta-a)\big[(b-a)^2-(\xi-a)^2\big]}{2\bar{p}}+\frac{RE_2(b-a)}{\bar{p}}\Big)\Big(\frac{\beta_4\mu_2(b-a)}{|q(b)|}\Big)\Big]\Big\} \\
&\times\|\mathcal{B}_2\|\big(\|u-\bar{u}\|+\|v-\bar{v}\|\big) \\
\le&\ (\mathcal{D}_1\|\mathcal{B}_1\|+\mathcal{D}_2\|\mathcal{B}_2\|)(\|u-\bar{u}\|+\|v-\bar{v}\|),
\end{aligned}
$$

which implies that

$$|h_1(u,v)(t)-h_2(u,v)(t)| \le (\mathcal{D}_1\|\mathcal{B}_1\|+\mathcal{D}_2\|\mathcal{B}_2\|)(\|u-\bar{u}\|+\|v-\bar{v}\|).$$

In a similar manner, one can be establish that

$$|\tilde{h}_1(u,v)(t) - \tilde{h}_2(u,v)(t)| \le (\mathcal{D}_3\|\mathcal{B}_1\| + \mathcal{D}_4\|\mathcal{B}_2\|)(\|u-\bar{u}\| + \|v-\bar{v}\|).$$

In consequence, we get

$$\begin{aligned}\|(h_1,h_2),(\tilde{h}_1,\tilde{h}_2)\| &\le [(\mathcal{D}_1+\mathcal{D}_3)\|\mathcal{B}_1\| + (\mathcal{D}_2+\mathcal{D}_4)\|\mathcal{B}_2\|)](\|u-\bar{u}\| + \|v-\bar{v}\|)\\ &\le [(\mathcal{E}_1\|\mathcal{B}_1\| + \mathcal{E}_2\|\mathcal{B}_2\|)](\|u-\bar{u}\| + \|v-\bar{v}\|).\end{aligned}$$

Similarly, by interchanging the roles of (u,v) and $(\bar{u},\bar{v})$, we can obtain that

$$H_d(\Theta(u,v),\Theta(\bar{u},\bar{v})) \le [(\mathcal{E}_1\|\mathcal{B}_1\| + \mathcal{E}_2\|\mathcal{B}_2\|)](\|u-\bar{u}\| + \|v-\bar{v}\|).$$

Therefore, it follows by the assumption: $\mathcal{E}_1\|\mathcal{B}_1\| + \mathcal{E}_2\|\mathcal{B}_2\| < 1$ that Θ is a contraction, So, by Lemma 4.1, Θ has a fixed point (u,v), which is a solution of the problem (1.1). The proof is finished. □

5. Examples

Example 5.1. *Consider the following self-adjoint coupled system of second-order ordinary differential inclusions with boundary conditions*

$$\begin{cases}\Big(\Big(\dfrac{1}{t+13}\Big)u'(t)\Big)' \in \mu_1 F(t,u,v),\ t\in[0,3],\\ \Big(\dfrac{8}{4t^2+2t+12}\,v'(t)\Big)' \in \mu_2 G(t,u,v),\ t\in[0,3],\\ \dfrac{7}{3}u(0)+\dfrac{5}{3}u(3) = \dfrac{1}{7}\displaystyle\int_0^{\frac{1}{2}} v(s)ds,\ \ \dfrac{4}{3}u'(0)+u'(3) = \dfrac{2}{7}\displaystyle\int_0^{\frac{1}{2}} v'(s)ds,\\ \dfrac{1}{9}v(0)+\dfrac{2}{9}v(3) = \dfrac{3}{7}\displaystyle\int_{\frac{5}{2}}^{3} u(s)ds,\ \ \dfrac{3}{9}v'(0)+\dfrac{4}{9}v'(3) = \dfrac{4}{7}\displaystyle\int_{\frac{5}{2}}^{3} u'(s)ds.\end{cases} \tag{5.1}$$

Here $p(t)=1/(t+13), q(t)=8/(4t^2+2t+12), \mu_1=3/36, \mu_2=2/93, a=0, b=3, \eta=1/2, \xi=5/2, \lambda_1=1/7, \lambda_2=2/7, \lambda_3=3/7, \lambda_4=4/7, \alpha_1=7/3, \alpha_2=5/3, \alpha_3=4/3, \alpha_4=1, \beta_1=1/9, \beta_2=2/9, \beta_3=3/9, \beta_4=4/9$, and $F(t,u,v), G(t,u,v)$ will be fixed later.

Using the given data, we find that $|R|\approx 1.323129 \neq 0$, $|E|\approx 115.6354\neq 0$, $\bar{p}\approx 0.0625$, $\bar{q}=0.148148$, $\mathcal{D}_1\approx 17.1389708$, $\mathcal{D}_2\approx 0.06036034$, $\mathcal{D}_3\approx 38.2023705$, $\mathcal{D}_4\approx 4.565128967$, $\mathcal{E}_1\approx 17.19933114$ and $\mathcal{E}_2\approx 42.76749946$ ($\bar{p},\bar{q}$ and $\mathcal{D}_i$ $(i=1,2,3,4)$ are defined in (2.5), while $\mathcal{E}_1,\mathcal{E}_2$ are given in (2.4)).

(a) For illustration of Theorem 3.3, we choose

$$F(t,u,v) = \Big(\frac{t}{108t^2+32}\Big)\Big[\frac{|u(t)|}{\sqrt{|u(t)|^2+65}}, \frac{|v(t)|^2}{|v(t)|^2+1}\Big],$$

and

$$G(t,u,v) = \Big(\frac{t^2+1}{t^3+120}\Big)\Big[\frac{|u(t)|}{|u(t)|+1}, \frac{|v(t)|^3}{1+|v(t)|^3}\Big].$$

For $f\in F$, we have

$$|f| \le \max\Big\{\Big(\frac{t}{108t^2+32}\Big[\frac{|u(t)|}{\sqrt{|u(t)|^2+65}}, \frac{|v(t)|^2}{|v(t)|^2+1}\Big]\Big\} \le 2\Big\{\frac{t}{108t^2+32}\Big\}, u,v\in\mathbb{R}, t\in[0,3],$$

and for $g\in G$, we have

$$|g| \le \max\Big\{\Big(\frac{t^2+1}{t^3+120}\Big)\Big[\frac{|u(t)|}{|u(t)|+1}, \frac{|v(t)|^3}{1+|v(t)|^3}\Big]\Big\} \le 2\Big\{\frac{t^2+1}{t^3+120}\Big\}, u,v\in\mathbb{R}, t\in[0,3].$$

Thus

$$\|F(t,u,v)\|_{\mathcal{P}} := \sup\{|f| : f\in F(t,u,v)\} \le 2\Big[\frac{t}{108t^2+32}\Big] = p_1(t)[\psi_1(\|u\|)+\phi_1(\|v\|)],$$

and

$$\|G(t,u,v)\|_{\mathcal{P}} := \sup\{|g| : g \in G(t,u,v)\} \leq 2\left[\frac{t^2+1}{t^3+120}\right] = p_2(t)[\psi_2(\|u\|)+\phi_2(\|v\|)],$$

with $p_1(t) = \dfrac{t}{108t^2+32}$, $p_2(t) = \dfrac{t^2+1}{t^3+120}$, $\psi_1(\|u\|) = \phi_1(\|v\|) = \psi_2(\|u\|) = \phi_2(\|v\|) = 1$. Furthermore, it is found that $N > N_1$, where $N_1 = 0.81272506$ (N is given in (H_3)). Clearly all the hypotheses of Theorem 3.3 are satisfied. Thus, there exists at least one solution for the problem (5.1) on $[0,3]$.

(b) For illustrating Theorem 4.3, we take the multivalued maps $F, G : [0,3] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ as

$$F(t,u,v) = \left[\left(\frac{1}{4t+150}\right)\left(\frac{|u(t)|}{|u(t)|+1}, \sin v(t)\right) + \frac{1}{175}, 0\right],$$

$$G(t,u,v) = \left[\left(\frac{1}{3t^2+140}\right)\left(\tan^{-1} u(t), \frac{|v(t)|}{1+|v(t)|}\right) + \frac{1}{170}, 0\right]. \tag{5.2}$$

Letting $\mathcal{B}_1(t) = \dfrac{1}{4t+150}$ and $\mathcal{B}_2(t) = \dfrac{1}{3t^2+140}$, we find that $H_d(F(t,u,v), F(t,\bar{u},\bar{v})) \leq \mathcal{B}_1(t)(|u-\bar{u}|+|v-\bar{v}|)$ and $H_d(G(t,u,v), G(t,\bar{u},\bar{v})) \leq \mathcal{B}_2(t)(|u-\bar{u}|+|v-\bar{v}|)$. Observe that $d(0, F(t,0,0)) = \dfrac{1}{175} \leq \mathcal{B}_1(t)$ and $d(0, G(t,0,0)) = \dfrac{1}{170} \leq \mathcal{B}_2(t)$ for almost all $t \in [0,3]$. Obviously $\|\mathcal{B}_1\| = 1/150$ and $\|\mathcal{B}_2\| = 1/140$ and

$$\mathcal{E}_1\|\mathcal{B}_1\| + \mathcal{E}_2\|\mathcal{B}_2\| \approx 0.4201443466 < 1.$$

Consequently, all the assumptions of Theorem 4.3 hold true. Therefore, by conclusion of Theorem 4.3, the problem (5.1) with F, G given by (5.2), has at least one solution on $[0,3]$.

6. Conclusions

We have developed the existence theory for a self-adjoint coupled system of nonlinear second-order ordinary differential inclusions supplemented with nonlocal integral multi-strip coupled boundary conditions on an arbitrary domain. Our study includes the cases of convex as well as non-convex multi-valued maps. Nonlinear alternative of Leray-Schauder type for multi-valued maps and Covitz and Nadler fixed point theorem for contractive multi-valued maps are applied to prove the main results. Numerical examples are constructed for the illustration of the obtained results. Our results are new in the given configuration and enrich the related literature. Moreover, several new results can be recorded as special cases of the present work by fixing the parameters appearing in the system. For example, we obtain the existence results for an anti-periodic multi-valued boundary value problem of self-adjoint coupled second-order ordinary differential inclusions by fixing $\alpha_i = 1, \beta_i = 1, \lambda_i = 0, i = 1,2,3,4$ in the results of this paper, which are indeed new.

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On vertex-edge corona of graphs and its spectral polynomial

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Abstract. Given a graph G_1, the vertex-corona (corona) and the edge-corona focus only on vertices and edges respectively, in forming the corona product with other graphs. In the present work, we define a new corona by considering both vertices and edges simultaneously in forming the corona aproduct with other graphs, called vertex-edge corona. Further, we study the spectral polynomial for the vertex-edge corona of three arbitrary graphs, followed by some corollaries related to regular graphs for their spectrum, energy and equienergetic graphs.

AMS Subject Classifications: 05C50.

Keywords: vertex-corona, edge-corona, vertex-edge corona, spectral polynomial.

Contents

1. Introduction

In 1969, R. Frucht et al. defined a new operation on two graphs G_1 and G_2, called their corona [9] while studying the isomorphism between the group associated with the new graph and the wreath product of the groups G_1 and G_2. The corona of two graphs so defined, focus only on the vertices in forming the corona product with the other graph, hance can be called as vertex-corona. Graph is associated with many concepts like: edges, neighbours, subdivision of edges and more. With the advent of reseacrh various corona products are defined, namely:

1. edge corona (2010) [12],
2. neighbourhood corona (2011) [13],
3. subdivision-vertex and subdivision-edge corona (2013) [18],
4. subdivision-vertex and subdivision-edge neighbourhood coronae (2013) [16],
5. R-vertex, R-edge, R-vertex neighbourhood and R-edge neighbourhood corona (2015) [14],
6. N-vertex, N-edge, C-vertex and C-edge corona (2015) [1],

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7. Extended and extended neighbourhood corona (2016) [2].

The work related to their spectrum and various polynomials can be seen in [4–7, 15, 17, 20].

It is observed that, for a given graph G_1, the vertex-corona (corona) and the edge-corona focus only on vertices and edges, respectively, in forming the corona product with other graphs. We thought to focus both on vertices and edges simultaneously in forming the corona product with other graphs, hence define new corona, called vertex-edge corona, which involves two more graphs G_2 and G_3 one corresponding to vertices and other to edges of G_1. Further, we study the spectral polynomial for the vertex-edge corona of three arbitrary graphs, followed by some corollaries related to regular graphs for their spectrum, energy and equienergetic graphs.

Remarkable observation is that, vertex-edge corona can be considered as the generalization of: vertex-corona, edge-corona, R-vertex corona and C-edge corona, which is possible with the suitable selection of the graphs G_2 and G_3.

2. Preliminaries

All graphs considered here are simple, finite and undirected. If G is a graph on n vertices $v_1, v_2, \ldots, v_n$ and m edges $e_1, e_2, \ldots, e_m$ then its adjacency matrix, $A(G) = [a_{ij}]_{n\times n}$ in which $a_{ij} = 1$ if the vertices v_i and v_j are adjacent, and 0 otherwise, and the vertex-edge incidence (incidence) matrix $R(G) = [b_{ij}]_{n\times m}$ in which $b_{ij} = 1$ if the vertex v_i and edge e_j are incident, and 0 otherwise. The polynomial $\phi\Big(A(G);x\Big) = det\Big(xI_n - A(G)\Big)$ associated with $A(G)$ is called the spectral polynomial and the roots of the equation, $\phi\Big(A(G);x\Big) = 0$ are the eigenvalues of G, which constitute the spectrum of G. If G has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with multiplicities $n_1, n_2, \ldots, n_k$ respectively, then we can write: $\begin{pmatrix}\lambda_1 & \lambda_2 & \ldots & \lambda_k\\ n_1 & n_2 & \ldots & n_k\end{pmatrix}$ for the spectrum of G, where $\sum_{i=1}^{k} n_i = n$. The aggregate of the absolute values of these graph eigenvalues, called energy[10] of G is defined as: $\mathcal{E}(G) = \sum\limits_{i=1}^{k} n_i \mid \lambda_i \mid$. The degree of a vertex v_i in G denoted by d_i is the number of edges incident to it, if $d_i = r$ (a constant) for all the vertices v_i then G is called an r-regular graph. If G is r-regular graph, then $R(G)R(G)^T = A(G) + rI_n$. The Kronecker product $C \otimes D$ of two matrices $C = [c_{ij}]_{m\times n}$ and $D = [d_{ij}]_{p\times q}$ is the $mp \times nq$ matrix obtained from C by replacing each entry c_{ij} by $c_{ij}D$ [11]. For matrices C, D, E and F such that products CE and DF exist, $(C \otimes D)(E \otimes F) = CE \otimes DF$, $(C \otimes D)^{-1} = C^{-1} \otimes D^{-1}$ and $(C \otimes D)^T = C^T \otimes D^T$. 1_n denotes the column vector of dimension n. $K_n, K_{p,q}$ denotes complete graph and complete bipartite graphs respectively. Zero order graph is a graph with no vertices. For undefined graph theoretical terminologies and notations, we follow the book [8].

Proposition 2.1. *(Schur Complement [3]) Suppose that the order of all four matrices D_{11}, D_{12}, D_{21} and D_{22} satisfy the rules of operations on matrices. Then we have,*

$$\begin{vmatrix} D_{11} & D_{12}\\ D_{21} & D_{22}\end{vmatrix} = \begin{cases} |D_{22}|\ |D_{11} - D_{12}D_{22}^{-1}D_{21}|, & \textit{if } D_{22} \textit{ is a non-singular matrix,}\\ |D_{11}|\ |D_{22} - D_{21}D_{11}^{-1}D_{12}|, & \textit{if } D_{11} \textit{ is a non-singular matrix.}\end{cases}$$

Definition 2.2. *[9] Given a graph G_1 on n_1 vertices, the vertex-corona (corona) $G_1 \circ G_2$ of G_1 with the graph G_2 is the graph obtained by taking one copy of G_1 and n_1 copies of G_2, then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.*

Definition 2.3. *[12] Given a graph G_1 with m_1 edges, the edge-corona $G_1 \diamond G_2$ of G_1 with the graph G_2 is the graph obtained by taking one copy of G_1 and m_1 copies of G_2, then joining two end vertices of the i^{th} edge of G_1 to every vertex in the i^{th} copy of G_2.*

Definition 2.4. *[19] Given a graph G on n vertiecs with the graph matrix M, where M is viewed as a matrix over the field of rational functions $\mathbb{C}(x)$ with $det(xI_n - M)$ non zero. The M-coronal $\Gamma_M(x) \in \mathbb{C}(x)$ of G is, $\Gamma_M(x) = 1_n^T\Big(xI_n - M\Big)^{-1}1_n$. If M has a constant row sum r, $\Gamma_M(x) = \dfrac{n}{x-r}$.*

3. Vertex-edge corona of graphs

Given a graph G_1, the vertex-corona (corona) and the edge-corona focus only on vertices and edges, respectively, in forming the corona product with other graphs. Our prime purpose here is to focus both on vertices and edges simultaneously in forming the corona product with other graphs, hence define a new corona, called vertex-edge corona, which involves two more graphs G_2 and G_3 one corresponding to vertices and other to edges of G_1.

Definition 3.1. *Let G_1, G_2, G_3 be any three graphs on n_1, n_2, n_3 vertices and m_1, m_2, m_3 edges, respectively. The vertex-edg corona $G_1 \circ G_2 \diamond G_3$ of G_1, G_2 and G_3 is the graph obtained by taking one copy of G_1, n_1 copies of G_2 and m_1 copies of G_3, then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2 and two end vertices of the i^{th} edge of G_1 to every vertex in the i^{th} copy of G_3.*

It is noted that $G_1 \circ G_2 \diamond G_3$ has $n_1 + n_1n_2 + m_1n_3$ vertices and $m_1 + n_1m_2 + m_1m_3 + n_1n_2 + 2m_1n_3$ edges.

Example 3.2. *Let P_n denotes the path on n vertices. Figure 1 depicts $P_4 \circ P_3 \diamond P_2$.*

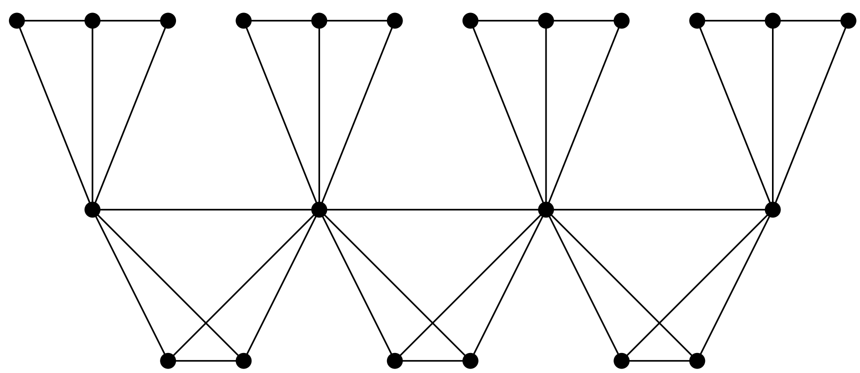

Figure 1:

In the following section, we study the spectral polynomial for the vertex-edge corona of three arbitrary graphs, followed by some corollaries related to regular graphs for their spectrum and energy. We also construct infinitely many pairs of cospectral graphs by applying these results.

4. Spectral polynomial of vertex-edge corona of three graphs

Theorem 4.1. *Let G_1, G_2, G_3 be any there graphs on n_1, n_2, n_3 vertices respectively. If G_1 has m_1 edges then the spectral polynomial of the vertex-edge corona $G_1 \circ G_2 \diamond G_3$ of three graphs is*

$$\begin{aligned}&\phi\Big(A\left(G_1 \circ G_2 \diamond G_3\right); x\Big)\\&= \phi\Big(A\left(G_2\right); x\Big)^{n_1} \phi\Big(A\left(G_3\right); x\Big)^{m_1} det\Big(xI_{n_1} - A\left(G_1\right) - \Gamma_{A(G_3)}\left(x\right) R\left(G_1\right) R\left(G_1\right)^T - \Gamma_{A(G_2)}\left(x\right) I_{n_1}\Big).\end{aligned}$$

Proof. The general adjacency matrix of the vertex-edge corona $G_1 \circ G_2 \diamond G_3$ of G_1, G_2, G_3 on n_1, n_2, n_3 vertices respectively with m_1 edges in G_1 is,

$$A\left(G_1 \circ G_2 \diamond G_3\right) = \begin{pmatrix} A(G_1) & I_{n_1} \otimes 1_{n_2}^T & R(G_1) \otimes 1_{n_3}^T \\ I_{n_1} \otimes 1_{n_2} & I_{n_1} \otimes A(G_2) & O_{n_1n_2 \times n_3m_1} \\ R(G_1)^T \otimes 1_{n_3} & O_{n_3m_1 \times n_1n_2} & I_{m_1} \otimes A(G_3) \end{pmatrix}.$$

The spectral polynomial is,

$$\begin{aligned}
&\phi\Big(A\left(G_1 \circ G_2 \diamond G_3\right);\, x\Big)\\
&= det\Big(x\, I_{n_1+n_1n_2+m_1n_3} - A\left(G_1 \circ G_2 \diamond G_3\right)\Big)\\
&= det\left(\begin{array}{cc|c} xI_{n_1} - A(G_1) & -I_{n_1} \otimes 1_{n_2}^T & -R(G_1) \otimes 1_{n_3}^T \\ -I_{n_1} \otimes 1_{n_2} & I_{n_1} \otimes \Big(xI_{n_2} - A(G_2)\Big) & O_{n_1n_2\times n_3m_1} \\ \hline -R(G_1)^T \otimes 1_{n_3} & O_{n_3m_1\times n_1n_2} & I_{m_1} \otimes \Big(xI_{n_3} - A(G_3)\Big) \end{array}\right).
\end{aligned}$$

Applying Proposition 2.1, we have

$$\phi\Big(A\left(G_1 \circ G_2 \diamond G_3\right);\, x\Big) = \phi\Big(A\left(G_3\right);\, x\Big)^{m_1} det\left(\begin{pmatrix} xI_{n_1} - A(G_1) & -I_{n_1} \otimes 1_{n_2}^T \\ -I_{n_1} \otimes 1_{n_2} & I_{n_1} \otimes \Big(xI_{n_2} - A(G_2)\Big) \end{pmatrix} - S\right)$$

where,

$$\begin{aligned}
S &= \begin{pmatrix} -R(G_1) \otimes 1_{n_3}^T \\ O_{n_1n_2\times n_3m_1} \end{pmatrix} \Big(I_{m_1} \otimes \Big(xI_{n_3} - A(G_3)\Big)\Big)^{-1} \left(-R(G_1)^T \otimes 1_{n_3}\ O_{n_3m_1\times n_1n_2}\right)\\
&= \begin{pmatrix} R(G_1)R(G_1)^T \otimes 1_{n_3}^T \Big(xI_{n_3} - A(G_3)\Big)^{-1} 1_{n_3} & O \\ O & O \end{pmatrix}.
\end{aligned}$$

Therefore,

$$\phi\Big(A\left(G_1 \circ G_2 \diamond G_3\right);\, x\Big) = \phi\Big(A\left(G_3\right);\, x\Big)^{m_1} det\left(\begin{array}{c|c} xI_{n_1} - A(G_1) - S_{11} & -I_{n_1} \otimes 1_{n_2}^T \\ \hline -I_{n_1} \otimes 1_{n_2} & I_{n_1} \otimes \Big(xI_{n_2} - A(G_2)\Big) \end{array}\right),$$

where $S_{11} = R(G_1)R(G_1)^T \otimes 1_{n_3}^T \Big(xI_{n_3} - A(G_3)\Big)^{-1} 1_{n_3}$.
Again applying Proposition 2.1,

$$\begin{aligned}
\phi\Big(A\left(G_1 \circ G_2 \diamond G_3\right);\, x\Big) &= \phi\Big(A\left(G_3\right);\, x\Big)^{m_1} \phi\Big(A\left(G_2\right);\, x\Big)^{n_1}\\
&\quad det\Big(xI_{n_1} - A(G_1) - S_{11} - I_{n_1} \otimes 1_{n_2}^T \Big(xI_{n_2} - A(G_2)\Big)^{-1} 1_{n_2}\Big)\\
&= \phi\Big(A\left(G_3\right);\, x\Big)^{m_1} \phi\Big(A\left(G_2\right);\, x\Big)^{n_1}\\
&\quad det\Big(xI_{n_1} - A(G_1) - R(G_1)R(G_1)^T \Gamma_{A(G_3)}(x) - I_{n_1}\Gamma_{A(G_2)}(x)\Big),
\end{aligned}$$

on re-arrangement result follows. ■

Corollary 4.2. *If G_1, G_2, G_3 are all r_1, r_2, r_3 regular graphs, respectively. If $r_1 = \lambda_1, \lambda_2, \ldots, \lambda_{n_1}$ are the eigenvalues of G_1, then*

$$\begin{aligned}
\phi\Big(A\left(G_1 \circ G_2 \diamond G_3\right);\, x\Big) &= \frac{\phi\Big(A\left(G_3\right);\, x\Big)^{m_1} \phi\Big(A\left(G_2\right);\, x\Big)^{n_1}}{(x - r_3)^{n_1}(x - r_2)^{n_1}}\\
&\quad \prod_{i=1}^{n_1} \Big[x^3 - (\lambda_i + r_2 + r_3)x^2 + (\lambda_i r_2 + \lambda_i r_3 - \lambda_i n_3 + r_2 r_3 - r_1 n_3 - n_2)x\\
&\quad +(\lambda_i r_2 n_3 - \lambda_i r_2 r_3 + r_1 r_2 n_3 + n_2 r_3)\Big].
\end{aligned}$$

Proof. Substituting $R(G_1)R(G_1)^T = A(G_1) + r_1 I_{n_1}$, $\Gamma_{A(G_3)}(x) = \dfrac{n_3}{x - r_3}$ and $\Gamma_{A(G_2)}(x) = \dfrac{n_2}{x - r_2}$ in Theorem 4.1, and expanding the determinant interms of λ_i, result follows. ■

Corollary 4.3. *If G_1, G_2, G_3 are all r_1, r_2, r_3 regular graphs, respectively. If $r_1 = \lambda_1, \lambda_2, \ldots, \lambda_{n_1}$, $r_2 = \mu_1, \mu_2, \ldots, \mu_{n_2}$, and $r_3 = \delta_1, \delta_2, \ldots, \delta_{n_3}$ are the eigenvalues of G_1,G_2 and G_3 respectively. Then*

$$\begin{aligned}\mathcal{E}\Big(G_1 \circ G_2 \diamond G_3\Big) &= m_1 \mathcal{E}(G_3) + n_1 \mathcal{E}(G_2) - n_1(r_2 + r_3)\\ &+ \sum_{i=1}^{n_1} \Big[\mid \gamma_{1i} \mid + \mid \gamma_{2i} \mid + \mid \gamma_{3i} \mid \Big]\end{aligned}$$

where $\gamma_{1i}, \gamma_{2i}, \gamma_{3i}$ are roots of the polynomial,

$$\begin{aligned}&\Big[x^3 - (\lambda_i + r_2 + r_3)x^2 + (\lambda_i r_2 + \lambda_i r_3 - \lambda_i n_3 + r_2 r_3 - r_1 n_3 - n_2)x\\ &+(\lambda_i r_2 n_3 - \lambda_i r_2 r_3 + r_1 r_2 n_3 + n_2 r_3)\Big].\end{aligned}$$

Proof. Equating the polynomial in Corollary 4.2 for eigenvalues and applying the definition of energy, result follows. ■

Corollary 4.4. *If G_1 is an r_1-regular graph, $G_2 = G_3$ is an r_2-regular graph, then*

$$\begin{aligned}\phi\Big(A\,(G_1 \circ G_2 \diamond G_2)\,;\; x\Big) &= \frac{\phi\Big(A\,(G_2)\,;\; x\Big)^{m_1+n_1}}{(x - r_2)^{n_1}}\\ &\prod_{i=1}^{n_1} \Big[x^2 - (\lambda_i + r_2)x + (\lambda_i r_2 - r_1 n_2 - \lambda_i n_2 - n_2)\Big].\end{aligned}$$

where $r_1 = \lambda_1, \lambda_2, \ldots, \lambda_{n_1}$ are the eigenvalues of G_1.

Proof. Substituting $G_2 = G_3$, $R(G_1)R(G_1)^T = A(G_1) + r_1 I_{n_1}$, $\Gamma_{A(G_2)}(x) = \dfrac{n_2}{x - r_2}$ in Theorem 4.1 and expanding the determinant interms of λ_i, result follows. ■

Corollary 4.5. *If G_1 is an r_1-regular graph, $G_2 = G_3$ is an r_2-regular graph, then spectrum of $G_1 \circ G_2 \diamond G_2$ is:*

$$\begin{pmatrix} r_2 & \mu_2 & \mu_3 & \ldots & \mu_{n_2} & \dfrac{r_2 + \lambda_i \pm \sqrt{(r_2 - \lambda_i)^2 + 4n_2(r_1 + \lambda_i + 1)}}{2} \\ m_1 & m_1 + n_1 & m_1 + n_1 & \ldots & m_1 + n_1 & 1 \end{pmatrix}$$

for $i = 1, 2, \ldots, n_1$. Hence, energy

$$\mathcal{E}\Big(G_1 \circ G_2 \diamond G_2\Big) = (n_1 + m_1)\mathcal{E}(G_2) - n_1 r_2 + \sum_{i=1}^{n} \left| \frac{r_2+\lambda_i \pm \sqrt{(r_2-\lambda_i)^2 + 4n_2(r_1+\lambda_i+1)}}{2} \right|.$$

Proof. Equating the polynomial in Corollary 4.4 to zero for the eigenvalues and applying the definition of energy, result follows. ■

Corollary 4.6. *If G_1 is an r_1-regular graph, and $G_2 = G_3 = K_{p,q}$ with $p \neq q$ then*

$$\begin{aligned}\phi\Big(A\,(G_1 \circ K_{p,q} \diamond K_{p,q})\,;\; x\Big) &= x^{(p+q-2)(n_1+m_1)}(x^2 - pq)^{m_1}\\ &\prod_{i=1}^{n_1} \Big[x^3 - \lambda_i x^2 - (pq + pr + qr + p\lambda_i + q\lambda_i + p + q)x\\ &-pq(\lambda_i + 2r + 2)\Big].\end{aligned}$$

Remark

1. If H_1, H_2 be a pair of cospectral graphs with same order and of same regularity, then for two regular graphs G_2, G_3, the graphs $H_1 \circ G_2 \diamond G_3$ and $H_2 \circ G_2 \diamond G_3$ are also cospectral.
2. In $G_1 \circ G_2 \diamond G_3$:
   - if G_3 is a zero order garph, then resulting corona is vertex-corona (corona).
   - if G_2 is a zero order garph, then resulting corona is edge-corona.
   - if G_3 is K_1, then the resulting corona is R-vertex corona.
   - if G_2 is K_1, then the resulting corona is C-edge corona.

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Modeling and optimal control of the dynamics of narcoterrorism in the Sahel

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Abstract. This work explores some aspects of modeling and controlling narcoterrorism in the Sahel. We examine the multidimensional factors underlying this dynamic, identifying interactions and recruitment within the narcoterrorist class. We then develop a preventive model and decision-support tools to optimize resource allocation and formulate more effective counter-narcotics and brigandage policies. This research will certainly contribute to the fight against narcoterrorism in the Sahel by proposing solutions based on rigorous scientific approaches and assessing the benefits and limitations of optimal modeling and control.

AMS Subject Classifications: 49K15, 93B05, 93C15, 93D23.

Keywords: narcoterrorism, local and global asymptotic stability, global threshold, optimal control, and numerical simulation.

Contents

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1. Introduction and Background

The African continent offers Latin American and South American drug traffickers an uncontrolled transit route, with its porous borders, ideal location close to Europe, and fragile, corrupted states. According to the United Nations Office on Drugs and Crime (UNODC), the market value of cocaine transiting West Africa each year was estimated at US dollars 1.25 billion in 2013. The map below illustrates drug trafficking and transit zones from Latin and South America to Europe via West Africa and the Sahel, updated in February 2013 by the United Nations Office on Drugs and Crime (UNODC).

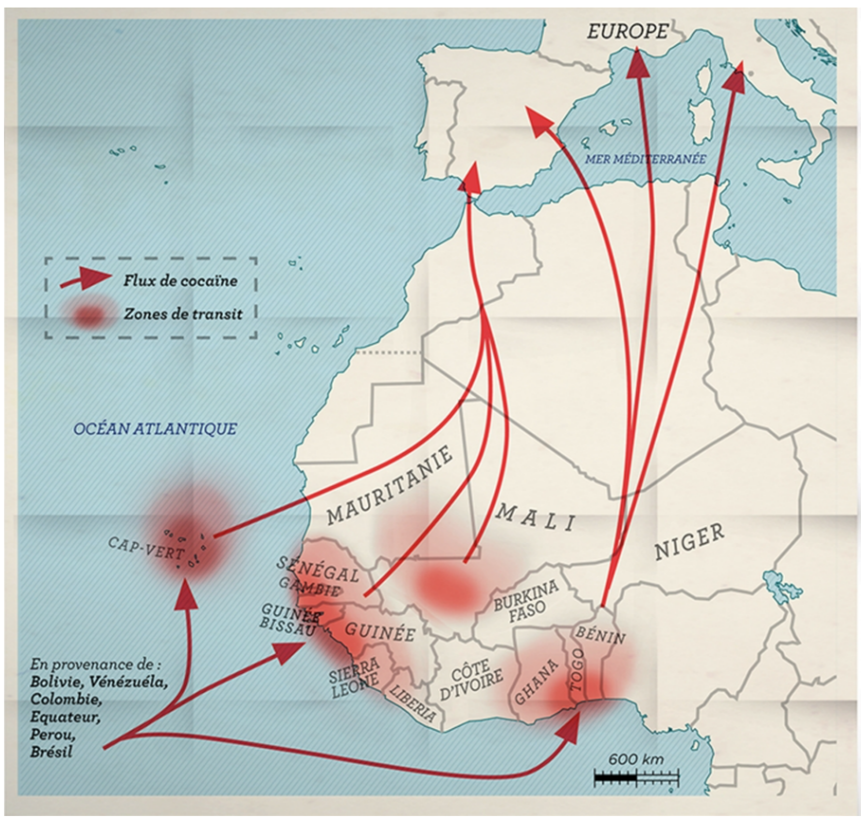

Figure 1: Map of drug trafficking and transit zones to Europe via West Africa and the Sahel.

In recent years, narcoterrorism has become a major problem in the Sahel. This deadly combination of drug trafficking and terrorist activity creates a complex and constantly evolving security and humanitarian crisis, requiring innovative approaches to understanding and controlling the threat. What are the dynamics of narcoterrorism in the Sahel? What factors have encouraged the development of narcoterrorism in the Sahel over the last few decades? Are there effective, targeted, and optimal strategies for eradicating narcoterrorism in the Sahel? It would be interesting to find answers to these questions and develop decision-making tools for political decision-makers and defense and security forces in the fight against drug trafficking, terrorism, and insecurity in general. In this spirit, we have decided to tackle this problem using a mathematical approach that is intended to be modest, as we do not claim to be able to say that mathematics can answer all these questions.

The rise of narcoterrorism in the Sahel can be explained by several factors. These include geographical and demographic factors. The Sahel's vast, sparsely populated territory, porous borders, and proximity to major drug-producing regions such as Latin America and West Africa make it an attractive transit route for drug traffickers. Added to this is the weakness of governance and security structures in the Sahel, which is said to benefit transnational criminal networks transporting illicit drugs, notably cocaine, heroin, and cannabis, across the region. We also have ideological terrorism and insurgency movements in the Sahel. The Sahel is indeed experiencing an increase in terrorist and insurgent activity, mainly perpetrated by groups such as Al-Qaeda in the Islamic Maghreb (AQIM), Boko Haram, and the Islamic State in the Greater Sahara (ISGS). These extremist groups exploit the region's socio-economic and political vulnerabilities, including poverty, unemployment, poor governance, and community tensions, to recruit fighters, finance their activities, and carry out attacks. The presence of drug-trafficking networks is an additional source of revenue for these terrorist groups. Another factor would be the financing of terrorism, as terrorist groups engage in a variety of criminal activities, including protecting drug convoys, taxing drug traffickers, and drug trafficking itself. Profits from the drug trade would enable these groups to continue their operations, buy weapons and recruit new members. The convergence of these criminal and terrorist activities creates a complex and dangerous environment that challenges the security forces and governments of the Sahel countries.

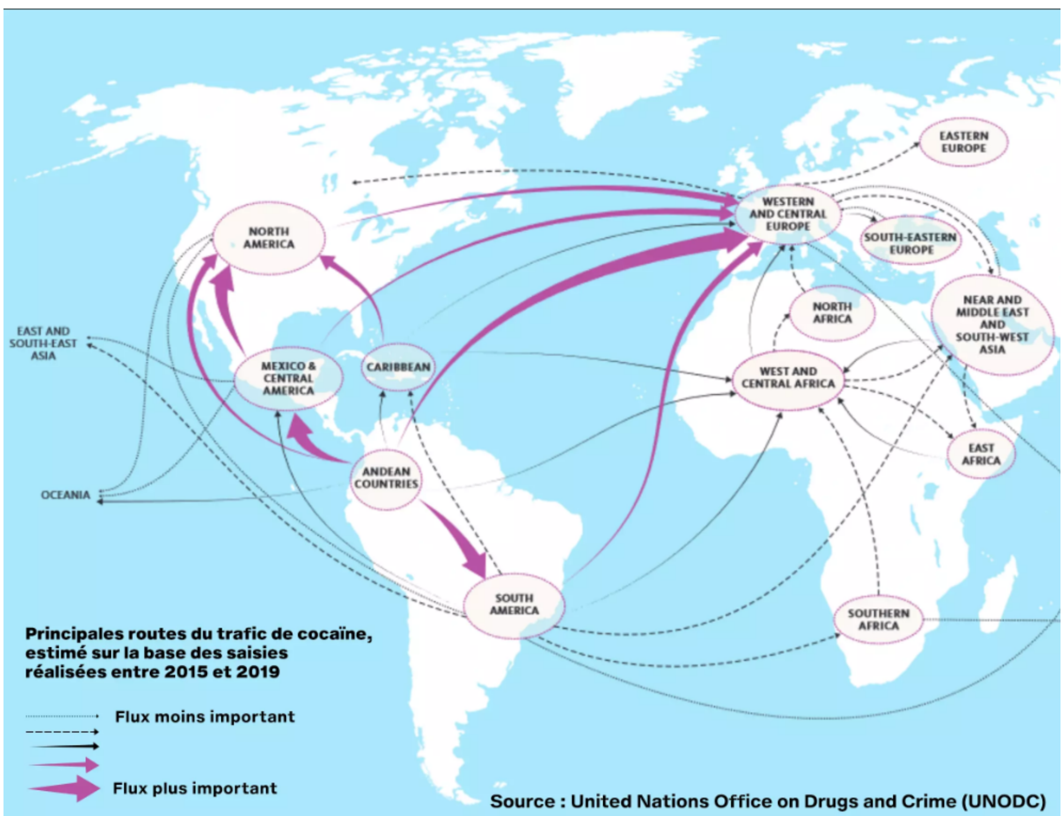

Figure 2: Map of the main cocaine trafficking flows.

The map above shows the scale of the threat. In November 2009, the image of the charred wreckage of a Boeing 727 found north of Gao in Mali revealed the scale of a hitherto unknown phenomenon. The plane, coming from Venezuela near the Colombian border, was carrying a cargo of several tonnes of cocaine. The media went so far as to popularise the concept of ”air cocaine”, while government intelligence services became aware of the imminence of the new threat looming on the horizon as a result of the convergence between extremist movements in the Sahel and drug traffickers in South America.

There is growing interest in the modelling and optimal control of the dynamics of narcoterrorism in the Sahel. These approaches, which combine mathematical tools, advanced simulation methods, and empirical data, provide a better understanding of the mechanisms underlying this complex dynamic. They also offer the possibility of formulating more effective and targeted control strategies. In this study, we seek to explore the different aspects of modeling and optimal control of narcoterrorism in the Sahel. We examine the multidimensional factors that drive this dynamic. By identifying the interactions as in evolution studies [3], [11], [6], [5] and recruitment within the narcoterrorist class we can better understand the mechanisms by which narcoterrorism spreads in the region. Building on this knowledge and adapting it to the specific context of the Sahel, we are developing a preventive model and decision-support tools to optimize resource allocation and formulate more effective counter-narcotics and counter-brigandage policies. This research aims to contribute to the fight against narcoterrorism in the Sahel by proposing solutions based on rigorous scientific approaches. Finally, by assessing the advantages and limitations of modelling and optimal control, we hope that this work will be useful to political decision-makers, security forces, and international players involved in the region. The specifics of the model are described in more detail in the next paragraph.

2. Model formulation

In order to facilitate the description of this model, we have divided the total population (N), into seven classes. Thus, we have the class of non-combatant civilians (C), the class of volunteers for the defence of the homeland and self-defence groups (V), the class of defence and security forces (A), the class of people discharged from the ranks of the defence and security forces (R), the class of brigands (B), the class of narcoterrorists (T), and the class of prisoners (P). The sum of the fighting classes $(A + V + B + T)$ is also referred to as I.

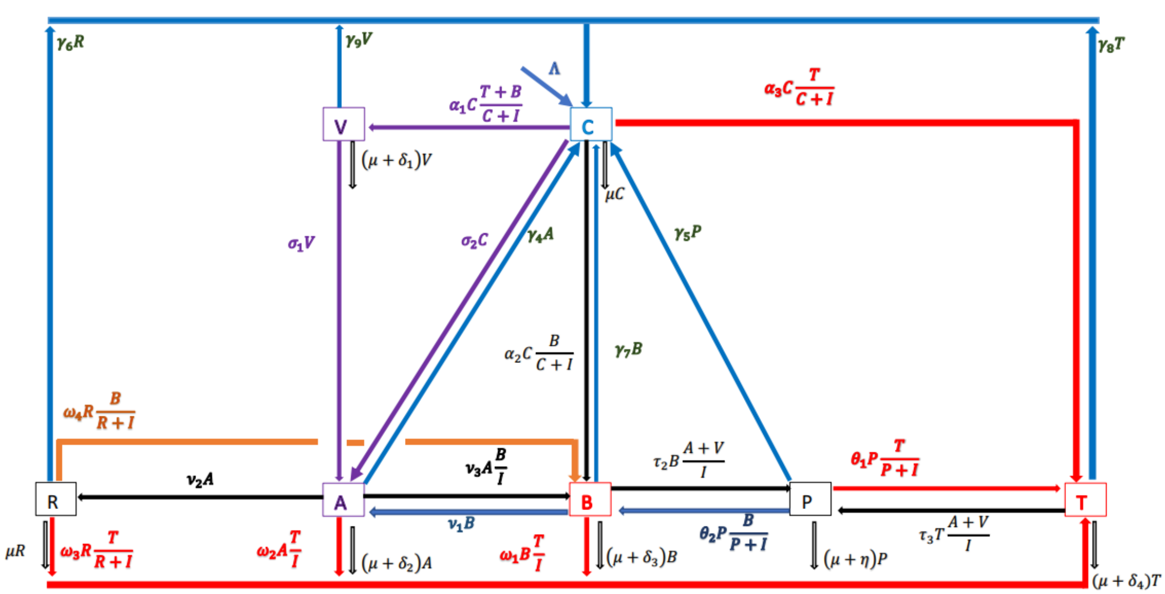

Figure 3: Diagram of the dynamics of narcoterrorism in the Sahel.

Here Λ denotes the population renewal constant, γ_i the rate of return to non-combatant civilian life for individuals in classes A, P, R, B, T, and V respectively for i ranging from 4 to 8. The probability of dying as a result of combat is denoted by δ_i, i ranging from 1 to 4 for individuals in classes V, A, B, and T respectively and ζ_i the intensity of the combat or nuisance force of individuals in classes B and T over those in classes V and A for i ranging from 1 to 2 respectively but also those of individuals in classes V and A over individuals in classes B and T respectively for i ranging from 3 to 4, η is the probability of dying as a result of the conditions of detention and μ is the natural mortality rate for all individuals in the population. The strength or capacity of recruitment into the narcoterrorist class of individuals in classes B, A, and R is respectively by ω_i, where i ranges from 1 to 3, and the strength of recruitment into brigandage of individuals in class R by ω_4. In the same way, for individuals in class C, α_1 denotes the intensity of the force of determination in defense of the homeland, α_2 that of the force of attraction in brigandage, α_3 the intensity of the force of attraction in narcoterrorism activities, σ_1 and σ_2 are the rates of recruitment into class A of individuals in classes V and C respectively. It is assumed that these rates (σ_1 and σ_2) are fixed by a given State in its defense strategy, but it is also assumed that a slight disturbance could occur during this recruitment which would mean that individuals from class B could be recruited with a probability ν_1. Furthermore, ν_2 designates the rate of radiation or desertion in class A and ν_3 the intensity of the conversion force in the brigandage of individuals in class A. The parameters, θ_1 and θ_2 are the capacities of recruitment of prisoners by the narcoterrorists and the brigands respectively, τ_2 and τ_3 the operational capacities of the classes V and A to be able to put in prison the individuals of the classes B and T respectively. It is assumed that these prisoners can be recruited as a result of prison breaks, prison attacks, or just contacts before the end of their sentence. Last but not least, it should be noted that recruitment is modelled on a contact or contagion process in epidemiology, taking into account in some cases the dissuasive presence of defence and security forces as well as self-defence groups and volunteers for the defence of the homeland. The equation of the model is formulated as follows:

$$\frac{dC}{dt} = \Lambda + \gamma_4 A + \gamma_5 P + \gamma_6 R + \gamma_7 B + \gamma_8 T + \gamma_9 V - \left(\alpha_1 \frac{T+B}{C+I} + \alpha_2 \frac{B}{C+I} + \alpha_3 \frac{T}{C+I} + \sigma_2 + \mu\right) C \tag{2.1}$$

$$\frac{dR}{dt} = \nu_2 A - \left(\omega_3 \frac{T}{R+I} + \omega_4 \frac{B}{R+I} + \gamma_6 + \mu\right) R \tag{2.2}$$

$$\frac{dA}{dt} = \sigma_1 V + \sigma_2 C + \nu_1 B - \left(\nu_3 \frac{B}{I} + \omega_2 \frac{T}{I} + \gamma_4 + \nu_2 + \mu + \zeta_1 \frac{T+B}{I}\right) A \tag{2.3}$$

$$\frac{dV}{dt} = \alpha_1 C \frac{T+B}{C+I} - \left(\gamma_9 + \sigma_1 + \mu + \zeta_2 \frac{T+B}{I}\right) V \tag{2.4}$$

$$\frac{dB}{dt} = \alpha_2 \frac{CB}{C+I} + \omega_4 \frac{RB}{R+I} + \nu_3 \frac{AB}{I} + \theta_2 \frac{PB}{P+I} - \left(\omega_1 \frac{T}{I} + \tau_2 \frac{A+V}{I} + \gamma_7 + \nu_1 + \mu + \zeta_3 \frac{A+V}{I}\right) B \tag{2.5}$$

$$\frac{dT}{dt} = \alpha_3 C \frac{T}{C+I} + \omega_1 B \frac{T}{I} + \omega_2 A \frac{T}{I} + \omega_3 R \frac{T}{R+I} + \theta_1 P \frac{T}{P+I} - \left(\tau_3 \frac{A+V}{I} + \gamma_8 + \mu + \zeta_4 \frac{A+V}{I}\right) T \tag{2.6}$$

$$\frac{dP}{dt} = \tau_2 B \frac{A+V}{I} + \tau_3 T \frac{A+V}{I} - \left(\theta_1 \frac{T}{P+I} + \theta_2 \frac{B}{P+I} + \gamma_5 + \mu + \eta\right) P \tag{2.7}$$

with non-negative initial conditions given by:

$$C(0) > 0; V(0) \geq 0; A(0) > 0; R(0) \geq 0; B(0) \geq 0; P(0) \geq 0; T(0) \geq 0, N(0) \leqslant \frac{\Lambda}{\mu}. \tag{2.8}$$

The parameters of the system $(2.1) - (2.7)$ are assumed to be all non-negative.

3. Mathematical analysis of the model

3.1. Existence and uniqueness of solution

The $(2.1) - (2.7)$ model is described by a system of first order nonlinear differential equations. It is rewritten as follows:

$$X'(t) = f(X(t)) \tag{3.1}$$

where $X(t)$ is a column vector of the number of individuals by class, and $f:\mathbb{R}^7\to\mathbb{R}^7$ is a function. More precisely,

$$X(t)=\begin{bmatrix}C(t)\\R(t)\\A(t)\\V(t)\\B(t)\\T(t)\\P(t)\end{bmatrix} \tag{3.2}$$

and

$$f(x)=\begin{bmatrix}\Lambda+\gamma_4x_3+\gamma_5x_7+\gamma_6x_2+\gamma_7x_5+\gamma_8x_6+\gamma_9x_4-\Big(\alpha_1\dfrac{x_6+x_7}{x_1+x_8}+\alpha_2\dfrac{x_5}{x_1+x_8}+\alpha_2\dfrac{x_5}{x_1+x_8}+\sigma_2+\mu\Big)x_1\\ \nu_2x_3-\Big(\omega_3\dfrac{x_6}{x_2+x_8}+\omega_4\dfrac{x_5}{x_2+x_8}+\gamma_6+\mu\Big)x_2\\ \sigma_1x_4+\sigma_2x_1+\nu_1x_5-\Big(\nu_3\dfrac{x_5}{x_8}+\omega_2\dfrac{x_6}{x_8}+\gamma_4+\nu_2+\mu+\zeta_1\dfrac{x_6+x_5}{x_8}\Big)x_3\\ \alpha_1x_1\dfrac{x_6+x_5}{x_1+x_8}-\Big(\gamma_9+\sigma_1+\mu+\zeta_2\dfrac{x_6+x_5}{x_8}\Big)x_4\\ \alpha_2\dfrac{x_1x_5}{x_1+x_8}+\omega_4\dfrac{x_2x_5}{x_2+x_8}+\nu_3\dfrac{x_3x_5}{x_8}+\theta_2\dfrac{x_7x_5}{x_7+x_8}-\Big(\omega_1\dfrac{x_6}{x_8}+\tau_2\dfrac{x_3+x_4}{x_8}+\gamma_7+\nu_1+\mu+\zeta_3\dfrac{x_3+x_4}{x_8}\Big)x_5\\ \alpha_3x_1\dfrac{x_6}{x_1+x_8}+\omega_1x_5\dfrac{x_6}{x_8}+\omega_2x_3\dfrac{x_6}{x_8}+\omega_3x_2\dfrac{x_6}{x_2+x_8}+\theta_1x_7\dfrac{x_6}{x_7+x_8}-\Big(\tau_3\dfrac{x_3+x_4}{x_8}+\gamma_8+\mu+\zeta_4\dfrac{x_3+x_4}{x_8}\Big)x_6\\ \tau_2x_5\dfrac{x_3+x_4}{x_8}+\tau_3x_6\dfrac{x_3+x_4}{x_8}-\Big(\theta_1\dfrac{x_6}{x_7+x_8}+\theta_2\dfrac{x_5}{x_7+x_8}+\gamma_5+\mu+\eta\Big)x_7\end{bmatrix} \tag{3.3}$$

with

$$x=(x_1,x_2,x_3,x_4,x_5,x_6,x_7)\in\mathbb{R}^7$$

and

$$\begin{cases}x_8=x_3+x_4+x_5+x_6\\ x_9=x_1+x_2+x_3+x_4+x_5+x_6+x_7.\end{cases}$$

The function f is clearly locally lipschitzian with respect to x. We then deduce the existence and the uniqueness of the maximal solution to the Cauchy problem associated to the differential equation $(2.1)-(2.7)$ related to the initial condition (2.8).

3.2. Positivity of the solutions

For this model of the dynamics of mafia terrorism to be realistic, it is necessary to show that all state variables remain positive at all times.

Proposition 3.1. ***(Positivity)*** *The positive orthan* $\mathbb{R}^7_{\geq 0}$ *is positively invariant for the system* $(2.1)-(2.7)$*, and the initial condition* (2.8) *ensures the positivity of the solutions of the system* $(2.1)-(2.7)$ *for any time* $t>0$*.*

Proof: We use the barrier theorem [2].

Let us show that the set $\Big\{C\geq 0\Big\}$ is positively invariant. Let $x=\Big(C,R,A,V,B,T,P\Big)$ and consider L an application defined by

$$L(x)=-C \tag{3.4}$$

The application L thus defined is differentiable and we have:

$$\nabla L(x)=(-1,0,0,0,0,0,0)\neq 0_{\mathbb{R}^7}. \tag{3.5}$$

The vector field for $\{C=0\}$ is given by

$$X(x)=\begin{bmatrix}\Lambda+\gamma_4A+\gamma_5P+\gamma_6R+\gamma_7B+\gamma_8T+\gamma_9V\\ \nu_2A-\Big(\omega_3\frac{T}{R+I}+\omega_4\frac{B}{R+I}+\gamma_6+\mu\Big)R\\ \sigma_1V+\nu_1B-\Big(\nu_3\frac{B}{I}+\omega_2\frac{T}{I}+\gamma_4+\nu_2+\mu+\zeta_1\frac{T+B}{I}\Big)A\\ -\Big(\gamma_9+\sigma_1+\mu+\zeta_2\frac{T+B}{I}\Big)V\\ \omega_4R\frac{B}{R+I}+\nu_3A\frac{B}{I}+\theta_2P\frac{B}{P+I}-\Big(\omega_1\frac{T}{I}+\tau_2\frac{A+V}{I}+\gamma_7+\nu_1+\mu+\zeta_3\frac{A+V}{I}\Big)B\\ \omega_1B\frac{T}{I}+\omega_2A\frac{T}{I}+\omega_3R\frac{T}{R+I}+\theta_1P\frac{T}{P+I}-\Big(\tau_3\frac{A+V}{I}+\gamma_8+\mu+\zeta_4\frac{A+V}{I}\Big)T\\ \tau_2B\frac{A+V}{I}+\tau_3T\frac{A+V}{I}-\Big(\theta_1\frac{T}{P+I}+\theta_2\frac{B}{P+I}+\gamma_5+\mu+\eta\Big)P\end{bmatrix}\qquad(3.6)$$

From (3.5) and (3.6), we have

$$\langle X(x),\nabla L(x)\rangle=-\Big(\Lambda+\gamma_4A+\gamma_5P+\gamma_6R+\gamma_7B+\gamma_8T+\gamma_9V\Big)\leq 0\qquad(3.7)$$

From (3.5) and (3.7) we deduce that $\{C\geq 0\}$ is positively invariant by application of the barrier theorem. Similarly, we show that $\{R\geq 0\}$, $\{A\geq 0\}$, $\{V\geq 0\}$, $\{B\geq 0\}$, $\{T\geq 0\}$, and $\{P\geq 0\}$ are positively invariant. Therefore, $\mathbb{R}^7_{\geq 0}$ is positively invariant.
Also by the initial condition (2.8), we have $x(0)\in\mathbb{R}^7_{\geq 0}$. Since $\mathbb{R}^7_{\geq 0}$ is positively invariant, then this ensures that all solutions of the system $(2.1)-(2.7)$ stay positive for all time $t>0$ □.

3.3. Invariant region

Theorem 3.2. *For initial conditions* (2.8)*, the solutions of the system* $(2.1)-(2.7)$ *are contained in the positively invariant, compact and attractive region*

$$\Psi=\left\{\Big(C,R,A,V,B,T,P\Big)\in\mathbb{R}^7_{\geq 0}:N(t)\leq\frac{\Lambda}{\mu}\right\}\qquad(3.8)$$

Proof: Summing the equations (2.1) to (2.7), we find :

$$\frac{dN}{dt}=\Lambda-\mu N-\delta_1V-\delta_2A-\delta_3B-\delta_4T-\eta P,$$

with $\delta_1=\zeta_1\frac{T+B}{I}$, $\delta_2=\zeta_2\frac{T+B}{I}$, $\delta_3=\zeta_3\frac{A+V}{I}$, and $\delta_4=\zeta_4\frac{A+V}{I}$.
Since A,V,B,T,F,P are positive functions and using the positivity of the functions δ_1, δ_2, δ_3, δ_4, given that the constants ζ_1, ζ_2, ζ_3, ζ_4 and η are strictly positive as well, we get:

$$\frac{dN}{dt}\leq\quad\Lambda-\mu N.$$

Then

$$\frac{d}{dt}\Big(N-\frac{\Lambda}{\mu}\Big)\leq\quad-\mu\Big(N-\frac{\Lambda}{\mu}\Big).$$

So the Gromwall inequality gives

$$N(t) - \frac{\Lambda}{\mu} \leq \left(N(0) - \frac{\Lambda}{\mu}\right) e^{-\mu t}.$$

Thus

$$N(t) \leq \frac{\Lambda}{\mu} + \left(N(0) - \frac{\Lambda}{\mu}\right) e^{-\mu t}.$$

Since $N(0) \leq \dfrac{\Lambda}{\mu}$, then $0 \leq N(t) \leq \dfrac{\Lambda}{\mu}$.

Therefore, all feasible solutions of the model $(2.1)-(2.7)$ converge in the region Ψ. □

4. Equilibrium without terrorist, nor brigand (x^*), and basic reproduction rumber $\mathcal{R}_0$

4.1. Equilibrium without terrorist, nor brigand x^*

The uninfected compartments are C, R, A, V and the infected compartments are B, T, P. Given that we are at equilibrium without narcoterrorist nor brigand then we can discard the P compartment and the infected compartments being B, T, then an equilibrium solution with B=T=0 has the form:

$$x^* = \Big(C^*, R^*, A^*, 0, 0, 0\Big) \tag{4.1}$$

with

$$\begin{cases} C^* = \dfrac{\Lambda(\gamma_6+\mu)(\gamma_4+\nu_2+\mu)}{\mu\big[(\gamma_6+\mu)(\gamma_4+\mu+\nu_2+\sigma_2)+\sigma_2\nu_2\big]} \\[2ex] R^* = \dfrac{\Lambda\nu_2\sigma_2}{\mu\big[(\gamma_6+\mu)(\gamma_4+\mu+\nu_2+\sigma_2)+\sigma_2\nu_2\big]} \\[2ex] A^* = \dfrac{\Lambda\sigma_2(\gamma_6+\mu)}{\mu\big[(\gamma_6+\mu)(\gamma_4+\mu+\nu_2+\sigma_2)+\sigma_2\nu_2\big]} \end{cases}$$

4.2. Matrix of next generation $\mathcal{K}$, and basic reproduction number $\mathcal{R}_0$

The Jacobian matrix of the system $(2.1)-(2.7)$ is decomposed into $J_x(x^*) = D\mathcal{F}(x^*) + D\mathcal{V}(x^*)$ with

$$\mathcal{F} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \alpha_2 C\dfrac{B}{C+I} + \omega_4 R\dfrac{B}{R+I} + \nu_3 A\dfrac{B}{I} \\ \alpha_3 C\dfrac{T}{C+I} + \omega_1 B\dfrac{T}{I} + \omega_2 A\dfrac{T}{I} + \omega_3 R\dfrac{T}{R+I} \end{bmatrix}$$

and

$$\mathcal{V}=\begin{bmatrix}\Lambda+\gamma_4A+\gamma_6R+\gamma_7B+\gamma_8T+\gamma_9V-\left(\alpha_3\dfrac{T}{C+I}+\alpha_1\dfrac{T+B}{C+I}+\alpha_2\dfrac{B}{C+I}+\sigma_2+\mu\right)C\\ \nu_2A-\left(\omega_3\dfrac{T}{R+I}+\omega_4\dfrac{B}{R+I}+\gamma_6+\mu\right)R\\ \sigma_1V+\sigma_2C+\nu_1B-\left(\nu_3\dfrac{B}{I}+\omega_2\dfrac{T}{I}+\gamma_4+\nu_2+\mu+\zeta_1\dfrac{T+B}{I}\right)A\\ \alpha_1C\dfrac{T+B}{C+I}-\left(\gamma_9+\sigma_1+\mu+\zeta_2\dfrac{T+B}{I}\right)V\\ -\left(\omega_1\dfrac{T}{I}+\tau_2\dfrac{A+V}{I}+\gamma_7+\nu_1+\mu+\zeta_3\dfrac{A+V}{I}\right)B\\ -\left(\tau_3\dfrac{A+V}{I}+\gamma_8+\mu+\zeta_4\dfrac{A+V}{I}\right)T\end{bmatrix}$$

$$D\mathcal{F}(x^*)=\begin{bmatrix}0&0\\0&\mathbb{F}\end{bmatrix}\quad;\quad D\mathcal{V}(x^*)=\begin{bmatrix}J_1&J_2\\0&\mathbb{V}\end{bmatrix}\quad with\quad \mathbb{F}=\left[\frac{\partial\mathcal{F}_i(x^*)}{\partial x_j}\right]_{5\le i,j\le 6};$$

$$J_1=\left[\frac{\partial\mathcal{V}_i(x^*)}{\partial x_j}\right]_{1\le i,j\le 4}\quad;\quad J_2=\left[\frac{\partial\mathcal{V}_i(x^*)}{\partial x_j}\right]_{1\le i\le 4;5\le j\le 6}\quad and\ \ \mathbb{V}=\left[\frac{\partial\mathcal{V}_i(x^*)}{\partial x_j}\right]_{5\le i,j\le 6}.$$

Let:

$$g=\alpha_2\frac{C^*}{C^*+A^*}+\omega_4\frac{R^*}{R^*+A^*}+\nu_3;$$

$$h=\alpha_3\frac{C^*}{C^*+A^*}+\omega_2+\omega_3\frac{R^*}{R^*+A^*}.$$

We get

$$\mathbb{F}=\begin{bmatrix}g&0\\0&h\end{bmatrix};$$

$$J_1=\begin{bmatrix}-(\sigma_2+\mu)&\gamma_6&\gamma_4&\gamma_9\\0&-(\gamma_6+\mu)&\nu_2&0\\\sigma_2&0&-(\gamma_4+\mu+\nu_2)&\sigma_1\\0&0&0&-(\gamma_9+\mu+\sigma_1)\end{bmatrix}$$

and

$$J_2=\begin{bmatrix}\varpi_1&\varpi_2\\-\omega_4\dfrac{R^*}{R^*+A^*}&\varpi_3\\\nu_1-\nu_3&-\omega_2\\\alpha_1\dfrac{C^*}{C^*+A^*}&\alpha_1\dfrac{C^*}{C^*+A^*}\end{bmatrix},$$

with

$$\begin{aligned}
\varpi_1 &= \gamma_7 - \alpha_1 \frac{C^*}{C^* + A^*} - \alpha_2 \frac{C^*}{C^* + A^*} \\
\varpi_2 &= \gamma_8 - \alpha_3 \frac{C^*}{C^* + A^*} - \alpha_1 \frac{C^*}{C^* + A^*} \\
\varpi_3 &= -\omega_3 \frac{R^*}{R^* + A^*}
\end{aligned}$$

Note that J_1 is a non-singular Metzler matrix $(see \quad [1])$.

$$\mathbb{V} = \begin{bmatrix} -d & 0 \\ 0 & -e \end{bmatrix}$$

with

$$\begin{aligned}
d &= \gamma_7 + \mu + \tau_2 + \nu_1 + \zeta_3 \\
e &= \gamma_8 + \mu + \tau_3 + \zeta_4
\end{aligned}$$

We also note that $\mathbb{V}$ is a Metzler-Hurwitz matrix and

$$\mathbb{V}^{-1} = \begin{bmatrix} -\frac{1}{d} & 0 \\ 0 & -\frac{1}{e} \end{bmatrix}$$

$$\mathbb{V}^{-1} = \begin{bmatrix} -\frac{1}{d} & 0 \\ 0 & -\frac{1}{e} \end{bmatrix} \Rightarrow \mathcal{K} = -\mathbb{F}\mathbb{V}^{-1} = \begin{bmatrix} \frac{g}{d} & 0 \\ 0 & \frac{h}{e} \end{bmatrix}$$

where

$$\begin{cases}
\dfrac{g}{d} = \left(\dfrac{1}{\gamma_7 + \tau_2 + \nu_1 + \mu + \zeta_3} \right) \left(\alpha_2 \dfrac{\gamma_4 + \nu_2 + \mu}{\gamma_4 + \nu_2 + \sigma_2 + \mu} + \omega_4 \dfrac{\nu_2}{\gamma_6 + \nu_2 + \mu} + \nu_3 \right) \\[2em]
\dfrac{h}{e} = \left(\dfrac{1}{\gamma_8 + \tau_3 + \mu + \zeta_4} \right) \left(\alpha_3 \dfrac{\gamma_4 + \nu_2 + \mu}{\gamma_4 + \nu_2 + \sigma_2 + \mu} + \omega_2 + \omega_3 \dfrac{\nu_2}{\gamma_6 + \nu_2 + \mu} \right)
\end{cases}$$

and

$$\mathcal{R}_0 = \rho(\mathcal{K}) = \max \left\{ \frac{g}{d}; \frac{h}{e} \right\} \tag{4.2}$$

Theorem 4.1. *The equilibrium without terrorist, nor brigand x^*, is locally asymptotically stable if $\mathcal{R}_0 < 1$ and is unstable if $\mathcal{R}_0 > 1$.*

See [14, 16].

Theorem 4.2. *The equilibrium without terrorist, nor brigand x^*, is globally asymptotically stable if $\mathcal{R}_0 < 1$ and is unstable if $\mathcal{R}_0 > 1$.*

Proof: From Theorem 4.1 when $\mathcal{R}_0 < 1$ the states $B, T \to 0$ when $t \to \infty$. Identifying B and T with zero, it comes that $(C, R, A, V, B, T, P) \to x^*$ when $t \to \infty$ since x^* is the unique point in the positively invariant, compact and attractive solution region Ψ, such that $B = T = 0$. □

5. Global thresholds

5.1. A sufficient condition for the eradication of narcoterrorism

The result we set out in this section highlights the fact that, when the recruitment capacity or the sum of the forces of association with individuals in the narcoterrorist class is lower than the forces of exit from this class, then we will see an eradication of narcoterrorism. It's worth noting that when we talk about the forces of attraction in narcoterrorism activities, we're alluding in this study to the ability of narcoterrorists to offer a certain improvement in living conditions in financial terms.

Theorem 5.1. *Let $\lambda_2 = \alpha_3 + \omega_1 + \omega_2 + \omega_3 + \theta_1$, $\lambda_3 = (\tau_3 + \zeta_4)\kappa + \gamma_8$ with κ the infimum of $\dfrac{A+V}{I}$. So for all $\mathcal{R}_2 = \dfrac{\lambda_2}{\lambda_3} < 1$, we have $\lim\limits_{t\to\infty} T(t) = 0$.*

Proof: From the equation (2.6) we have:

$$\begin{aligned}
\frac{dT}{dt} &= \alpha_3 \frac{C}{C+I} T + \omega_1 B \frac{T}{I} + \omega_2 A \frac{T}{I} + \omega_3 R \frac{T}{R+I} + \theta_1 P \frac{T}{P+I} - \left(\tau_3 \frac{A+V}{I} + \gamma_8 + \zeta_4 \frac{A+V}{I}\right) T \\
&= (\alpha_3 \frac{C}{C+I} + \omega_1 \frac{B}{I} + \omega_2 \frac{A}{I} + \omega_3 \frac{R}{R+I} + \theta_1 \frac{P}{P+I}) T - \left(\tau_3 \frac{A+V}{I} + \gamma_8 + \zeta_4 \frac{A+V}{I}\right) T \\
&\leq (\alpha_3 \frac{C}{C+I} + \omega_1 \frac{B}{I} + \omega_2 \frac{A}{I} + \omega_3 \frac{R}{R+I} + \theta_1 \frac{P}{P+I}) T - \left(\tau_3 \kappa + \gamma_8 + \zeta_4 \kappa\right) T \\
&\leq (\alpha_3 + \omega_1 + \omega_2 + \omega_3 + \theta_1) T - \left((\tau_3 + \zeta_4)\kappa + \gamma_8\right) T = (\lambda_2 - \lambda_3) T.
\end{aligned}$$

It follows from the last inequality that T decreases exponentially to zero as soon as $\lambda_2 < \lambda_3$.□
Thus $\mathcal{R}_2 = \dfrac{\lambda_2}{\lambda_3} < 1$, gives a sufficient condition of the stabilization or eradication of narcoterrorism. This result reflects the fact that the greater the nuisance capacity of the defense and security forces, as well as their attractiveness in other legal activities, the more the narcoterrorist class tends towards elimination.

5.2. A sufficient condition of the eradication of brigandage.

The result that we also present in this section highlights the fact that, when the recruitment capacity or the sum of the forces of association with individuals in the bandit class is less than the forces of exit from this class, banditry is eradicated or stabilized.

Theorem 5.2. *Let $\lambda_5 = \alpha_2 + \omega_4 + \nu_3 + \theta_2$ and $\lambda_6 = \tau_2\kappa + \gamma_7 + \nu_1 + \zeta_3\kappa$ with κ respective infimum of $\dfrac{A+V}{I}$ and of. So for all $\mathcal{R}_4 = \dfrac{\lambda_5}{\lambda_6} < 1$, we have $\lim\limits_{t\to\infty} B(t) = 0$.*

Proof: From the equation (2.5) we have:

$$\begin{aligned}\frac{dB}{dt} &= \alpha_2 C\frac{B}{C+I} + \omega_4 R\frac{B}{R+I} + \nu_3 A\frac{B}{I} + \theta_2 P\frac{B}{P+I} - \left(\tau_2\frac{A+V}{I} + \gamma_7 + \nu_1 + \zeta_3\frac{A+V}{I}\right)B \\ &\leq \big(\alpha_2 + \omega_4 + \nu_3 + \theta_2\big)B - \Big(\tau_2\kappa + \gamma_7 + \nu_1 + \zeta_3\kappa\Big)B \\ &\leq \big(\lambda_5 - \lambda_6\big)B.\end{aligned}$$

6. Numerical simulation

To highlight the results of our analysis, we carry out a numerical simulation in this section. This simulation is carried out in Matlab using the difference method, in particular an explicit Euler scheme. Figure 4 shows that for a value of $\mathcal{R}_0$ less than 1 we have a complete elimination or stabilization at zero of classes T, B, and P but also of class V. The latter result can be explained by the fact that, in this model, class V is linked to classes T and B. On the other hand, Figure 5 shows the persistence of narcoterrorism and brigandage for a value of $\mathcal{R}_0$ strictly greater than 1. For this simulation, we consider the initial states C(0)=100000, R(0)=80, A(0)=1000, V(0)=2000, B(0)=110, T(0)=110, P(0)=80 and the parameter values defined in the table below:

Table 1: Parameter values estimeted

Parameters	value for extinction	value for persistence
Λ	36900	36900
γ_4	0.047	0.047
γ_5	0.0016	0.0016
γ_6	0.00149	0.00149
γ_7	0.0046	0.00046
γ_8	0.0000011	0.0000011
γ_9	0.011	0.011
θ_1	0.0032	0.22
θ_2	0.0032	0.24
η	0.19	0.19
ζ_1	0.27	0.27
ζ_2	0.27	0.27
ζ_3	0.37	0.37
ζ_4	0.37	0.37
μ	0.148	0.148
ν_1	0.02	0.02
ν_2	0.01	0.001
ν_3	0.02	0.02
τ_1	0.2	0.02
τ_2	0.125	0.0125
τ_3	0.125	0.125
σ_1	0.012	0.012
σ_2	0.006	0.006
α_1	0.2	0.2
α_2	0.31	0.78
α_3	0.31	0.48
ω_1	0.02	0.1
ω_2	0.02	0.147
ω_3	0.02	0.58
ω_4	0.04	0.5

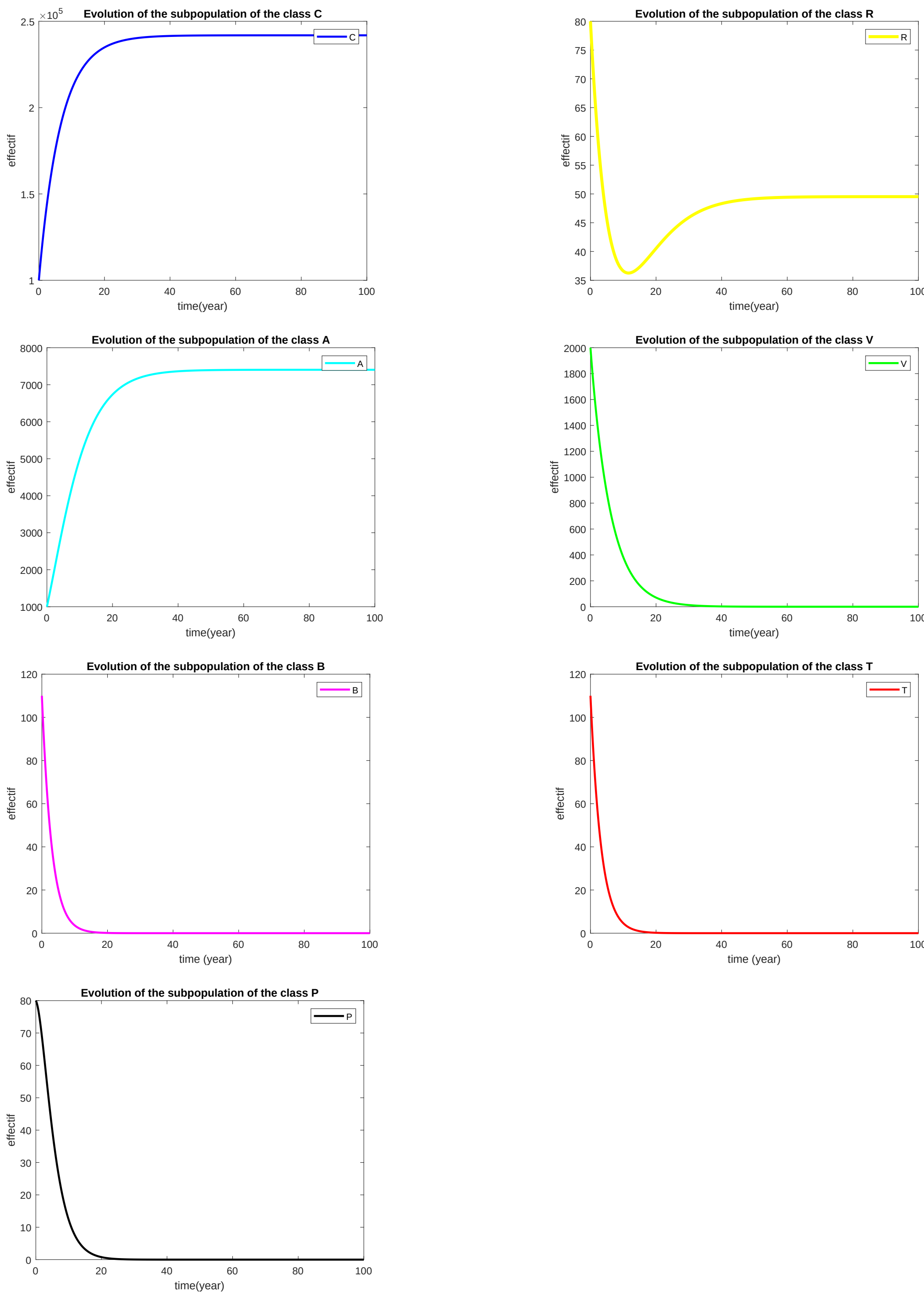

Figure 4: Evolution of the different classes of the model $(2.1)-(2.7)$ with the extinction values. We get $\mathcal{R}_0 = 0.7656$, which is less than unity.

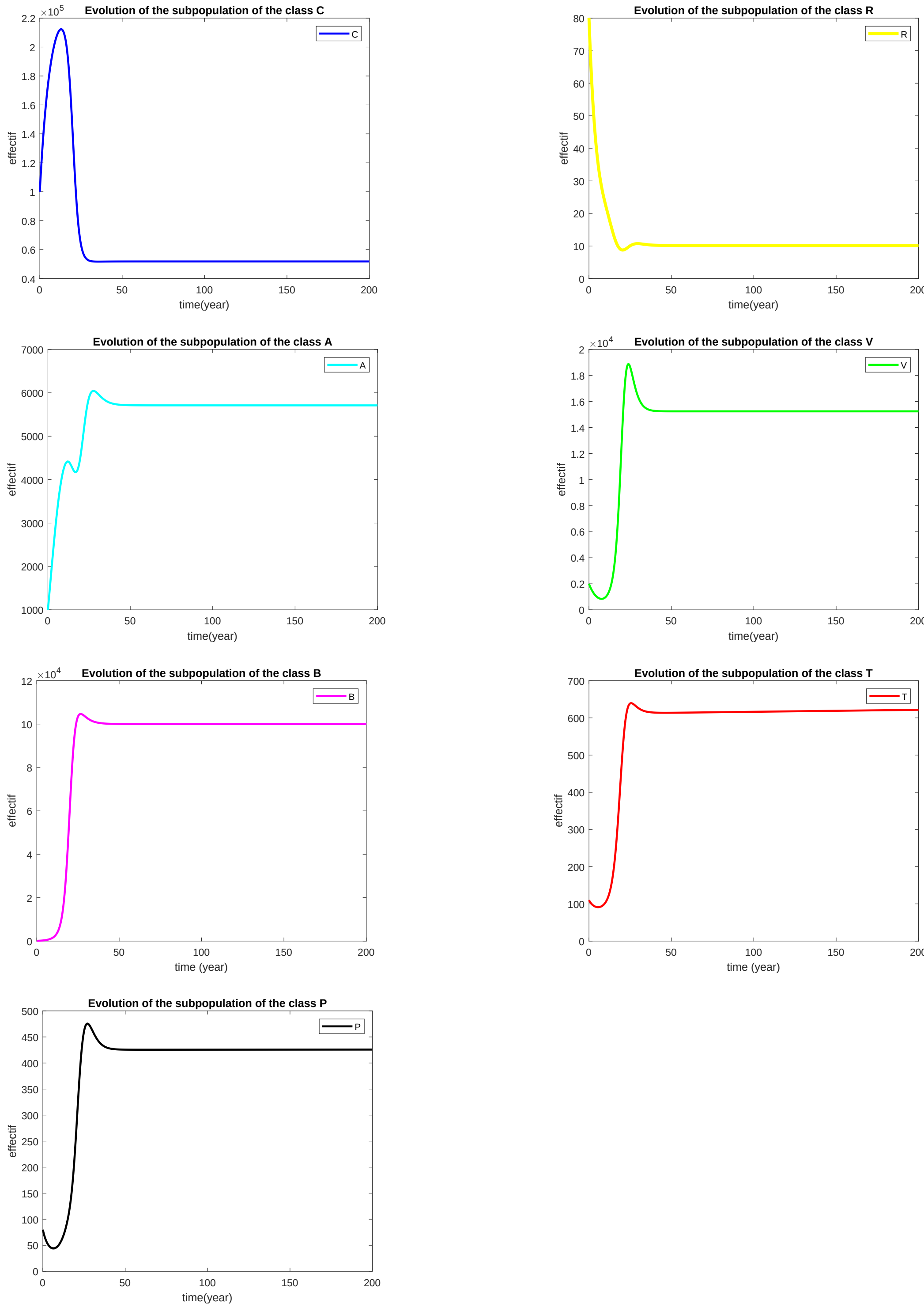

Figure 5: Evolution of the different classes of the model $(2.1) - (2.7)$ with persistence values. We obtain $\mathcal{R}_0 = 1.4966$, which is greater than unity.

7. Optimal control analysis

7.1. Strategy to fight against narcoterrorism and brigandage

In light of the results of the analysis, optimal control theory is applied to the $(2.1) - (2.7)$ model to fight narcoterrorism and banditry. Thus, two time-dependent control variables are introduced: $u_1(t)$ and $u_2(t)$, which are several strategies described in detail as follows:

(*i*) $u_1(t)$ is a strategy to fight against drug trafficking, organized crime, brigandage, and corruption. It also integrates all police actions of proximity, investigation, and protection. By ensuring a better territorial network and better training and equipment for the defense and security forces, as well as for volunteers for the defense of the country. In addition to these actions, this strategy could also integrate all the actions of accompaniment and reintegration of the accused or prisoners into active life. Note that the closer the u_2 strategy is to 1, the more efficient it is.

(*ii*) $u_2(t)$ is a strategy to combat narcoterrorism. It places particular emphasis on the fight against drug trafficking, which is the main source of funding for this type of terrorism. In addition, this strategy integrates all actions aimed at increasing the firepower of defense and security forces, while developing operational intelligence that is better adapted and better than that of narco-terrorists, so as to be able to carry out well-coordinated and well-calculated actions to minimize narco-terrorist attacks. Note that the closer u_2 is to 1, the more efficient it is.

7.2. Mathematical analysis of strategy optimality

Let's put

$$c_i(t) = 1 - u_i(t), \qquad \forall i \in \{1, 2\}. \tag{7.1}$$

Consequently, the optimal control model with the two aforementioned time-dependent variables is given by the following differential equations

$$\begin{cases} \dfrac{dC}{dt} &= \Lambda + \gamma_4 A + \gamma_5 P + \gamma_6 R + \gamma_7 B + \gamma_8 T + \gamma_9 V - \left(\alpha_1 \dfrac{T+B}{C+I} + c_1\alpha_2 \dfrac{B}{C+I} + c_2\alpha_3 \dfrac{T}{C+I} + \sigma_2 + \mu\right) C \\ \dfrac{dR}{dt} &= \nu_2 A - \left(c_2\omega_3 \dfrac{T}{R+I} + c_1\omega_4 \dfrac{B}{R+I} + \gamma_6 + \mu\right) R \\ \dfrac{dA}{dt} &= \sigma_1 V + \sigma_2 C + \nu_1 B - \left(c_1\nu_3 \dfrac{B}{I} + c_2\omega_2 \dfrac{T}{I} + \gamma_4 + \nu_2 + \mu + \zeta_1 \dfrac{T+B}{I}\right) A \\ \dfrac{dV}{dt} &= \alpha_1 C \dfrac{T+B}{C+I} - \left(\gamma_9 + \sigma_1 + \mu + \zeta_2 \dfrac{T+B}{I}\right) V \\ \dfrac{dB}{dt} &= c_1 \left(\alpha_2 \dfrac{CB}{C+I} + \omega_4 \dfrac{RB}{R+I} + \nu_3 \dfrac{AB}{I} + \theta_2 \dfrac{PB}{P+I}\right) - \left(c_2\omega_1 \dfrac{T}{I} + \tau_2 \dfrac{A+V}{I} + \gamma_7 + \nu_1 + \mu + \zeta_3 \dfrac{A+V}{I}\right) B \\ \dfrac{dT}{dt} &= c_2 \left(\alpha_3 C \dfrac{T}{C+I} + \omega_1 B \dfrac{T}{I} + \omega_2 A \dfrac{T}{I} + \omega_3 R \dfrac{T}{R+I} + \theta_1 P \dfrac{T}{P+I}\right) - \left(\tau_3 \dfrac{A+V}{I} + \gamma_8 + \mu + \zeta_4 \dfrac{A+V}{I}\right) T \\ \dfrac{dP}{dt} &= \tau_2 B \dfrac{A+V}{I} + \tau_3 T \dfrac{A+V}{I} - \left(c_2\theta_1 \dfrac{T}{P+I} + c_1\theta_2 \dfrac{B}{P+I} + \gamma_5 + \mu + \eta\right) P \end{cases} \tag{7.2}$$

with initial conditions given by (2.8). This system can be rewritten in matrix form as follows:

$$X'(t) = g(t, X, c) \tag{7.3}$$

where X is defined in (3.2), $c = (c_1(t), c_2(t)) \in \mathbb{R}^2$ verifies (7.1), and $g : \mathbb{R} \times^7 \times \mathbb{R}^2 \to \mathbb{R}^7$ is a non-linear function written as in (3.3) but introducing the control c in order to verify (7.2). The purpose of introducing the two control variables is to find the optimal solution required to minimize the number of individuals in both the narcoterrorists class and the brigands class. Consequently, the objective function for this control problem is given by

$$\mathscr{J}(u_1,u_2) = \min_{0\leqslant u_1,u_2\leqslant 1} \int_0^{T_f} \left(j(t) + \frac{1}{2}k(t) \right) dt \tag{7.4}$$

where

$$j(t) = w_1 B(t) + w_2 T(t) + w_3 P(t)$$
$$k(t) = \left[w_4 u_1^2(t) + w_5 u_2^2(t) \right]$$

where the constants $w_i, i = 1, 2, ..., 5$ are positive weights required to balance the corresponding terms of the objective function. We choose quadratic costs on the controls, where $\frac{1}{2}w_4 u_1^2(t)$, $\frac{1}{2}w_5 u_2^2(t)$, are the total costs of implementing the preventive measure and the military-police response to manage the active cases of narcoterrorism and brigands over the time interval $[0, T_f]$. More precisely, we are looking for the optimal dual control $u^* = \left(u_1^*, u_2^* \right)$ such that

$$\mathscr{J}(u_1^*, u_2^*) = \min \left\{ \mathscr{J}(u_1,u_2) : u_1, u_2 \in \mathcal{U} \right\}, \tag{7.5}$$

where, $\mathcal{U}$ is the non-empty control set defined by

$$\mathcal{U} = \left\{ (u_1,u_2) \;\middle|\; \begin{array}{c} u_i(t) \text{ is a piecewise continuous function on } [0,T_f] \\ \text{and} \quad 0 \leqslant u_i \leqslant 1, \quad \forall \in t \in [0,T_f], \quad i = 1,2 \end{array} \right\} \tag{7.6}$$

Thus, to determine the necessary conditions that the optimal control must satisfy, we use the Pontryagin maximum principle [12], which transforms the control problem (7.5) subject to the model (7.2) into a problem of pointwise minimization of a Hamiltonian $\mathcal{H}$. This Hamiltonian is given by

$$\begin{aligned}
\mathcal{H} =\; & w_1 B + w_2 T + w_3 P + \frac{1}{2}\left[w_4 u_1^2(t) + w_5 u_2^2(t) \right] \\
& + \lambda_1 \left[\Lambda + \gamma_4 A + \gamma_5 P + \gamma_6 R + \gamma_7 B + \gamma_8 T + \gamma_9 V - \left(\alpha_1 \frac{T+B}{C+I} + c_1 \alpha_2 \frac{B}{C+I} + c_2 \alpha_3 \frac{T}{C+I} + \sigma_2 + \mu \right) C \right] \\
& + \lambda_2 \left[\nu_2 A - \left(c_2 \omega_3 \frac{T}{R+I} + c_1 \omega_4 \frac{B}{R+I} + \gamma_6 + \mu \right) R \right] \\
& + \lambda_3 \left[\sigma_1 V + \sigma_2 C + \nu_1 B - \left(c_1 \nu_3 \frac{B}{I} + c_2 \omega_2 \frac{T}{I} + \gamma_4 + \nu_2 + \mu + \zeta_1 \frac{T+B}{I} \right) A \right] \\
& + \lambda_4 \left[\alpha_1 C \frac{T+B}{C+I} - \left(\gamma_9 + \sigma_1 + \mu + \zeta_2 \frac{T+B}{I} \right) V \right] \\
& + \lambda_5 \left[c_1 \left(\alpha_2 \frac{CB}{C+I} + \omega_4 \frac{RB}{R+I} + \nu_3 \frac{AB}{I} + \theta_2 \frac{PB}{P+I} \right) - \left(c_2 \omega_1 \frac{T}{I} + \tau_2 \frac{A+V}{I} + \gamma_7 + \nu_1 + \mu + \zeta_3 \frac{A+V}{I} \right) B \right] \\
& + \lambda_6 \left[c_2 \left(\alpha_3 C \frac{T}{C+I} + \omega_1 B \frac{T}{I} + \omega_2 A \frac{T}{I} + \omega_3 R \frac{T}{R+I} + \theta_1 P \frac{T}{P+I} \right) - \left(\tau_3 \frac{A+V}{I} + \gamma_8 + \mu + \zeta_4 \frac{A+V}{I} \right) T \right] \\
& + \lambda_7 \left[\tau_2 B \frac{A+V}{I} + \tau_3 T \frac{A+V}{I} - \left(c_2 \theta_1 \frac{T}{P+I} + c_1 \theta_2 \frac{B}{P+I} + \gamma_5 + \mu + \eta \right) P \right]
\end{aligned} \tag{7.7}$$

where $\lambda_i, i = 1, 2, ..., 7$, represent the adjoint variables associated with the state variables of the model (7.2). The standard existence results for the minimizing control problem, as they appeared in [7] are adapted as follows.

Theorem 7.1. *There exists an optimal control* $\left(u_1^*, u_2^*\right) \in \mathcal{U}$ *satisfying* (7.4) *subject to the control system* (7.2) *with non-negative initial conditions given by* (2.8).

Proof: The existence of optimal control is obtained thanks to Fleming and Rishel [7]. Thanks to a result of Lukes's [10] which ensures the existence of solutions for the state system (7.2) with constant coefficients, the set of controls and corresponding solutions is non-empty. In addition, the set of controls $\mathcal{U}$ is a closed convex set by definition, and the vector field of the system (7.2) is bounded. Also, the integrand of the objective function is convex, and $g(t, X, c)$ in (7.3) is convex concerning c. On the other hand, there exist $a_1, a_2 > 0$ and $\beta > 1$ such that

$$w_1 B + w_2 T + w_3 P + \frac{1}{2}\left[w_4 u_1^2(t) + w_5 u_2^2(t)\right] \geq a_1 \left(|u_1|^2 + |u_2|^2\right)^{\dfrac{\beta}{2}} - a_2$$

since the state variables are bounded. Then, we deduce the existence of an optimal control (u_1^*, u_2^*) that minimizes the objective function $\mathscr{J}(u_1, u_2)$. □

Theorem 7.2. *Given that* $\left(u_1^*, u_2^*\right)$ *minimizes the objective functional* (7.4) *subject to the corresponding state system* (7.2)*, then the adjoint variables* $\lambda_i, i = 1, 2, ..., 7$*, satisfy the following system:*

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= (\lambda_1 - \lambda_4)\alpha_1 \frac{(T+B)I}{(C+I)^2} + (\lambda_1 - \lambda_5)c_1\alpha_2 \frac{BI}{(C+I)^2} + (\lambda_1 - \lambda_6)c_2\alpha_3 \frac{TI}{(C+I)^2} + (\lambda_1 - \lambda_3)\sigma_2 + \lambda_1\mu \\
\frac{d\lambda_2}{dt} &= (\lambda_2 - \lambda_5)c_1\omega_4 \frac{BI}{(R+I)^2} + (\lambda_2 - \lambda_6)c_2\omega_3 \frac{TI}{(R+I)^2} + (\lambda_2 - \lambda_1)\gamma_6 + \lambda_2\mu \\
\frac{d\lambda_3}{dt} &= (\lambda_3 - \lambda_1)\gamma_4 + (\lambda_4 - \lambda_1)\alpha_1 \frac{(T+B)C}{(C+I)^2} + (\lambda_5 - \lambda_1)c_1\alpha_2 \frac{BC}{(C+I)^2} + (\lambda_6 - \lambda_1)c_2\alpha_3 \frac{TC}{(C+I)^2} \\
&\quad + (\lambda_6 - \lambda_2)c_2\omega_3 \frac{TR}{(R+I)^2} + (\lambda_5 - \lambda_2)c_1\omega_4 \frac{BR}{(R+I)^2} + (\lambda_3 - \lambda_5)c_1\nu_3 \frac{B(V+T+B)}{I^2} \\
&\quad + (\lambda_3 - \lambda_6)c_2\omega_2 \frac{T(V+T+B)}{I^2} + \lambda_3\zeta_1 \frac{(T+B)(V+T+B)}{I^2} + \lambda_3\mu - \lambda_4\zeta_2 \frac{(T+B)V}{I^2} \\
&\quad + (\lambda_5 - \lambda_7)c_1\theta_2 \frac{PB}{(P+I)^2} + (\lambda_5 - \lambda_7)\tau_2 \frac{B(T+B)}{I^2} + (\lambda_6 - \lambda_5)c_2\omega_1 \frac{TB}{I^2} + (\lambda_3 - \lambda_2)\nu_2 \\
&\quad + \lambda_5\zeta_3 \frac{B(T+B)}{I^2} + (\lambda_6 - \lambda_7)\theta_1 \frac{TP}{(P+I)^2} + (\lambda_6 - \lambda_7)\tau_3 \frac{T(T+B)}{I^2} + \lambda_6\zeta_4 \frac{T(T+B)}{I^2}
\end{aligned} \tag{7.8}$$

$$\begin{aligned}\frac{d\lambda_4}{dt} &= (\lambda_3-\lambda_1)\gamma_9+(\lambda_4-\lambda_1)\alpha_1\frac{(T+B)C}{(C+I)^2}+(\lambda_5-\lambda_1)c_1\alpha_2\frac{BC}{(C+I)^2}+(\lambda_6-\lambda_1)c_2\alpha_3\frac{TC}{(C+I)^2}\\&+(\lambda_6-\lambda_2)c_2\omega_3\frac{TR}{(R+I)^2}+(\lambda_5-\lambda_2)c_1\omega_4\frac{BR}{(R+I)^2}+(\lambda_5-\lambda_3)c_1\nu_3\frac{BA}{I^2}+(\lambda_4-\lambda_3)\sigma_1\\&+(\lambda_6-\lambda_3)c_2\omega_2\frac{TA}{I^2}-\lambda_3\zeta_1\frac{(T+B)A}{I^2}+\lambda_4\mu+\lambda_4\zeta_2\frac{(T+B)(A+T+B)}{I^2}\\&+(\lambda_5-\lambda_7)c_1\theta_2\frac{PB}{(P+I)^2}+(\lambda_5-\lambda_7)\tau_2\frac{B(T+B)}{I^2}+(\lambda_6-\lambda_5)c_2\omega_1\frac{TB}{I^2}\\&+\lambda_5\zeta_3\frac{B(T+B)}{I^2}+(\lambda_6-\lambda_7)\theta_1\frac{TP}{(P+I)^2}+(\lambda_6-\lambda_7)\tau_3\frac{T(T+B)}{I^2}+\lambda_6\zeta_4\frac{T(T+B)}{I^2}\end{aligned}$$

$$\begin{aligned}\frac{d\lambda_5}{dt} &= -w_1+(\lambda_5-\lambda_1)\gamma_7+(\lambda_6-\lambda_2)c_2\omega_3\frac{TR}{(R+I)^2}+(\lambda_6-\lambda_1)c_2\alpha_3\frac{TC}{(C+I)^2}+(\lambda_5-\lambda_3)\nu_1\\&+(\lambda_1-\lambda_4)\alpha_1\frac{C(C+A+V)}{(C+I)^2}+(\lambda_1-\lambda_5)c_1\alpha_2\frac{C(C+A+V+T)}{(C+I)^2}+(\lambda_2-\lambda_5)c_1\omega_4\frac{R(R+A+V+T)}{(R+I)^2}\\&+\lambda_5\mu+(\lambda_3-\lambda_6)c_2\omega_2\frac{TA}{I^2}+(\lambda_3-\lambda_5)c_1\nu_3\frac{A(A+V+T)}{I^2}+\lambda_3\zeta_1\frac{A(A+V)}{I^2}+\lambda_4\zeta_2\frac{V(A+V)}{I^2}\\&+(\lambda_5-\lambda_7)\tau_2\frac{(A+V)(A+V+T)}{I^2}+\lambda_5\zeta_3\frac{(A+V)(A+V+T)}{I^2}+(\lambda_5-\lambda_7)c_1\theta_2\frac{P(P+A+V+T)}{(P+I)^2}\\&+(\lambda_6-\lambda_5)c_2\omega_1\frac{TB}{I^2}+(\lambda_6-\lambda_7)\theta_1\frac{TP}{(P+I)^2}+(\lambda_7-\lambda_6)\tau_3\frac{T(A+V)}{I^2}-\lambda_6\zeta_4\frac{T(A+V)}{I^2}\\&+(\lambda_5-\lambda_6)c_2\omega_1\frac{T(A+V+T)}{I^2}\end{aligned}$$

$$\begin{aligned}\frac{d\lambda_6}{dt} &= -w_2+(\lambda_6-\lambda_1)\gamma_8+(\lambda_1-\lambda_4)\alpha_1\frac{C(C+A+V)}{(C+I)^2}+(\lambda_5-\lambda_1)c_1\alpha_2\frac{BC}{(C+I)^2}+(\lambda_5-\lambda_2)c_1\omega_4\frac{BR}{(R+I)^2}\\&+(\lambda_2-\lambda_9)c_2\omega_3\frac{R(R+A+V+B)}{(R+I)^2}+(\lambda_5-\lambda_3)c_1\nu_3\frac{BA}{I^2}+(\lambda_3-\lambda_6)c_2\omega_2\frac{A(A+V+B)}{I^2}+\lambda_3\zeta_1\frac{A(A+V)}{I^2}\\&+\lambda_4\zeta_2\frac{V(A+V)}{I^2}+(\lambda_5-\lambda_7)c_1\theta_2\frac{PB}{(P+I)^2}+(\lambda_5-\lambda_6)c_2\omega_1\frac{B(A+V+B)}{I^2}-\lambda_5\zeta_3\frac{B(A+V)}{I^2}\\&+(\lambda_7-\lambda_5)\tau_2\frac{B(A+V)}{I^2}+(\lambda_7-\lambda_6)c_2\theta_1\frac{P(P+A+V+B)}{(P+I)^2}+(\lambda_6-\lambda_7)\tau_3\frac{(A+V)(A+V+B)}{I^2}+\lambda_6\mu\\&+\lambda_6\zeta_4\frac{(A+V)(A+V+B)}{I^2}+(\lambda_1-\lambda_6)c_2\alpha_3\frac{C(C+A+V+B)}{(C+I)^2}\end{aligned}$$

$$\frac{d\lambda_7}{dt} = -w_3+(\lambda_7-\lambda_1)\gamma_5+(\lambda_7-\lambda_5)c_1\theta_2\frac{BI}{(P+I)^2}+(\lambda_7-\lambda_6)c_2\theta_1\frac{TI}{(P+I)^2}+\lambda_7\mu+\lambda_7\eta$$

with transversality conditions

$$\lambda_i(T_f)=0, \quad i=1,2,...,7.$$

Further, the optimal control $\left(u_1^*, u_2^*\right)$ is given as follows

$$\begin{cases} u_1^* = \max\left\{0, \min\left\{1, \dfrac{(\lambda_5-\lambda_1)\alpha_2\dfrac{BC}{C+I}+(\lambda_5-\lambda_2)\omega_4\dfrac{BR}{R+I}+(\lambda_5-\lambda_3)\nu_3\dfrac{BA}{I}+(\lambda_5-\lambda_7)\theta_2\dfrac{BP}{P+I}}{w_4}\right\}\right\} \\ \\ u_2^* = \max\left\{0, \min\left\{1, \dfrac{(\lambda_6-\lambda_1)\alpha_3\dfrac{TC}{C+I}+(\lambda_6-\lambda_2)\omega_3\dfrac{TR}{R+I}+(\lambda_6-\lambda_3)\omega_2\dfrac{TA}{I}+(\lambda_6-\lambda_5)\omega_1\dfrac{TB}{I}+(\lambda_6-\lambda_7)\theta_1\dfrac{TP}{P+I}}{w_5}\right\}\right\} \end{cases} \tag{7.9}$$

Proof:

As mentioned earlier, the characterization of the optimal solution is obtained by applying the Pontryagin's maximum principle to the Hamiltonian of the system $\mathcal{H}$. The system of ordinary differential equations (7.8) governing the adjoint variables is derived by differentiating the Hamiltonian. Further, the control characterizations in (7.9) are derived by solving, on the interior of the control set $\mathcal{U}$, the partial differentials of the Hamiltonian $\mathcal{H}$ with respect to each of the controls u_1 and u_2. Hence, by standard arguments involving control bounds, it follows that:

$$u_1^* = \begin{cases} 0 & \text{if } \ r_1^* \le 0 \\ r_1^* & \text{if } \ 0 < r_1^* < 1 \\ 1 & \text{if } \ r_1^* \ge 1 \end{cases}$$

$$u_2^* = \begin{cases} 0 & \text{if } \ r_2^* \le 0 \\ r_2^* & \text{if } \ 0 < r_2^* < 1 \\ 1 & \text{if } \ r_2^* \ge 1 \end{cases}$$

where,

$$\begin{cases} r_1^* = \dfrac{(\lambda_5-\lambda_1)\alpha_2\dfrac{BC}{C+I}+(\lambda_5-\lambda_2)\omega_4\dfrac{BR}{R+I}+(\lambda_5-\lambda_3)\nu_3\dfrac{BA}{I}+(\lambda_5-\lambda_7)\theta_2\dfrac{BP}{P+I}}{w_4} \\ \\ r_2^* = \dfrac{(\lambda_6-\lambda_1)\alpha_3\dfrac{TC}{C+I}+(\lambda_6-\lambda_2)\omega_3\dfrac{TR}{R+I}+(\lambda_6-\lambda_3)\omega_2\dfrac{TA}{I}+(\lambda_6-\lambda_5)\omega_1\dfrac{TB}{I}+(\lambda_6-\lambda_7)\theta_1\dfrac{TP}{P+I}}{w_5} \end{cases}$$

This puts an end to the proof. □

7.3. Numerical simulation

In this section, we use numerical simulation to illustrate the effect of control on the dynamics of the controlled compartments, in particular compartments B and T respectively. For reasons of clarity, the color red has been chosen for the curves of the classes with no control over the persistence parameters of brigandage and narcoterrorism, while the color blue has been chosen for the curves with control. Figure 6 shows that if u_1 control is very weak and u_2 control is effective, banditry persists and narcoterrorism stabilizes. Figure 7 shows that when u_1 control is effective and u_2 control is weak, banditry and narcoterrorism stabilize. Finally, there is a very quick stabilization in classes B and T when the u_1 and u_2 controls approach 1, as shown in Figure 8.

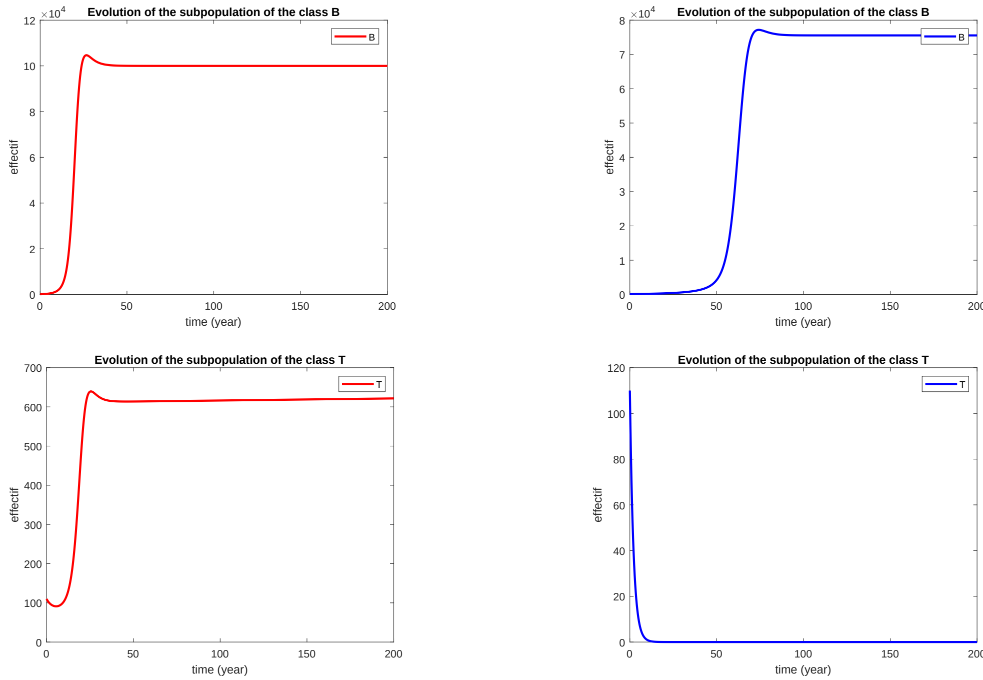

Figure 6: Dynamics of evolution of classes B and T illustrating the effect of control with $u_1 = 0.25$, and $u_2 = 0.75$

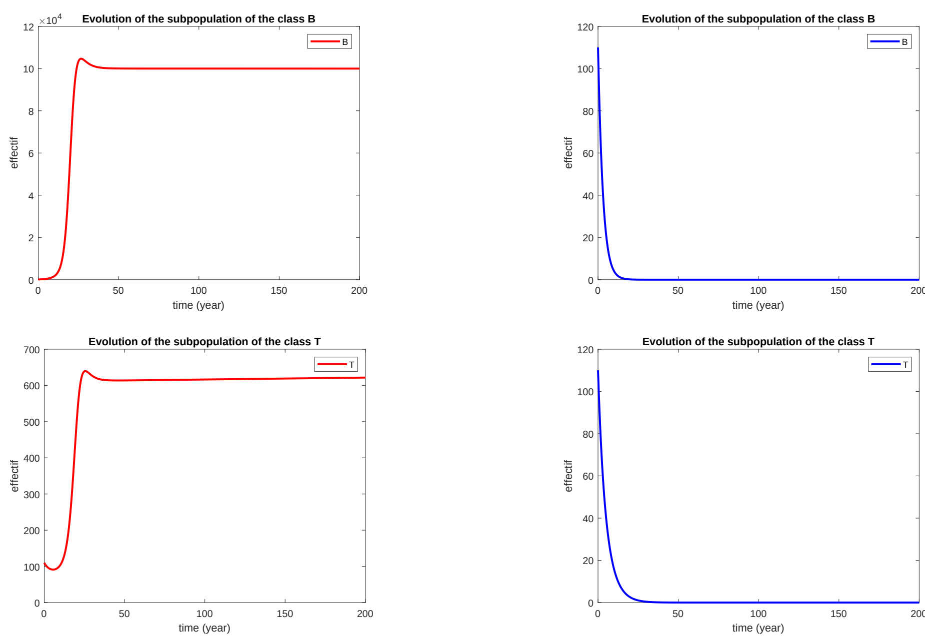

Figure 7: Dynamics of classes B and T illustrating the effect of control with $u_1 = 0.75$, and $u_2 = 0.25$

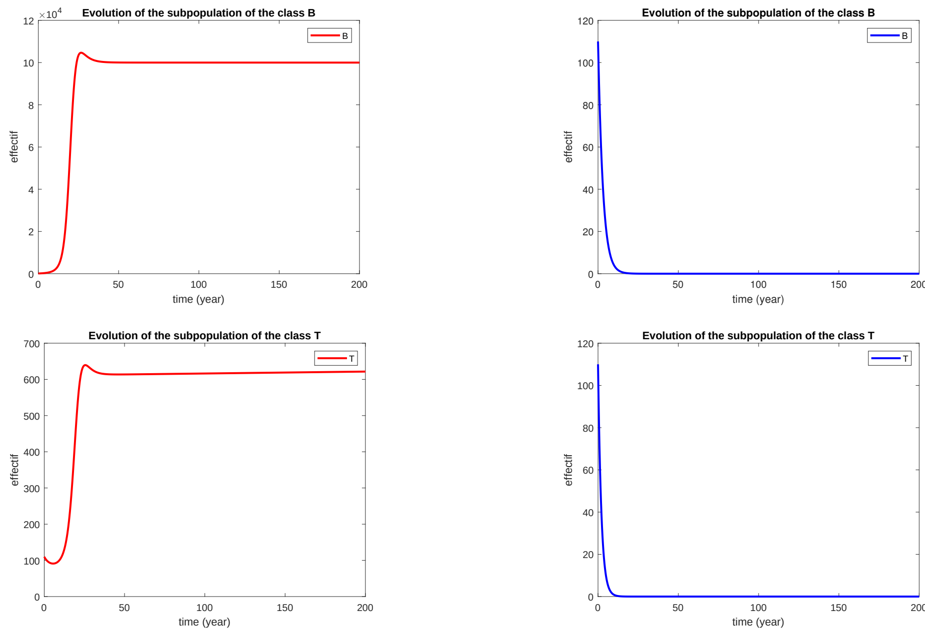

Figure 8: Dynamics of classes B and T illustrating the effect of control with $u_1 = 0.75$, and $u_2 = 0.75$

8. Conclusion

In this study, we first designed a mathematical model to illustrate the dynamics of narcoterrorism, based on the situation in certain Sahelian countries. The proposed mathematical model focused on the dynamics of recruitment into the narcoterrorist and brigand classes, showing the importance of contact and the deterrent presence of certain classes. We then carried out a rigorous mathematical analysis of the model. We then defined a first threshold $\mathcal{R}_0$ for this model, which designates the number of basic reproductions in the brigand or narcoterrorist class. In other words, the average number of people that a brigand or narcoterrorist manages to recruit into his class. From this threshold, we give asymptotic stability conditions for the equilibrium without brigands or terrorists. We also define two global thresholds, which are sufficient conditions for the eradication of narcoterrorism. Based on the results of the analysis, a strategy for combating narcoterrorism and banditry was proposed through a model check. The effectiveness of the strategy was then assessed using an optimality study based essentially on the Pontryagin maxima principle and Fleming's theorem. To make this study more readable, we carried out a numerical simulation of the analysis and control results. On the strength of some of the results of this study, we are convinced that to fight narcoterrorism and banditry more effectively, the Sahel and West African states must work to strengthen their systems of governance adapted to their realities. This strengthening of governance could be achieved through a better administrative and security network, as well as the development of local production activities and the promotion of local products. It is still time for the countries of the Sahel to take their destiny into their own hands. They will need to strengthen their cooperation on security, economic and social issues. There is still time for the Sahel countries to apply measures of good governance and virtuous governance adapted to their reality, all within a framework of faultless social cohesion and a local security system that is effective against violent extremism, narcoterrorism, and all forms of organized crime.

9. Acknowledgement

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Exploring new proofs for three important trigonometric inequalities

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Abstract. In this concise article, we present alternative proofs of three significant inequalities relating to various trigonometric functions. The key ingredients of these proofs are well-known series expansions defined with Bernoulli numbers. We are thus contributing to the development of this technique to establish precise inequalities. In some sense, our results provide a simplified overview of these fundamental mathematical relationships.

AMS Subject Classifications: 33B10, 26D05.

Keywords: Trigonometric functions; Bernoulli's numbers

Contents

1. Introduction

There are different methods for determining the characteristics of an inequality, each offering a valuable strategy for understanding the behavior of mathematical functions. One approach, often used in calculus, is the derivative test. This method involves examining the sign of the derivative of a function to determine whether it is increasing or decreasing over a given interval. By analyzing critical points and concavity intervals, the derivative test provides valuable information about the behavior of functions and their associated inequalities.

Another contemporary method of studying inequality is to use series expansion techniques. By expanding a function into an integer series, one can better understand its behavior and derive inequalities based on the properties of the series. This method is particularly useful for exploring inequalities in the context of complex functions and their convergent properties. For a more comprehensive understanding of these methods and their applications in inequality analysis, we may refer to the following references: [6], [7], and [8]. In them, detailed explanations and examples to aid in the study of inequalities and their associated mathematical concepts are provided.

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In particular, Guo et al. [11] presented an alternative proof for the following double-sided inequalities, contributing to the examination of trigonometric inequalities and their properties:

$$2+\frac{8}{45}t^3\tan t<\left(\frac{\sin t}{t}\right)^2+\frac{\tan t}{t}<2+\left(\frac{2}{\pi}\right)^4 t^3\tan t,\qquad 0<t<\frac{\pi}{2}.$$

Still in the spirit of proposing an alternative proof, Nantomah [12] reestablished the following hyperbolic inequalities:

$$\left(\frac{\sinh t}{t}\right)^{2a}+\left(\frac{\tanh t}{t}\right)^{a}>\left(\frac{t}{\sinh t}\right)^{2a}+\left(\frac{t}{\tanh t}\right)^{a}>2,\qquad t>0\ and\ a\geq 1.$$

On the other hand, Zhu [10] gives the very simple alternative proof of the following inequality:

$$\left(\frac{\sin t}{t}\right)^2+\frac{\tan t}{t}>2,\qquad 0<t<\frac{\pi}{2}.$$

This inequality is known as Wilker's inequality.
Later, Zhu and Zhang [9] gave a new concise proof of the following inequalities with the help of power series expansion of trigonometric functions:

$$\frac{16}{\pi^2}t^3\tan t<\left(\frac{\sin t}{t}\right)^2+\frac{\tan t}{t}-2<\frac{8}{45}t^3\tan t,\qquad 0<t<\frac{\pi}{2}.$$

Thus, the principle of alternative proof is central to understanding all the mathematical facets of inequalities. In order to explain our contribution in this direction, some existing results need to be presented. As remarkable advances in the field, the following three inequalities (in theorem form) are elucidated in [13]:

Theorem 1.1. *For $0<t<\pi/2$, the following inequalities*

$$\left[\sqrt{\frac{1+\cos t}{2}}\right]^{4/3}<\frac{\sin t}{t}<\left[\sqrt{\frac{1+\cos t}{2}}\right]^{\gamma}$$

hold true, where $\gamma=2\ln(\pi/2)/\ln 2\approx 1.30299$.

Theorem 1.2. *For $0<t<\pi/2$, we have*

$$t\tan\frac{t}{2}<\ln\left(\frac{1}{\cos t}\right).$$

Theorem 1.3. *For $0<t<\pi/2$, we have*

$$\ln\left(\frac{t}{\sin t}\right)<\frac{\sin t-t\cos t}{2\sin t}.$$

In [13], Bhayo, Ali, and Sándor established the validity of these inequalities using the concept of monotonicity, thus demonstrating their importance in mathematical analysis. The main goal of this concise article is to provide alternative proofs for Theorems 1.1, 1.2 and 1.3. Our approach relies on power series expansions to demonstrate their validity, which remain new in the literature to the best of our knowledge. We hope that this approach sheds more light on the fundamental principles underlying these inequalities and will be inspirational in future proofs.

The remainder of the paper is organized into three distinct sections. Section 2 provides essential preliminaries and introduces a pivotal lemma, while Section 3 details the alternative proofs. A conclusion is given in Section 4.

2. Preliminaries and Key Lemma

The Bernoulli numbers represent a crucial sequence of rational numbers. Several researchers are actively studying them to solve various mathematical problems. These numbers have found applications in various areas of mathematics, including number theory, combinatorics, and mathematical analysis. In this context, researchers explore the properties and relationships of Bernoulli numbers to discover deeper insights into mathematical structures and phenomena. A notable application of Bernoulli numbers is in their role in integer series expansions for various trigonometric functions. Through the study of Bernoulli numbers, researchers have discovered elegant expressions and relationships that facilitate the derivation of such expansions. These expansions play a fundamental role in mathematical analysis, allowing the representation of trigonometric functions as infinite series. The importance of these results is underlined in several references. In particular, [1], [2], [4], and [5] provide comprehensive information on integer series expansions derived from Bernoulli numbers. These works offer detailed explanations and mathematical proofs to support their claims.

In our current work, we exploit these established results as essential tools in our main proofs. Using integer series expansions derived from Bernoulli numbers, we aim to elucidate key mathematical relationships and advance our understanding of the underlying mathematical structures. Especially, the following inequalities will be used in our main proofs:

$$\cot t = \frac{1}{t} - \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|t^{2n-1}}{(2n)!}, \qquad 0 < |t| < \pi, \tag{2.1}$$

$$\ln\left(\frac{\sin t}{t}\right) = -\sum_{n=1}^{\infty} \frac{2^{2n-1}|B_{2n}|t^{2n}}{n(2n)!}, \qquad 0 < |t| < \pi, \tag{2.2}$$

$$\ln\cos t = -\sum_{n=1}^{\infty} \frac{2^{2n-1}(2^{2n}-1)|B_{2n}|t^{2n}}{n(2n)!}, \qquad |t| < \frac{\pi}{2}, \tag{2.3}$$

$$\frac{1}{\sin^2 t} = \frac{1}{t^2} + \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n}|B_{2n}|t^{2n-2}}{(2n)!}, \qquad 0 < |t| < \pi, \tag{2.4}$$

$$\tan t = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)|B_{2n}|t^{2n-1}}{(2n)!}, \qquad |t| < \frac{\pi}{2} \tag{2.5}$$

and

$$\sec^2 t = \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|t^{2n-2}}{(2n)!}, \qquad |t| < \frac{\pi}{2}. \tag{2.6}$$

In addition, the following technical lemma will play an important role in one of our proofs.

Lemma 2.1. *[3] For $0 < R \leq \infty$, let $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $B(t) = \sum_{n=0}^{\infty} b_n t^n$ be two real power series converging on the interval $(-R, R)$. If the sequence $(a_n/b_n)_n$ is increasing(decreasing) and $b_n > 0$ for all n, then the ratio function $A(t)/B(t)$ is also increasing(decreasing) on $(0, R)$.*

We are now in a position to prove Theorems 1.1, 1.2 and 1.3 in alternative manners through the use of derivatives, Bernoulli's series expansions, and Lemma 2.1 when appropriate.

3. Proofs

Let us prove Theorems 1.1, 1.2 and 1.3, in turns.

3.1. Proof of Theorem 1.1

To prove this result, let us consider the function

$$F(t)=\frac{\ln\left(\frac{t}{\sin t}\right)}{\ln\left(\frac{1}{\sqrt{(1+\cos t)/2}}\right)}=\frac{\ln\left(\frac{t}{\sin t}\right)}{\ln\left(\frac{1}{\sqrt{\cos^2\frac{t}{2}}}\right)}=\frac{\ln\left(\frac{t}{\sin t}\right)}{\ln\left(\frac{1}{\cos\frac{t}{2}}\right)}=\frac{A(t)}{B(t)}, \tag{3.1}$$

where

$$A(t)=\ln\left(\frac{t}{\sin t}\right)=\sum_{n=1}^{\infty}\frac{2^{2n-1}|B_{2n}|t^{2n}}{n(2n)!}=\sum_{n=1}^{\infty}a_n t^{2n},$$

where

$$a_n=\frac{2^{2n-1}|B_{2n}|}{n(2n)!}$$

and

$$B(t)=\ln\left(\frac{1}{\cos\frac{t}{2}}\right)=\sum_{n=1}^{\infty}\frac{2^{2n-1}(2^{2n}-1)|B_{2n}|t^{2n}}{2^{2n}n(2n)!}=\sum_{n=1}^{\infty}b_n t^{2n}$$

where

$$b_n=\frac{2^{2n-1}(2^{2n}-1)|B_{2n}|}{2^{2n}n(2n)!}.$$

Let us now set

$$\begin{aligned}c_n&=\frac{b_n}{a_n}\\&=\frac{2^{2n-1}(2^{2n}-1)|B_{2n}|}{2^{2n}n(2n)!}\Big/\frac{2^{2n-1}|B_{2n}|}{n(2n)!}=\frac{2^{2n}-1}{2^{2n}}.\end{aligned}$$

Clearly c_n is increasing for $n\geq 1$.
Therefore, by Lemma 2.1, $B(t)/A(t)$ is strictly increasing and $A(t)/B(t)$ is strictly decreasing, so $F(t)$ too. This implies that $F(\pi/2)<F(t)<F(0)$. Since $\lim_{t\to 0}F(t)=\frac{4}{3}$ and

$$\lim_{t\to\pi/2}F(t)=\frac{2\ln\left(\frac{\pi}{2}\right)}{\ln 2}=1.30299=\gamma.$$

It follows from Equation (3.1) that

$$\left[\sqrt{\frac{1+\cos t}{2}}\right]^{4/3}<\frac{\sin t}{t}<\left[\sqrt{\frac{1+\cos t}{2}}\right]^{\gamma}.$$

This ends this alternative proof. □

3.2. Proof of Theorem 1.2

Let us set

$$f(t)=\ln\left(\frac{1}{\cos t}\right)-t\tan\frac{t}{2}=-\ln(\cos t)-t\tan\frac{t}{2}.$$

Therefore, we have

$$f'(t)=\tan t-\tan\frac{t}{2}-\frac{t}{2}\sec^2\frac{t}{2}.$$

Owing to Equations (2.5) and (2.6), we have

$$\begin{aligned}
f'(t)&=\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)|B_{2n}|t^{2n-1}}{(2n)!}-\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)|B_{2n}|t^{2n-1}}{2^{2n-1}(2n)!}\\
&-\frac{t}{2}\sum_{n=1}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|t^{2n-2}}{2^{2n-2}(2n)!}\\
&=\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)|B_{2n}|t^{2n-1}}{(2n)!}-\sum_{n=1}^{\infty}\frac{2(2^{2n}-1)|B_{2n}|t^{2n-1}}{(2n)!}\\
&-\sum_{n=1}^{\infty}\frac{(2n-1)2(2^{2n}-1)|B_{2n}|t^{2n-1}}{(2n)!}\\
&=\sum_{n=1}^{\infty}\frac{(2^{2n}-1)|B_{2n}|t^{2n-1}}{(2n)!}\left[2^{2n}-2-2(2n-1)\right]\\
&=\sum_{n=1}^{\infty}\frac{(2^{2n}-1)|B_{2n}|t^{2n-1}}{(2n)!}\left(2^{2n}-4n\right).
\end{aligned}$$

For $n\geq 1$, it is clear that $2^{2n}\geq 4n$. This implies that $f'(t)>0$, so $f(t)$ is strictly increasing. In particular, we have $f(t)>f(0)=0$, which is equivalent to

$$t\tan\frac{t}{2}<\ln\left(\frac{1}{\cos t}\right).$$

This ends this alternative proof. □

3.3. Proof of Theorem 1.3

Let us set

$$f(t)=\ln\left(\frac{t}{\sin t}\right)-\frac{\sin t-t\cos t}{2\sin t}.$$

Hence, after some developments, we establish that

$$\begin{aligned}
f'(t)&=\frac{\sin t}{t}\left[\frac{\sin t-t\cos t}{\sin^2 t}\right]-\frac{1}{2}\left[\frac{\sin t(\cos t-\cos t+t\sin t)-\cos t(\sin t-t\cos t)}{\sin^2 t}\right]\\
&=\frac{\sin t-t\cos t}{t\sin t}-\frac{1}{2}\left[\frac{t\sin^2 t-\sin t\cos t+t\cos^2 t}{\sin^2 t}\right]\\
&=\frac{1}{t}-\cot t-\frac{1}{2}\left[\frac{t-\sin t\cos t}{\sin^2 t}\right]=\frac{1}{t}-\cot t-\frac{t}{2\sin^2 t}+\frac{\cot t}{2}\\
&=\frac{1}{t}-\frac{\cot t}{2}-\frac{t}{2\sin^2 t}.
\end{aligned}$$

Owing to Equations (2.1) and (2.4), we have

$$f'(t) = \frac{1}{t} - \frac{1}{2}\left[\frac{1}{t} - \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|t^{2n-1}}{(2n)!}\right] - \frac{t}{2}\left[\frac{1}{t^2} + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)|B_{2n}|t^{2n-2}}{(2n)!}\right]$$

$$= \sum_{n=2}^{\infty} \frac{2^{2n}|B_{2n}|t^{2n-1}}{2(2n)!} - \sum_{n=2}^{\infty} \frac{2^{2n}(2n-1)|B_{2n}|t^{2n-1}}{2(2n)!}$$
$$= \sum_{n=2}^{\infty} \frac{2^{2n}|B_{2n}|t^{2n-1}}{(2n)!}(1-n).$$

It is clear that $1-n<0$ for $n \geq 2$, implying that $f'(t)<0$. Hence $f(t)$ is a strictly decreasing function and, in particular, $f(t)<f(0)=0$, so

$$\ln\left(\frac{t}{\sin t}\right) < \frac{\sin t - t\cos t}{2\sin t}.$$

This ends this alternative proof. □

4. Conclusion

In this concise article, we have reestablished important existing theorems in the area of trigonometric inequalities, with an approach involving series expansions based on Bernoulli numbers. In some sense, this extends the applicability of such series expansions to explore comprehensive mathematical results beyond conventional methodologies.

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Polynomial stability of nonlinear Timoshenko beam with distributed delay

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Abstract. In this work, we consider a Nonlinear Timoshenko system with distributed delay-time. We prove the polynomial stability of the system for the case of nonequal speeds of wave propagation. This is after verifying the exponential stability in the opposite one.

AMS Subject Classifications: 35B40, 35L70, 93D15, 93D20, 74F05.

Keywords: Porous system, distributed delay, polynomial stability, Lyapunov functional.

Contents

1. Introduction

In this work, we consider a nonlinear Timoshenko system with distributed delay term,

$$\begin{cases} \rho_1\phi_{\mathbf{tt}} - k(\phi_\varkappa + \psi)_\varkappa = 0, \\ \rho_2\psi_{\mathbf{tt}} - b\psi_{\varkappa\varkappa} + k(\phi_\varkappa + \psi) + \mu_1\psi_{\mathbf{t}} + \int_{\iota_1}^{\iota_2} \eta_2(\tau)\psi_{\mathbf{t}}(\varkappa, \mathbf{t} - \tau)d\tau + f(\psi) = 0, \end{cases} \tag{1.1}$$

where $(\varkappa, \mathbf{t}) \in (0,1) \times \mathbb{R}^+$. The system (1.1) with $\mu_1 = \eta_2 = f = 0$, was first proposed by Timoshenko [24] as a model that describes the impact of vibrations on a thin elastic beam of length. The functions $\phi = \phi(\varkappa, \mathbf{t})$ and $\psi = \psi(\varkappa, \mathbf{t})$ describe the small transverse displacement of the beam and the rotation angle of the beam's filament. The parameters ρ_1,ρ_2, k and b are positive constants. The function $f(\psi)$ is a forcing term and $\mu_1\psi_{\mathbf{t}}$ designate a frictional damping. The distributed delay is given by $\int_{\iota_1}^{\iota_2} \eta_2(\tau)\psi_{\mathbf{t}}(\varkappa, \mathbf{t} - \tau)d\tau$, where, $\iota_1, \iota_2 > 0$.
We provide the system (1.1) with the initial data

$$\begin{cases} \phi(\varkappa, 0) = \phi_0,\ \phi_{\mathbf{t}}(\varkappa, 0) = \phi_1,\ \psi(\varkappa, 0) = \psi_0,\ \psi_{\mathbf{t}}(\varkappa, 0) = \psi_1 \\ \psi_{\mathbf{t}}(\varkappa, -\mathbf{t}) = f_0(\varkappa, \mathbf{t}),\ \varkappa \in (0,1), \end{cases} \tag{1.2}$$

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and the boundary conditions

$$\phi(0,\mathbf{t}) = \phi(1,\mathbf{t}) = \psi(0,\mathbf{t}) = \psi(1,\mathbf{t}) = 0. \tag{1.3}$$

We remember that Timoshenko system without delay has been considered by many authors. Their goal was to achieve the asymptotic behaviour of the solutions of these systems by introducing different types of damping. See for instance [1–3, 9, 11, 12, 15, 19] and references therein.

In recent years, including the delay term makes the problems of EDPs more interesting. In fact, delays can cause destabilization of a system which is stable without the delays. Datko et al. [7] studied the the destabilizing effect of arbitrarily small delays in the boundary control of a wave equation. In [17], the authors proved an exponential decay result of the solution under suitable assumptions of the delayed wave equation where the delay is considered both in the boundary condition and in the internal feedback. Later [18] the same authors introduced a distributed delay on a part of the boundary, and they proved an exponential stability under some assumptions, they also studied the following problem with internal feedback

$$\begin{cases} u_{\mathbf{tt}} - \Delta u + \mu_0 u_{\mathbf{t}} + \int_{\iota_1}^{\iota_2} a(\varkappa)\mu(\tau) u_{\mathbf{t}}(\mathbf{t}-\tau)d\tau \\ u = 0 \qquad on \quad \Gamma_0(0,\alpha) \\ \dfrac{\partial u}{\partial \nu} = 0 \qquad on \quad \Gamma_1(0,\alpha) \\ u(\varkappa,0) = u_0(\varkappa) \;\; and \quad u_{\mathbf{t}}(\varkappa,0) = u_1(\varkappa) \qquad in\;\Omega \\ u_{\mathbf{t}}(\varkappa,-\mathbf{t}) = f_0(\varkappa,-\mathbf{t}) \qquad in \;\; \Omega(0,\iota_2) \end{cases} \tag{1.4}$$

where $a \in L^2(\Omega)$ is a function satisfies

$$\mu_0 > ||a||_\alpha \int_{\iota_1}^{\iota_2} \mu(\tau)d\tau.$$

They obtained an exponential decay result for the energy.

In [22], the authors discussed the stability of a linear Timoshenko system with a constant delay

$$\begin{cases} \rho_1\phi_{\mathbf{tt}} - k(\phi_\varkappa + \psi)_\varkappa = 0, \\ \rho_2\psi_{\mathbf{tt}} - b\psi_{\varkappa\varkappa} + k(\phi_\varkappa + \psi) + \mu_1\psi_{\mathbf{t}} + \eta_2\psi_{\mathbf{t}}(\varkappa,\mathbf{t}-\tau) = 0. \end{cases} \tag{1.5}$$

a necessary condition which made the solutions of (1.5) exponentially stable is

$$\frac{k}{\rho_1} = \frac{b}{\rho_2}. \tag{1.6}$$

It is most important to report that most of results on Timoshenko types systems is based on the above condition, otherwise, only a polynomial stability was proved for the case of nonequal speeds (see [2, 9, 11, 12, 15]).

For a non linear Timoshenko system , Feng and Pelicer [8] added to (1.5) a forcing term $f(\psi)$ in the second equation and proved an exponential decay under an appropriate condition between the weights of the the delay term and frictional damping, their result was extended by Hao and Wei [12] to nonlinear heatTimoshenko system of based on the energy method. In the case where the speeds are nonequal, they established a polynomial decay estimate.

System (1.1) was recently investigated by Bouzettouta et al. [4] and they proved an exponential decay result of the energy when (1.6) holds, in this paper our goal is to complete their study for the case of non equal wave speeds.

2. Preliminaries

The necessary assumptions and transformations needed to obtain the desired results were presented in this section. As in [17], we use the following notation

$$\chi(\varkappa,\rho,\tau,\mathbf{t}) = \psi_{\mathbf{t}}(\varkappa,\mathbf{t}-\rho\tau), \; \varkappa \in (0,L), \; \rho \in (0,L), \; \mathbf{t},\tau \in (\iota_1,\iota_2).$$

The new variable χ satisfies the following differential equation

$$\tau\chi_{\mathbf{t}}(\varkappa,\rho,\tau,\mathbf{t})+\chi_{\rho}(\varkappa,\rho,\tau,\mathbf{t})=0,\ \ (\varkappa,\rho,\tau,\mathbf{t})\in(0,L)\times(0,L)\times(\iota_1,\iota_2)\times(0,+\infty).$$

Therefore, the problem (1.1) becomes

$$\begin{cases}\rho_1\phi_{\mathbf{tt}}-k(\phi_\varkappa+\psi)_\varkappa=0,\ \varkappa\in(0,L),\ \mathbf{t}>0,\\ \rho_2\psi_{\mathbf{tt}}-b\psi_{\varkappa\varkappa}+k(\phi_\varkappa+\psi)+\mu_1\psi_{\mathbf{t}}\\ +\int_{\iota_1}^{\iota_2}\eta_2(\tau)\chi(\varkappa,1,\tau,\mathbf{t})d\tau+f(\psi)=0,\ \varkappa\in(0,L),\ \mathbf{t}>0,\\ \tau\chi_{\mathbf{t}}(\varkappa,\rho,\tau,\mathbf{t})+\chi_\rho(\varkappa,\rho,\tau,\mathbf{t})=0,\ \rho\in(0,L),\ \tau\in(\iota_1,\iota_2),\ \mathbf{t}>0,\end{cases}\tag{2.1}$$

with the initial data and boundary conditions

$$\begin{cases}\phi(\varkappa,0)=\phi_0,\ \phi_{\mathbf{t}}(\varkappa,0)=\phi_1,\ \varkappa\in(0,L),\\ \psi(\varkappa,0)=\psi_0,\ \psi_{\mathbf{t}}(\varkappa,0)=\psi_1,\ \varkappa\in(0,L),\\ \chi(\varkappa,\rho,\tau,0)=f_0(\varkappa,\rho\tau),\ \varkappa\in(0,1),\ \rho\in(0,L),\ \tau\in(0,\iota_2),\\ \phi(0,\mathbf{t})=\phi(1,\mathbf{t})=\psi(0,\mathbf{t})=\psi(1,\mathbf{t})=0,\ \mathbf{t}>0.\end{cases}\tag{2.2}$$

In what follows, we assume that

$$\int_{\iota_1}^{\iota_2}|\eta_2(\tau)|\,d\tau<\mu_1.\tag{2.3}$$

We assume that $f:\mathbb{R}\to\mathbb{R}$ satisfies

$$\left|f(\psi^1)-f(\psi^2)\right|\le k_0\left(\left|\psi^1\right|^\theta-\left|\psi^2\right|^\theta\right)\left|\psi^1-\psi^2\right|\tag{2.4}$$

for all $\psi^1,\psi^2\in\mathbb{R}$, where $k_0>0$, $\theta>0$. Also

$$0\le\tilde{f}(\psi)\le f(\psi)\psi,\ \text{for all}\ \psi\in\mathbb{R},\tag{2.5}$$

with

$$\tilde{f}(y)=\int_0^y f(\tau)d\tau.$$

Let $\mathbf{H}$ the Hilbert space,

$$\mathbf{H}=\mathbf{H}_0^1(0,L)\times L^2(0,L)\times\mathbf{H}_0^1(0,L)\times L^2(0,L)\times L^2((0,L)\times(0,L)\times(\iota_1,\iota_2)),$$

and for any $U=(\phi,u,\psi,v,\chi)^{\mathbf{t}}\in\mathbf{H}$, $\tilde{U}=\left(\widetilde{\phi},\widetilde{u},\widetilde{\psi},\widetilde{v},\widetilde{\chi}\right)^{\mathbf{t}}\in\mathbf{H}$, we equip the space$\mathbf{H}$ with the inner product

$$\begin{aligned}\left\langle U,\tilde{U}\right\rangle_{\mathbf{H}}&=\int_0^L\left[\rho_1u\widetilde{u}+\rho_2v\widetilde{v}+k(\phi_\varkappa+\psi)\left(\widetilde{\phi}_\varkappa+\widetilde{\psi}\right)+b\psi_\varkappa\widetilde{\psi}_\varkappa\right]d\varkappa\\&\quad+\int_0^L\int_{\iota_1}^{\iota_2}\tau|\eta_2(\tau)|\int_0^L\chi(\varkappa,\rho,\tau,\mathbf{t})\,\widetilde{\chi}(\varkappa,\rho,\tau,\mathbf{t})\,d\rho d\tau d\varkappa.\end{aligned}$$

By introducing the variables $\phi_{\mathbf{t}}=u$ and $\psi_{\mathbf{t}}=v$, then the system (2.1)-(2.2) is equivalent to

$$\begin{cases}U_{\mathbf{t}}=AU+F,\ \mathbf{t}>0\\ U(\varkappa,0)=U^0(\varkappa)=\left(\phi^0,\phi^1,\psi^0,\psi^1,f_0\right)^{\mathbf{t}},\end{cases}\tag{2.6}$$

and

$$AU=\begin{pmatrix}u\\ \frac{k}{\rho_1}(\phi_{\varkappa\varkappa}+\psi_\varkappa)\\ v\\ \frac{b}{\rho_2}\psi_{\varkappa\varkappa}-\frac{k}{\rho_2}(\phi_\varkappa+\psi)-\frac{\mu_1}{\rho_2}v-\frac{\mu_1}{\rho_2}\int_{\iota_1}^{\iota_2}\eta_2(\tau)\chi(\varkappa,\rho,\tau,\mathbf{t})\,d\tau\\ -\frac{1}{\tau}\chi_\rho(\varkappa,\rho,\tau,\mathbf{t})\end{pmatrix},\tag{2.7}$$

$$F=\begin{pmatrix}0\\0\\0\\\frac{-1}{\rho^2}f(\psi)\\0\end{pmatrix}$$

with the domain

$$D(A)=\left\{(\phi,u,\psi,v,\chi)^{\mathbf{t}}\in\mathbf{H}_1\right\},$$

with

$$\begin{aligned}\mathbf{H}_1=&\left(\mathbf{H}^2(0,L)\cap\mathbf{H}_0^1(0,L)\right)\times\mathbf{H}_0^1(0,L)\times\left(\mathbf{H}^2(0,L)\cap\mathbf{H}_0^1(0,L)\right)\\&\times\mathbf{H}_0^1(0,L)\times L^2\left((0,L)\times(0,L)\times(\iota_1,\iota_2)\right).\end{aligned}$$

We state the following well-posedness result (see [8]).

Theorem 2.1. *Let* $U_0\in$***H*** *and suppose that*(2.3)-(2.5) *hold. Then, the problem* (2.1)-(2.2) *has a unique weak solution* $U\in C(\mathbb{R}^+,\mathbf{H})$. *If* $U_0\in D(A)$, *then*

$$U\in C\left(\mathbb{R}^+,D(A)\right)\cap C\left(\mathbb{R}^+,\mathbf{H}\right).$$

3. Decay result

We exploit the multipliers technique, we show that the solution of (2.1)–(2.2) decays exponentially. First, we present the following lemmas.

Lemma 3.1. *The energy* $\mathbf{E}$ *of (2.1)–(2.2), defined by*

$$\begin{aligned}\mathbf{E}(\mathbf{t})=&\frac{1}{2}\int_0^L\left(\rho_1\phi_{\mathbf{t}}^2+\rho_2\psi_{\mathbf{t}}^2\right)d\varkappa+\frac{1}{2}\int_0^L\left\{K(\phi_\varkappa+\psi)^2+b\psi_\varkappa^2\right\}d\varkappa\\&+\int_0^L\int_0^L\int_{\iota_1}^{\iota_2}\tau\left|\eta_2(\tau)\right|\chi(\varkappa,\rho,\tau,\mathbf{t})\,d\tau d\rho d\varkappa+\int_0^L\tilde{f}(\psi)d\varkappa\end{aligned}\tag{3.1}$$

satisfies

$$\frac{d\mathbf{E}(\mathbf{t})}{d\mathbf{t}}\leq-m_1\int_0^L\psi_{\mathbf{t}}^2d\varkappa\leq0,\tag{3.2}$$

where $m_1=\mu_1-\int_{\iota_1}^{\iota_2}\left|\eta_2(\tau)\right|d\tau$.

Proof. Multiplying $(2.1)_1$ by $\phi_{\mathbf{t}}$, $(2.1)_2$ by $\psi_{\mathbf{t}}$, integrating and combining the results, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\mathbf{t}}\int_0^L\left(\rho_1\phi_{\mathbf{t}}^2+\rho_2\psi_{\mathbf{t}}^2\right)d\varkappa+\frac{1}{2}\frac{d}{d\mathbf{t}}\int_0^L\left\{K(\phi_\varkappa+\psi)^2+b\psi_\varkappa^2\right\}d\varkappa\\&=-\mu_1\int_0^L\psi_{\mathbf{t}}^2d\varkappa-\mu_1\int_0^Lf(\psi)\,\psi_{\mathbf{t}}d\varkappa-\int_0^L\int_{\iota_1}^{\iota_2}\psi_{\mathbf{t}}\eta_2(\tau)\,\chi(\varkappa,1,\tau,\mathbf{t})\,d\tau d\varkappa.\end{aligned}\tag{3.3}$$

Multiplying $(2.1)_3$ by $\left|\eta_2(\tau)\right|\chi(\varkappa,\rho,\tau,\mathbf{t})$, integrating over $(0,L)\times(0,L)\times(\iota_1,\iota_2)$, summing the result with (3.3) and applying Young's inequality, we have (3.1) and (3.2). ■

Lemma 3.2. *The functional*

$$I_1(\mathbf{t}):=-\int_0^L(\rho_1\phi\phi_{\mathbf{t}}+\rho_2\psi\psi_{\mathbf{t}})d\varkappa-\frac{\mu_1}{2}\int_0^L\psi^2d\varkappa.\tag{3.4}$$

satisfies

$$\frac{dI_1(\mathbf{t})}{d\mathbf{t}} \leq -\int_0^L (\rho_1\phi_{\mathbf{t}}^2 + \rho_2\psi_{\mathbf{t}}^2)d\varkappa + c_0\int_0^L \psi_{\varkappa}^2 d\varkappa + k\int_0^L (\phi_{\varkappa}+\psi)^2 d\varkappa + \frac{\mu_1}{4}\int_0^L\int_{\iota_1}^{\iota_2} |\eta_2(\tau)|\,\chi^2(\varkappa,1,\tau,\mathbf{t})d\tau d\varkappa, \tag{3.5}$$

Proof. Differentiating $I_1(\mathbf{t})$ with $(2.1)_1$, $(2.1)_2$ and Young's and Poincaré inequalities, we obtain (3.5). ■

Now, we introduce the following problem

$$-w_{\varkappa\varkappa} = \psi_{\varkappa}, \qquad w(0) = w(1) = 0, \tag{3.6}$$

where w the solution of the above problem is given by

$$w(\varkappa,\mathbf{t}) = -\int_0^{\varkappa} \psi(z,\mathbf{t})\,dz + \varkappa\left(\int_0^L \psi(z,\mathbf{t})\,dz\right).$$

Lemma 3.3. *The solution of* (3.6) *satisfies*

$$\int_0^L w_{\varkappa}^2 d\varkappa \leq \int_0^L \psi^2 d\varkappa \text{ and } \int_0^L w_{\mathbf{t}}^2 d\varkappa \leq \int_0^L \psi_{\mathbf{t}}^2 d\varkappa.$$

Proof. Multiplying (3.6) by w, integrating and introduce the Hölder inequality, we arrive at

$$\int_0^L w_{\varkappa}^2 d\varkappa \leq \int_0^L \psi^2 d\varkappa$$

Next, we differentiate (3.6) and using the same above technique, we get

$$\int_0^L w_{\mathbf{t}}^2 d\varkappa \leq \int_0^L \psi_{\mathbf{t}}^2 d\varkappa. \tag{3.7}$$

■

Lemma 3.4. *Let $\Phi = (\phi,\psi,\chi)$ be the solution of the system (2.1)–(2.2). Then, for any $\varepsilon_2 > 0$, the functional*

$$I_2(\mathbf{t}) := \int_0^L \left(\rho_2\psi_{\mathbf{t}}\psi + \rho_1\phi_{\mathbf{t}}w + \frac{\mu_1}{2}\psi^2\right)d\varkappa, \tag{3.8}$$

satisfies

$$\frac{dI_2(\mathbf{t})}{d\mathbf{t}} \leq -\frac{b}{2}\int_0^L \psi_{\varkappa}^2 d\varkappa + \left(\frac{\rho_1}{4\varepsilon_2}+\rho_2\right)\int_0^L \psi_{\mathbf{t}}^2 d\varkappa + \rho_1\varepsilon_2\int_0^L \phi_{\mathbf{t}}^2 d\varkappa + \frac{\mu_1}{4\varepsilon_2}\int_0^L\left(\int_{\iota_1}^{\iota_2} |\eta_2(\tau)|\,\chi^2(\varkappa,1,\tau,\mathbf{t})d\tau\right)d\varkappa - \int_0^L \tilde{f}(\psi)d\varkappa. \tag{3.9}$$

Proof. By differentiation $I_2(\mathbf{t})$ and using$(2.1)_1$, $(2.1)_2$, we obtain

$$\frac{dI_2(\mathbf{t})}{d\mathbf{t}} = \rho_2\int_0^L \psi_{\mathbf{t}}^2 d\varkappa - b\int_0^L \psi_{\varkappa}^2 d\varkappa + \rho_1\int_0^L \phi_{\mathbf{t}}w_{\mathbf{t}}d\varkappa - k\int_0^L \psi^2 d\varkappa + k\int_0^L w_{\varkappa}^2 d\varkappa - \int_0^L f(\psi)\psi d\varkappa - \int_0^L \psi\left(\int_{\iota_1}^{\iota_2} \eta_2(\tau)\,\chi(\varkappa,1,\tau,\mathbf{t})d\tau\right)d\varkappa. \tag{3.10}$$

Using (3.7), Young's, Cauchy-Schwarz, Poincaré inequalities, we have

$$\begin{aligned}\rho_1 \int_0^L \phi_{\mathbf{t}} w_{\mathbf{t}} d\varkappa &\leq \rho_1 \varepsilon_2 \int_0^L \phi_{\mathbf{t}}^2 d\varkappa + \frac{\rho_1}{4\varepsilon_2} \int_0^L w_{\mathbf{t}}^2 d\varkappa \\ &\leq \rho_1 \varepsilon_2 \int_0^L \phi_{\mathbf{t}}^2 d\varkappa + \frac{\rho_1}{4\varepsilon_2} \int_0^L \psi_{\mathbf{t}}^2 d\varkappa, \end{aligned} \tag{3.11}$$

$$\begin{aligned} & -\int_0^L \psi \left(\int_{\iota_1}^{\iota_2} \left| \eta_2(\tau) \chi(\varkappa, 1, \tau, \mathbf{t}) \right| d\tau \right) d\varkappa \\ & \leq \delta_1 \int_0^L \psi_\varkappa^2 d\varkappa + \frac{\mu_1}{4\delta_1} \int_0^L \left(\int_{\iota_1}^{\iota_2} \left| \eta_2(\tau) \right| \chi^2(\varkappa, 1, \tau, \mathbf{t}) d\tau \right) d\varkappa, \end{aligned} \tag{3.12}$$

$$\begin{aligned}\int_0^L |f(\psi)\psi| \, d\varkappa &\leq \int_0^L |\psi|^\theta \, |\psi| \, |\psi| \, d\varkappa \\ &\leq \|\psi\|_{2(\theta+1)}^\theta \, \|\psi\|_{2(\theta+1)} \, \|\psi\| \\ &\leq c_1 \int_0^L \psi_\varkappa^2 d\varkappa. \end{aligned} \tag{3.13}$$

By substituting (3.11), (3.12), (3.13) in (3.10), recalling (2.5) and letting $\delta_1 = \frac{b}{2}$, we obtain (3.9). ■

Lemma 3.5. *The functional*

$$I_3(\mathbf{t}) := \rho_2 \int_0^L \psi_{\mathbf{t}} (\phi_\varkappa + \psi) + \rho_2 \int_0^L \psi_\varkappa \phi_{\mathbf{t}} d\varkappa,$$

satisfies

$$\begin{aligned}\frac{dI_3(\mathbf{t})}{d\mathbf{t}} &\leq b \left[\psi_\varkappa \phi_\varkappa \right]_0^1 d\varkappa + \left(\rho_2 + \frac{\mu_1^2}{k} \right) \int_0^L \psi_{\mathbf{t}}^2 d\varkappa - \frac{k}{4} \int_0^L (\phi_\varkappa + \psi)^2 d\varkappa \\ &+ c_1 \int_0^L \psi_\varkappa^2 d\varkappa + \frac{\mu_1}{k} \int_0^L \int_{\iota_1}^{\iota_2} \left| \eta_2(\tau) \right| \chi^2(\varkappa, 1, \tau, \mathbf{t}) d\tau d\varkappa - \int_0^L \tilde{f}(\psi) d\varkappa \\ &+ \left(\frac{\rho_2 k - \rho_1 b}{\rho_1} \right) \int_0^L \psi_\varkappa \left(\phi_\varkappa + \psi \right)_\varkappa d\varkappa, \end{aligned} \tag{3.14}$$

where c_1 is a positive constant.

Proof. By differentiation $I_3(\mathbf{t})$ and exploiting $(2.1)_1$, $(2.1)_2$, we have

$$\begin{aligned}\frac{dI_3(\mathbf{t})}{d\mathbf{t}} &= \rho_2 \int_0^L \psi_{\mathbf{tt}} (\phi_\varkappa + \psi) d\varkappa + \rho_2 \int_0^L \psi_{\mathbf{t}} (\phi_\varkappa + \psi)_{\mathbf{t}} d\varkappa + \rho_2 \int_0^L \psi_{\varkappa \mathbf{t}} \phi_{\mathbf{t}} d\varkappa \\ &+ \rho_2 \int_0^L \psi_\varkappa \phi_{\mathbf{tt}} d\varkappa \\ &= b \left[\psi_\varkappa \phi_\varkappa \right]_0^1 + \rho_2 \int_0^L \psi_{\mathbf{t}}^2 d\varkappa - k \int_0^L (\phi_\varkappa + \psi)^2 d\varkappa - \mu_1 \int_0^L \psi_{\mathbf{t}} (\phi_\varkappa + \psi) d\varkappa \\ &- \int_0^L \int_{\iota_1}^{\iota_2} \eta_2(\tau) (\phi_\varkappa + \psi) \chi(\varkappa, 1, \tau, \mathbf{t}) d\tau d\varkappa - \int_0^L f(\psi)(\phi_\varkappa + \psi) d\varkappa. \end{aligned} \tag{3.15}$$

By functional inequalities, we arrive at

$$\mu_1 \int_0^L |\psi_{\mathbf{t}}(\phi_\varkappa + \psi)| \, d\varkappa \le \frac{k}{4} \int_0^L (\phi_\varkappa + \psi)^2 d\varkappa + \frac{\mu_1^2}{k} \int_0^L \psi_{\mathbf{t}}^2 d\varkappa, \tag{3.16}$$

$$\begin{aligned} &\int_0^L (\phi_\varkappa + \psi) \int_{\iota_1}^{\iota_2} |\eta_2(\tau) \, \chi(\varkappa, 1, \tau, \mathbf{t})| \, d\tau d\varkappa \\ &\le \frac{k}{4} \int_0^L (\phi_\varkappa + \psi)^2 d\varkappa + \frac{\mu_1^2}{k} \int_0^L \int_{\iota_1}^{\iota_2} |\eta_2(\tau)| \, \chi^2(\varkappa, 1, \tau, \mathbf{t}) d\tau d\varkappa, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \int_0^L f(\psi)\phi_\varkappa d\varkappa &\le \frac{\rho_0}{2b^2} \int_0^L \phi_\varkappa^2 d\varkappa + \frac{b^2}{2\rho_0\lambda_1} \int_0^L \psi_\varkappa^2 d\varkappa \\ &\le \frac{\rho_0}{2b^2} \int_0^L (\phi_\varkappa + \psi)^2 d\varkappa + \frac{\rho_0}{2b^2} \int_0^L \psi^2 d\varkappa + \frac{b^2}{2\rho_0\lambda_1} \int_0^L \psi_\varkappa^2 d\varkappa \\ &\le \frac{\rho_0}{2b^2} \int_0^L (\phi_\varkappa + \psi)^2 d\varkappa + \left(\frac{\rho_0}{2\lambda_1 b^2} + \frac{b^2}{2\rho_0\lambda_1} \right) \int_0^L \psi_\varkappa^2 d\varkappa. \end{aligned} \tag{3.18}$$

Inserting (3.16)-(3.18) in (3.15) and letting $\rho_0 = \dfrac{1}{2}kb^2$, we obtain (3.14). ∎

To manipulate the boundary terms appeared in (3.14), we introduce the function

$$q(\varkappa) = -4\varkappa + 2, \quad \varkappa \in (0, 1).$$

So, we were able to find the following result.

Lemma 3.6. *For any $\varepsilon_1 > 0$, we have*

$$\begin{aligned} b\left[\psi_\varkappa\phi_\varkappa\right]_0^1 &\le -\frac{b\rho_2}{4\varepsilon_1}\frac{d}{d\mathbf{t}} \int_0^L q\psi_{\mathbf{t}}\psi_\varkappa d\varkappa - \frac{\rho_1\varepsilon_1}{k}\frac{d}{d\mathbf{t}} \int_0^L q\phi_{\mathbf{t}}\phi_\varkappa d\varkappa + 3\varepsilon_1 \int_0^L \phi_\varkappa^2 d\varkappa \\ &+ \left(\frac{2\rho_1\varepsilon_1}{k} + \frac{b\rho_2}{2\varepsilon_1}\right) \int_0^L \psi_{\mathbf{t}}^2 d\varkappa + \left(\frac{k^2\varepsilon_1^2}{4} + \frac{\varepsilon_1}{4b^2}\right) \int_0^L (\phi_\varkappa + \psi)^2 d\varkappa \\ &+ \frac{b}{4\varepsilon_1} \int_0^L \int_{\iota_1}^{\iota_2} |\eta_2(\tau)| \, \chi^2(\varkappa, 1, \tau, \mathbf{t}) \, d\varkappa \\ &+ \left(\frac{b^2}{2\varepsilon_1^2} + \frac{1}{4\lambda_1 b^2} + \frac{b^2}{8\varepsilon_1^2\lambda_1} + \frac{\mu_1 b}{4\varepsilon_1} + \frac{b^2}{4\varepsilon_1^3} + \varepsilon_1\right) \int_0^L \psi_\varkappa^2 d\varkappa \end{aligned} \tag{3.19}$$

Proof. Young's inequality gives easily for $\varepsilon_1 > 0$,

$$b\left[\psi_\varkappa\phi_\varkappa\right]_0^1 \le \varepsilon_1 \left[\phi_\varkappa^2(1) + \phi_\varkappa^2(0)\right] + \frac{b^2}{4\varepsilon_1} \left[\psi_\varkappa^2(1) + \psi_\varkappa^2(0)\right], \tag{3.20}$$

we need the following fact

$$\frac{d}{d\mathbf{t}} \int_0^L b\rho_2 q\psi_{\mathbf{t}}\psi_\varkappa d\varkappa = b\rho_2 \int_0^L q\psi_{\mathbf{tt}}\psi_\varkappa d\varkappa + b\rho_2 \int_0^L q\psi_{\mathbf{t}}\psi_{\varkappa\mathbf{t}} d\varkappa.$$

On the other hand

$$
\begin{aligned}
b\rho_2 \int_0^L q\psi_{\mathbf{tt}}\psi_\varkappa d\varkappa &= b^2 \int_0^L q\psi_{\varkappa\varkappa}\psi_\varkappa d\varkappa - kb \int_0^L q(\phi_{\mathbf{t}}+\psi)\psi_\varkappa d\varkappa \\
&- b \int_0^L \int_{\iota_1}^{\iota_2} q\psi_\varkappa \eta_2(\tau)\chi(\varkappa,1,\tau,\mathbf{t})\, d\tau d\varkappa - b\int_0^L qf(\psi)\phi_\varkappa d\varkappa \\
&\le -b^2\left[\psi_\varkappa^2(1)+\psi_\varkappa^2(0)\right] + 2b^2\int_0^L \psi_\varkappa^2 d\varkappa \\
&+ (k^2\varepsilon_1^2 + \frac{\varepsilon_1}{b^2})\int_0^L (\phi_\varkappa+\psi)^2 d\varkappa \\
&+ (\frac{b^2}{\varepsilon_1^2} + \frac{\varepsilon_1}{2\lambda_1 b^2} + \frac{b^2}{2\varepsilon_1\lambda_1} + \mu_1 b)\int_0^L \psi_\varkappa^2 d\varkappa \\
&+ b\int_0^L \int_{\iota_1}^{\iota_2} |\eta_2(\tau)|\,\chi^2(\varkappa,1,\tau,\mathbf{t})\, d\tau d\varkappa. \qquad (3.21)
\end{aligned}
$$

Therefore

$$
b\rho_2 \int_0^L q\psi_{\mathbf{t}}\psi_{\varkappa\mathbf{t}} d\varkappa = 2\rho_2 b\int_0^L \psi_{\mathbf{t}}^2 d\varkappa
$$

Similarly

$$
\begin{aligned}
\frac{d}{d\mathbf{t}}\int_0^L \rho_1 q\phi_{\mathbf{t}}\phi_\varkappa d\varkappa &= \int_0^L q(\phi_{\mathbf{t}}+\psi)\phi_\varkappa d\varkappa + \int_0^L \rho_1 q\phi_{\mathbf{t}}\phi_{\varkappa\mathbf{t}} d\varkappa \\
&\le -k\left[\phi_\varkappa^2(1)+\phi_\varkappa^2(0)\right] + 3k\int_0^L \phi_\varkappa^2 d\varkappa \\
&+ k\int_0^L \psi_\varkappa^2 d\varkappa + 2\rho_1\int_0^L \psi_{\mathbf{t}}^2 d\varkappa
\end{aligned}
$$

which gives us (3.19) by exploiting (3.20)-(3.21). ■

Lemma 3.7. *([13]) For $\eta_1 > 0$, the functional*

$$
I_4(\mathbf{t}) = \int_0^L \int_0^L \int_{\iota_1}^{\iota_2} \tau \mathbf{e}^{-\tau\rho} |\eta_2(\tau)|\,\chi^2(\varkappa,\rho,\tau,\mathbf{t})\, d\tau d\rho d\varkappa, \qquad (3.22)
$$

satisfies

$$
\begin{aligned}
\frac{dI_4(\mathbf{t})}{d\mathbf{t}} &\le -\eta_1 \int_0^L \int_0^L \int_{\iota_1}^{\iota_2} \tau\,|\eta_2(\tau)|\,\chi^2(\varkappa,\rho,\tau,\mathbf{t}) d\tau d\rho d\varkappa \\
&- \eta_1 \int_0^L \int_{\iota_1}^{\iota_2} |\eta_2(\tau)|\,\chi^2(\varkappa,1,\tau,\mathbf{t})\, d\tau d\varkappa + \beta\int_0^L \phi_{\mathbf{t}}^2 d\varkappa. \qquad (3.23)
\end{aligned}
$$

where β is a positive constant.

Let $\mathcal{L}(\mathbf{t})$ the Lyapunov functional given by

$$
\mathcal{L}(\mathbf{t}) = N\mathbf{E}(\mathbf{t}) + \frac{1}{8}I_1(\mathbf{t}) + N_1 I_2(\mathbf{t}) + I_3(\mathbf{t}) + N_2 I_4(\mathbf{t}), \qquad (3.24)
$$

where $N_1, N_2, N > 0$.

Lemma 3.8. *There exist $\beta_1, \beta_2 > 0$, such that $\mathcal{L}(\mathbf{t})$ verifies*

$$\beta_1 \mathbf{E}(\mathbf{t}) \leq \mathcal{L}(\mathbf{t}) \leq \beta_2 \mathbf{E}(\mathbf{t}), \qquad \forall \mathbf{t} \geq 0, \tag{3.25}$$

and

$$\mathcal{L}'(\mathbf{t}) \leq -\lambda_1 \mathbf{E}(\mathbf{t}) + \left(\frac{\rho_2 k - \rho_1 b}{\rho_1}\right) \int_0^L \psi_\varkappa (\phi_\varkappa + \psi)_\varkappa \, d\varkappa. \tag{3.26}$$

Proof. Let

$$\mathcal{L}(\mathbf{t}) := N\mathbf{E}(\mathbf{t}) + \frac{1}{8} I_1(\mathbf{t}) + N_1 I_2(\mathbf{t}) + I_3(\mathbf{t}) + N_2 I_4(\mathbf{t}),$$

then

$$\begin{aligned}
|\mathcal{L}(\mathbf{t}) - N\mathbf{E}(\mathbf{t})| \leq & \frac{\rho_1}{8} \int_0^L |\phi\phi_\mathbf{t}| \, d\varkappa + \frac{\rho_2}{8} \int_0^L |\psi\psi_\mathbf{t}| \, d\varkappa + \frac{\mu_1}{16} \int_0^L \psi^2 d\varkappa \\
& + N_1 \rho_2 \int_0^L |\psi_\mathbf{t} \psi| \, d\varkappa + N_1 \rho_1 \int_0^L |\phi_\mathbf{t} w| \, d\varkappa + N_1 \frac{\mu_1}{2} \int_0^L \psi^2 d\varkappa \\
& + \rho_2 \int_0^L |\psi_\mathbf{t}(\phi_\varkappa + \psi)| \, d\varkappa + \rho_2 \int_0^L |\psi_\varkappa \phi_\mathbf{t}| \, d\varkappa \\
& + N_2 \int_0^L \int_0^L \int_{\iota_1}^{\iota_2} \tau \mathbf{e}^{-\tau\rho} |\eta_2(\tau)| \chi^2(\varkappa, \rho, \tau, \mathbf{t}) \, d\tau d\rho d\varkappa.
\end{aligned}$$

Exploiting some functional inequalities, we arrive at

$$\begin{aligned}
|\mathcal{L}(\mathbf{t}) - N\mathbf{E}(\mathbf{t})| \leq & \, C \int_0^L \left(\psi_\varkappa^2 + \psi_\mathbf{t}^2 + \phi_\mathbf{t}^2 + (\phi_\varkappa + \psi)^2\right) d\varkappa \\
& + \int_0^L \int_{\iota_1}^{\iota_2} \tau |\eta_2(\tau)| \chi^2(\varkappa, 1, \tau, \mathbf{t}) \, d\tau d\varkappa + \int_0^L \tilde{f}(\psi) d\varkappa \\
\leq & \, C\mathbf{E}(\mathbf{t}),
\end{aligned}$$

By (3.2), (3.5), (3.9), (3.14), (3.23) and (3.19), we get

$$\begin{aligned}
\frac{d\mathcal{L}(\mathbf{t})}{d\mathbf{t}} = & -\left(Nm_1 - N_1\left(\frac{\rho_1}{4\varepsilon_2} + \rho_2\right) - N_2\mu_1 - \left(\rho_2 + \frac{\mu_1^2}{k}\right) - (\frac{2\rho_1\varepsilon_1}{k} + \frac{b\rho_2}{2\varepsilon_1}) + \rho_2\right) \int_0^L \psi_\mathbf{t}^2 d\varkappa \\
& - \left(\frac{b}{2} N_1 - \frac{c_0}{8} - (\frac{b^2}{2\varepsilon_1^2} + \frac{1}{4b^2} + \frac{b^2}{8\varepsilon_1^2} + \frac{\mu_1 b}{4\varepsilon_1} + \frac{b^2}{4\varepsilon_1^3} + \varepsilon_1)\right) \int_0^L \psi_\varkappa^2 d\varkappa \\
& - \left(\frac{k}{8} - \varepsilon_1\left(k^2\varepsilon_1 + \frac{1}{b^2}\right)\right) \int_0^L (\phi_\varkappa + \psi)^2 d\varkappa \\
& - N_2\beta \int_0^L \int_0^L \int_{\iota_1}^{\iota_2} \tau |\eta_2(\tau)| \chi^2(\varkappa, \rho, \tau, \mathbf{t}) \, d\tau d\rho d\varkappa \\
& - \left(N_2\beta - N_1 \frac{\mu_1}{4\varepsilon_2} - \frac{\mu_1}{32} - \frac{\mu_1}{k} - \frac{b}{4\varepsilon_1}\right) \int_0^L \int_{\iota_1}^{\iota_2} |\eta_2(\tau)| \chi^2(\varkappa, 1, \tau, \mathbf{t}) d\tau d\varkappa \\
& - \left(\frac{\rho_1}{8} - \rho_1\varepsilon_2 N_1\right) \int_0^L \phi_\mathbf{t}^2 d\varkappa \\
& - (N_1 + 1) \int_0^L \tilde{f}(\psi) d\varkappa, \\
& + \left(\frac{\rho_2 k - \rho_1 b}{\rho_1}\right) \int_0^L \psi_\varkappa (\phi_\varkappa + \psi)_\varkappa \, d\varkappa.
\end{aligned}$$

By setting $\varepsilon_2 = \dfrac{\rho_1}{16N_1}$, we getFirst, we choose ε_1 small to hold

$$\frac{k}{8} - \varepsilon_1 \left(k^2 \varepsilon_1 + \frac{1}{b^2} \right) > 0.$$

Choosing N_1 large to verify

$$\frac{b}{2} N_1 - \frac{c_0}{8} - \left(\frac{b^2}{2\varepsilon_1^2} + \frac{1}{4b^2} + \frac{b^2}{8\varepsilon_1^2} + \frac{\mu_1 b}{4\varepsilon_1} + \frac{b^2}{4\varepsilon_1^3} + \varepsilon_1 \right) > 0.$$

Then, we select N_2 large to satisfies

$$N_2 \beta - N_1 \frac{\mu_1}{4\varepsilon_2} - \frac{\mu_1}{32} - \frac{\mu_1}{k} - \frac{b}{4\varepsilon_1} > 0.$$

Choosing N large such that

$$N m_1 - N_1 \left(\frac{\rho_1}{4\varepsilon_2} + \rho_2 \right) - N_2 \mu_1 - \left(\rho_2 + \frac{\mu_1^2}{k} \right) - \left(\frac{2\rho_1 \varepsilon_1}{k} + \frac{b\rho_2}{2\varepsilon_1} \right) + \rho_2 > 0,$$

and so that (3.25) remains valid. We obtain (3.26). ■

Here is the following polynomial stability result.

Lemma 3.9. *Let $\Phi = (\phi, \psi, \chi)$ be the solution of the system (2.1)–(2.2) and suppose that $\frac{k}{\rho_1} \neq \frac{b}{\rho_2}$ holds. Therefore, the solution (ϕ, ψ, χ) decays in polynomial manner, i.e. there exists $a > 0$ such that*

$$\mathbf{E}(\mathbf{t}) \leq \frac{a}{\mathbf{t}}, \ \mathbf{t} > 0.$$

Proof. Using (3.26) and $(1)_1$, we obtain

$$\mathcal{L}'(\mathbf{t}) \leq -\lambda_1 \mathbf{E}(\mathbf{t}) + \varsigma \int_0^L \psi_\varkappa \phi_{\mathbf{tt}} d\varkappa, \tag{3.27}$$

where $\varsigma = \dfrac{k}{\rho_1} \left(\dfrac{\rho_2 k - \rho_1 b}{\rho_1} \right)$, and by applying Young's inequality, we get

$$\varsigma \int_0^L \psi_\varkappa \phi_{\mathbf{tt}} d\varkappa \leq -\varsigma \frac{d}{d\mathbf{t}} \int_0^L (\phi_{\varkappa\mathbf{t}} \psi - \phi_\varkappa \psi_\mathbf{t}) \, d\varkappa + |\varsigma| \, \rho \int_0^L \phi_\varkappa^2 d\varkappa + \frac{|\varsigma|}{4\rho} \int_0^L \psi_{\mathbf{tt}}^2 d\varkappa, \ \rho > 0. \tag{3.28}$$

Because

$$\int_0^L \phi_\varkappa^2 d\varkappa \leq 2 \int_0^L (\phi_\varkappa + \psi)^2 \, d\varkappa + 2 \int_0^L \psi^2 d\varkappa, \tag{3.29}$$

we arrive at

$$\begin{aligned} \varsigma \int_0^L \psi_\varkappa \phi_{\mathbf{tt}} d\varkappa \leq & -\varsigma \frac{d}{d\mathbf{t}} \int_0^L (\phi_{\varkappa\mathbf{t}} \psi - \phi_\varkappa \psi_\mathbf{t}) \, d\varkappa + C_0 \, |\varsigma| \, \rho \mathbf{E}(\mathbf{t}) \\ & + \frac{|\varsigma|}{4\rho} \int_0^L \psi_{\mathbf{tt}}^2 d\varkappa. \end{aligned} \tag{3.30}$$

By substituting (3.30) in (3.27) and letting $\rho = \dfrac{\lambda_1}{2C_0 \, |\varsigma|}$, the inequality becomes as

$$\mathcal{L}'(\mathbf{t}) + \varsigma \frac{d}{d\mathbf{t}} \int_0^L (\phi_{\varkappa\mathbf{t}} \psi - \phi_\varkappa \psi_\mathbf{t}) \, d\varkappa \leq -\iota_1 \mathbf{E}(\mathbf{t}) + \iota_2 \int_0^L \psi_{\mathbf{tt}}^2 d\varkappa, \ \iota_1, \iota_2 > 0. \tag{3.31}$$

Now, let the functional

$$\mathcal{L}_1(\mathbf{t}) = \mathcal{L}(\mathbf{t}) + \varsigma \int_0^L (\phi_{\varkappa\mathbf{t}}\psi - \phi_\varkappa \psi_\mathbf{t})\, d\varkappa + N_3 (\mathbf{E}(\mathbf{t}) + \mathbf{E}_2(\mathbf{t})),$$

where $\mathbf{E}_2(\mathbf{t})$ is the second order energy of the problem (2.1)–(2.2) and we choice

$$N_3 > \max\left\{ \frac{2|\varsigma|}{K}, \frac{|\varsigma|}{\rho_2}, \frac{3|\varsigma|}{b}, \frac{\iota_2}{m_1} \right\}, \tag{3.32}$$

for that $\mathcal{L}_1(\mathbf{t})$ be equivalent to $\mathbf{E}(\mathbf{t}) + \mathbf{E}_2(\mathbf{t})$. Indeed,

$$|\mathcal{L}_1(\mathbf{t}) - N_3(\mathbf{E}(\mathbf{t}) + \mathbf{E}_2(\mathbf{t}))| \le \beta_2 \mathbf{E}(\mathbf{t}) + |\varsigma| \int_0^L |\phi_{\varkappa\mathbf{t}}\psi|\, d\varkappa + |\varsigma| \int_0^L |\phi_\varkappa \psi_\mathbf{t}|\, d\varkappa,$$

and by using some functional inequalities, we have

$$\begin{aligned} |\mathcal{L}_1(\mathbf{t}) - N_3(\mathbf{E}(\mathbf{t}) + \mathbf{E}_2(\mathbf{t}))| &\le \beta_2 \mathbf{E}(\mathbf{t}) + \frac{|\varsigma|}{2} \int_0^L \phi_{\varkappa\mathbf{t}}^2 d\varkappa + \frac{|\varsigma|}{2} \int_0^L \psi^2 d\varkappa \\ &\quad + \frac{|\varsigma|}{2} \int_0^L \phi_\varkappa^2 d\varkappa + \frac{|\varsigma|}{2} \int_0^L \psi_\mathbf{t}^2 d\varkappa. \end{aligned}$$

It's easily to show that

$$|\mathcal{L}_1(\mathbf{t}) - N_3(\mathbf{E}(\mathbf{t}) + \mathbf{E}_2(\mathbf{t}))| \le \max\left\{ \frac{2|\varsigma|}{K}, \frac{|\varsigma|}{\rho_2}, \frac{3|\varsigma|}{b} \right\} (\mathbf{E}(\mathbf{t}) + \mathbf{E}_2(\mathbf{t})),$$

and by recalling (3.32), we deduce that

$$\mathcal{L}_1(\mathbf{t}) \sim \mathbf{E}(\mathbf{t}) + \mathbf{E}_2(\mathbf{t}).$$

By using (3.31) and because $\mathbf{E}'(\mathbf{t}) \le 0$, we can conclude that

$$\mathcal{L}_1'(\mathbf{t}) \le -\iota_1 \mathbf{E}(\mathbf{t}) - (N_3 m_1 - \iota_2) \int_0^L \psi_{\mathbf{tt}}^2 d\varkappa,$$

the choice of N_3 given in (3.31) leads to

$$\mathcal{L}_1'(\mathbf{t}) \le -\iota_1 \mathbf{E}(\mathbf{t}). \tag{3.33}$$

Integrating the inequation (3.33), we obtain

$$\iota_1 \int_0^\mathbf{t} \mathbf{E}(\mathbf{t})\, d\mathbf{t} \le -\int_0^\mathbf{t} \mathcal{L}_1(\mathbf{t})\, d\mathbf{t},$$

and because $\mathbf{E}(\mathbf{t})$ is decreasing, we have

$$\iota_1 \mathbf{E}(\mathbf{t})\, \mathbf{t} \le \mathcal{L}_1(0),$$

which gives (3.33) by taking $a = \dfrac{\mathcal{L}_1(0)}{\iota_1}$. Which complete the proof. ■

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Conflict of interest

There is no conflict of interest.

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On derivations and Lie structure of semirings

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Abstract. In [9], Herstein introduced the notion of the Lie structure of associative rings and established the Lie type theory for rings. This paper extends these ring theoretical results and also extends some well known results of [3, 7, 8] in the framework of semirings, which are very important to investigate the Lie type theory of semirings and their higher commutators. Moreover, we characterize the Lie structure of semirings and thereby explore the action of derivations on Lie ideals of semirings.

AMS Subject Classifications: 16Y60, 16Y99.

Keywords: Semiring, Commutator, Higher commutator, Lie ideal, Additively Regular Semiring and Pseudo inverse.

Contents

1. Introduction

In past few years, various authors explored the brief structure of non commutative rings by developing the Lie type theory of associative rings. In 1969, Herstein [9] introduced the notion of Lie structure for rings and obtained several results which are helpful for rings of operators on a Hilbert space. The purpose of this study is to extend these results for some more general structure, e.g., the algebraic structure of non-commutative semirings. But the problem arises when we replace rings by semirings, as semirings do not have additive inverses, so we impose the weaker version of additive inverses, i.e., the pseudo inverse introduced by Karvellas [10]. Recently, semirings have been studied by various researchers (cf. [5, 11, 12]). In this paper, we generalized some of the Herstein's results in the framework of additively regular semirings which are further used to study the Lie structure of prime semirings and some of its subsets. Moreover, the behavior of derivations on Lie ideals of semirings is studied. Consequently, this enables us to measure the size of the centralizer of Lie ideals for the case of semirings. We

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also investigate some properties regarding Lie semirings which are very useful to investigate the Lie derivations and higher derivations of Lie ideals of semirings.

The prime motivation for this paper is not only the intense desire to extend these well known results, in the field of semirings, but to explore the close characterization of Lie theory and derivations in semirings. It is natural to point out that the ideal theory, homomorphisms and the Jordan theory are easily accessible to analyse in comparison to the Lie theory. As in the Lie case, the center of algebraic structure comes in our way and thereby various well known results are untouched in the corresponding Lie theory of semirings which are true in the aforementioned theories. Henceforth, this paper delivers the suitable techniques which can be efficiently used more and more to study the enormous structure of Lie semirings.

2. Preliminaries and some examples

Recall from [6] that a non empty set $\mathscr{S}$ is called a semiring if $(\mathscr{S},+)$ is a commutative monoid; $(\mathscr{S},\cdot)$ is a semigroup; and both distributive laws of multiplication over addition hold with $0\cdot t=0=t\cdot 0, \forall\, t\in\mathscr{S}$. Further, an element $t\in\mathscr{S}$ is called additively regular if and only if $\exists$ some element t' of $\mathscr{S}$ with $t+t+t'=t$ and $t'+t'+t=t'$ and $\mathscr{S}$ is known as an additively regular semiring if and only if $\mathscr{S}=\mathscr{S}'=reg(\mathscr{S})$, where $reg(\mathscr{S})$ represents the set of all additively regular elements of $\mathscr{S}$. The element s' is the pseudo inverse [10] of s. For instance, if $\mathbb{B}=\{0,1\}$ is a boolean semiring with binary operations as $0+0=0$; $0+1=1=1+0$; $1+1=1$ and $0\cdot 0=1\cdot 0=0\cdot 1=0$; $1\cdot 1=1$, then $\mathbb{B}$ is an additively regular semiring, where $t'=t, \forall\, t\in\mathbb{B}$ is the pseudo inverse of $t\in\mathbb{B}$. One can easily check that the pseudo inverse of an element is always unique. In 1982, Bandelt and Petrich [1] considered an additively regular semiring $\mathscr{S}$ with conditions:

$(A_1): a_1(a_1+a_1')=a_1+a_1', \forall\, a_1\in\mathscr{S}$; $(A_2): s_1(a_1+a_1')=(a_1+a_1')s_1, \forall\, a_1,s_1\in\mathscr{S}$; $(A_3): a_1+(a_1+a_1')s_1=1, \forall\, a_1,s_1\in\mathscr{S}$ and investigated various results for this class of semirings. In addition, every Bandelt semiring [6] is an additively regular semiring with A_2-condition.

Further, $\mathscr{S}$ is said to be prime if $\mathcal{H}\mathcal{K}=(0)$ infers that either $\mathcal{H}=(0)$ or $\mathcal{K}=(0)$, where $\mathcal{H}$ and $\mathcal{K}$ are any two ideals of $\mathscr{S}$. A semiring which does not have any nilpotent ideals is called a semiprime semiring. Note that every prime semiring is also a semiprime semiring.

For given $a,b\in\mathscr{S}$, then $[a,b]$ (the Lie bracket) symbolizes the element $ab+b'a$ or $ab+ba'$. Indeed, for $\mathcal{H},\mathcal{K}\subseteq\mathscr{S}$, the Lie bracket $[\mathcal{H},\mathcal{K}]$ is an additive submonoid of $\mathscr{S}$ which is generated by all elements of the form $hk+k'h$ or $hk+kh'$, for $h\in\mathcal{H}$ and $k\in\mathcal{K}$ and $(\mathcal{H})$ denotes the ideal generated by $\mathcal{H}$. However, an additive submonoid $\mathscr{L}$ of $\mathscr{S}$ is called a Lie ideal if $[\mathscr{L},\mathscr{S}]\subseteq\mathscr{L}$. Note that $[\mathscr{L}_1,\mathscr{L}_2]$ is also a Lie ideal of $\mathscr{S}$, for $\mathscr{L}_1$ and $\mathscr{L}_2$ are Lie ideals of $\mathscr{S}$, because of the existence of the Jacobi identity $[r_1,[s_1,t_1]]+[s_1,[t_1,r_1]]=[[r_1,s_1],t_1]$. Throughout this study, $\mathscr{S}$ represents an additively regular semiring with A_2-condition and $\mathscr{L}$ is a Lie ideal of $\mathscr{S}$, unless otherwise mentioned. We now delay the discussion of higher commutators of $\mathscr{S}$ until later and proceed with some results which will be used frequently in the sequel.

For simplicity, we denote $u_\circ=u+u'$ and by A_2- condition, $u_\circ\in\mathscr{Z}(\mathscr{S}), \forall\, u\in\mathscr{S}$, where $\mathscr{Z}(\mathscr{S})$ represents the center of $\mathscr{S}$.

Lemma 2.1 ([6]). *Let $\mathscr{S}$ be an additively regular semiring. Then the following hold:*
(i) $u_1''=u_1$; (ii) $u_1'v_1'=(u_1'v_1)'=(u_1v_1')'=(u_1v_1)''=u_1v_1$; (iii) $(u_1v_1)'=u_1'v_1=u_1v_1'$; (iv) $(u_1+v_1)'=u_1'+v_1'$; (v) If $u_1+v_1=0$, then $v_1=u_1'$; (vi) $u_{1\circ}+u_{1\circ}=u_{1\circ}=u_{1\circ}'$; (vii) $u_1+u_{1\circ}=u_1$; (viii) $u_1'+u_{1\circ}=u_1'$; (ix) $u_{1\circ}v_1=u_1v_{1\circ}=(u_1v_1)_\circ=u_{1\circ}v_{1\circ}=v_{1\circ}u_{1\circ}=(v_1u_1)_\circ, \forall\, u_1,v_1\in\mathscr{S}$.

Example 2.2. *Consider $\mathscr{S}=\{0,1,a\}$ having all additively idempotent elements and the binary operations in it can be illustrated with the help of the Cayley tables given below:*

$\oplus$	0	1	a
0	0	1	a
1	1	1	a
a	a	a	a

$\otimes$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	a

One can easily see that the pseudo inverse of a *is* $a' = a, \forall\, a \in \mathscr{S}$. *Obviously,* $\mathscr{S}$ *is an additively regular semiring with* A_2-*condition and* $\mathscr{L} = \{0, a\}$, *is a Lie ideal of* $\mathscr{S}$.

The forthcoming example demonstrates that every additively regular semiring may not satisfy A_2-condition.

Example 2.3. *Consider a Boolean semiring* $\mathbb{B}$ *and* $\mathscr{S} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{B}) : a, b, c, d \in \mathbb{B} \right\}$ *with usual addition and usual multiplication of matrices. Define pseudo inverse of an element of* $\mathscr{S}$ *by* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, \forall$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathscr{S}$. *Then* $\mathscr{S}$ *is an additively regular semiring which does not satisfy* A_2-*condition.*

The following lemma is easy to prove, so we omit it.

Lemma 2.4. *If* $u_1, v_1, w_1 \in \mathscr{S}$, *then*
(i) $[u_1, v_1w_1] = [u_1, v_1]w_1 + v_1[u_1, w_1]$; *(ii)* $[u_1v_1, w_1] = u_1[v_1, w_1] + [u_1, w_1]v_1$; *(iii)* $[u_1 + v_1, w_1] = [u_1, w_1] + [v_1, w_1]$; *(iv)* $[u_1, [v_1, w_1]] + [v_1, [w_1, u_1]] = [[u_1, v_1], w_1]$ *(Jacobi Identity).*

3. Lie Ideals and Higher Commutators

We hereby introduce the notion of higher commutators of semirings. Also, some results regarding higher commutators of $\mathscr{S}$ are proved which play a significant part in characterizing the Lie structure of semirings. Throughout this section, $\mathscr{S}$ represents a prime additively regular semiring satisfying A_2-condition.

Proposition 3.1. *If* $u_1 \in \mathscr{S}$ *with* $u_1[u_1, \mathscr{S}] = (0)$, *then* $u_1 \in \mathscr{Z}(\mathscr{S})$.

Proof. By hypothesis,

$$u_1[u_1, r] = 0,\ \forall\, r \in \mathscr{S}. \tag{1}$$

Again by hypothesis, we have $u_1[u_1, rs] = 0, \forall\, r, s \in \mathscr{S}$ which implies that $u_1(u_1rs + rs'u_1) = 0, \forall\, r, s \in \mathscr{S}$. Further, Lemma 2.1 implies that

$$\begin{aligned} 0 &= u_1(u_1rs + rs'u_1 + r_\circ su_1) = u_1(u_1rs + rs'u_1 + rs_\circ u_1) \\ &= u_1(u_1rs + rs'u_1 + rsu_{1\circ}) = u_1(u_1rs + rs'u_1 + ru_{1\circ}s), \text{by A}_2\text{−condition} \\ &= u_1(u_1rs + rs'u_1 + r_\circ u_1s) = u_1((u_1r + r'u_1)s + r(u_1s + s'u_1)). \end{aligned}$$

Then by equation (1), we have $u_1r(u_1s + s'u_1) = 0, \forall\, r, s \in \mathscr{S}$, that is, $u_1\mathscr{S}[u_1, s] = (0), \forall\, s \in \mathscr{S}$. Therefore, primeness of $\mathscr{S}$ infers that either $u_1 = 0$ or $[u_1, \mathscr{S}] = (0)$ and hence $u_1 \in \mathscr{Z}(\mathscr{S})$. ■

Lemma 3.2. *If char* $\mathscr{S} \neq 2$ *and* $[u_1, [u_1, \mathscr{S}]] = (0)$, *for* $u_1 \in \mathscr{S}$, *then* $u_1 \in \mathscr{Z}(\mathscr{S})$.

Proof. For any $x_1 \in \mathscr{S}$, we have $[u_1, [u_1, x_1]] = 0$ which gives that

$$u_1[u_1, x_1] + [u_1, x_1]u_1' = 0.$$

Then by adding both sides $[u_1, x_1]u_1$, we obtain

$$u_1[u_1, x_1] + [u_1, x_1]u_1' + [u_1, x_1]u_1 = [u_1, x_1]u_1.$$

This infers that

$$u_1[u_1, x_1] + [u_1, x_1]u_{1\circ} = [u_1, x_1]u_1.$$

In view of A_2-condition, we obtain $u_1[u_1,x_1]+u_{1\circ}[u_1,x_1]=[u_1,x_1]u_1$. Thus,

$$u_1(u_1x_1+x_1'u_1)=(u_1x_1+x_1'u_1)u_1,\ \forall\, x_1\in\mathscr{S}. \tag{1}$$

Again hypothesis leads to $[u_1,[u_1,x_1y]]=0, \forall\, x_1,y\in\mathscr{S}$. Equivalently,

$$u_1(u_1x_1y+x_1'yu_1)+(u_1x_1y+x_1'yu_1)u_1'=0,\ \textit{for any}\ x_1,y\in\mathscr{S}.$$

Thus, by Lemma 2.1 and A_2-condition, we can replace $u_1x_1y+x_1'yu_1$ by $(u_1x_1+x_1'u_1)y+x_1(u_1y+y'u_1)$ which gives that

$$u_1((u_1x_1+x_1'u_1)y+x_1(u_1y+y'u_1))+((u_1x_1+x_1'u_1)y+x_1(u_1y+y'u_1))u_1'=0,\ \text{for any}\ x_1,y\in\mathscr{S}.$$

Further, by applying equation (1), we obtain $2(u_1x_1+x_1'u_1)(u_1y+y'u_1)=0, \forall\, x_1,y\in\mathscr{S}$. As the characteristic of $\mathscr{S}$ is other than 2, so we are left with

$$[u_1,x_1][u_1,y]=0,\ \forall\, x_1,y\in\mathscr{S}. \tag{2}$$

Replacing x_1 by x_1z in (2), where $z\in\mathscr{S}$, then by Lemma 2.4 and equation (2), we get that $[u_1,x_1]\mathscr{S}[u_1,y]=(0), \forall\, x_1,y\in\mathscr{S}$. Now, since $\mathscr{S}$ is prime, therefore $[u_1,x_1]=0$ or $[u_1,y]=0, \forall\, x_1,y\in\mathscr{S}$ and hence in both cases $u_1\in\mathscr{Z}(\mathscr{S})$. ■

Proposition 3.3. *If char $\mathscr{S}\neq 2$ and $[\mathscr{L},\mathscr{L}]=(0)$, then $\mathscr{L}\subseteq\mathscr{Z}(\mathscr{S})$.*

Proof. For any $l_1\in\mathscr{L}, s_1\in\mathscr{S}$, we have $[l_1,s_1]\in\mathscr{L}$. By hypothesis, $[l_1,[l_1,s_1]]=0, \forall\, l_1\in\mathscr{L}, s_1\in\mathscr{S}$, that is, $[l_1,[l_1,\mathscr{S}]]=(0), \forall\, l_1\in\mathscr{L}$. Lemma 3.2 concludes that $\mathscr{L}\subseteq\mathscr{Z}(\mathscr{S})$. ■

One can easily prove the upcoming two lemmas by using the similar arguments of [9, Lemma 1.8 and Lemma 1.9] with necessary variations.

Lemma 3.4. *If $\mathscr{J}\neq(0)$ is a left ideal of $\mathscr{S}$, then $\mathscr{J}+[\mathscr{S},\mathscr{S}]=\mathscr{S}$.*

Lemma 3.5. *If $\mathscr{L}\neq(0)$ with $u_1\mathscr{L}=(0)$ or $\mathscr{L}u_1=(0)$, for any $u_1\in\mathscr{S}$, then $u_1=0$.*

Now, we turn our attention to define the higher commutator of $\mathscr{S}$ and prove some basic lemmas which we need later to prove results concerning the Lie structure of higher commutators.

Definition 3.6. *The higher commutator of $\mathscr{S}$ is defined inductively by:*
(1) $\mathscr{S}^{(0)}=\mathscr{S}$, of weight 1;
(2) $\mathscr{S}^{(1)}=[\mathscr{S},\mathscr{S}]$, of weight 2

and a higher commutator of $\mathscr{S}$ of weight n is defined by $[\mathcal{P},\mathcal{Q}]$, where $\mathcal{P}$ is a higher commutator of $\mathscr{S}$ of weight p, $\mathcal{Q}$ is of weight q, with $p+q=n$.

For convenience, we give notation $\mathscr{S}^{(k)}$ for the following series of $\mathscr{S}$ defined as: $\mathscr{S}^{(0)}=\mathscr{S}$, $\mathscr{S}^{(1)}=[\mathscr{S},\mathscr{S}]$,..., $\mathscr{S}^{(k)}=[\mathscr{S}^{(k-1)},\mathscr{S}^{(k-1)}]$.

It is pertinent to mention that the higher commutator of weight 2 is only $[\mathscr{S},\mathscr{S}]=\mathscr{S}^{(1)}$, whereas the higher commutator of weight 3 is only $\mathscr{S}^{(3)}=[[\mathscr{S},\mathscr{S}],\mathscr{S}]$; there are two higher commutators of weight 4 viz, $[\mathscr{S},[\mathscr{S},[\mathscr{S},\mathscr{S}]]]$ and $[[\mathscr{S},\mathscr{S}],[\mathscr{S},\mathscr{S}]]$; three of weight 5 viz, $[\mathscr{S},[\mathscr{S},[\mathscr{S},[\mathscr{S},\mathscr{S}]]]]$, $[\mathscr{S},[[\mathscr{S},\mathscr{S}],[\mathscr{S},\mathscr{S}]]$ and $[[\mathscr{S},\mathscr{S}],[[\mathscr{S},\mathscr{S}],\mathscr{S}]]$ and so on.

The next lemma follows verbatim as Lemma 3 in [8].

Lemma 3.7. *A higher commutator of $\mathscr{S}$ contains $\mathscr{S}^{(k)}$, for some k.*

An application of the above lemma is

Corollary 3.8. *If $\mathcal{H}$ is a higher commutator of $\mathscr{S}$, then $(\mathcal{H})$, the ideal generated by $\mathcal{H}$, contains $\mathscr{S}^{(k)}$, for some k.*

Lemma 3.9. *Let $\mathcal{H}$ be a higher commutator of $\mathscr{S}$. Then $\mathcal{H}$ is a Lie ideal of $\mathscr{S}$.*

Proof. Let $\mathcal{H}$ be a higher commutator of $\mathscr{S}$ of weight n. We shall prove the result by using induction on n. For $n = 1$, clearly $\mathscr{S}$ is a Lie ideal of $\mathscr{S}$. Now, we suppose that it is true for $n = k-1$, that is, $\mathcal{H} = [\mathcal{P}, \mathcal{Q}]$, where $\mathcal{P}$ is a higher commutator of $\mathscr{S}$ of weight p and $\mathcal{Q}$ is of weight q, with $p + q = k - 1$, is a Lie ideal of $\mathscr{S}$. Further, consider $\mathcal{K} = [\mathcal{H}, \mathscr{S}]$ is a higher commutator of $\mathscr{S}$ of weight k. Then obviously, $\mathcal{K}$ is an additive submonoid of $\mathscr{S}$ and $[\mathcal{K}, \mathscr{S}] = [[\mathcal{H}, \mathscr{S}], \mathscr{S}] \subseteq [\mathcal{H}, \mathscr{S}] = \mathcal{K}$, as $[\mathcal{H}, \mathscr{S}] \subseteq \mathcal{H}$. Therefore, $\mathcal{K}$ is a Lie ideal of $\mathscr{S}$. This finishes the proof. ■

Theorem 3.10. *If u_1 is any element of $\mathscr{S}$ which satisfies $[u_1, [\mathscr{S}, \mathscr{S}]] = (0)$, then $u_1 \in \mathscr{Z}(\mathscr{S})$.*

Proof. For any $x_1, y \in \mathscr{S}$, we have $[u_1, [x_1, y]] = 0$ leading to

$$u_1[x_1, y] + [x_1, y]u_1' + [x_1, y]u_1 = [x_1, y]u_1.$$

Now, by applying A_2- condition on this equality, we have

$$u_1[x_1, y] + (u_1 + u_1')[x_1, y] = [x_1, y]u_1$$

which infers that

$$u_1x_1y + u_1yx_1' = x_1yu_1 + y'x_1u_1, \ \forall\, x_1, y \in \mathscr{S}. \tag{1}$$

Again by hypothesis, we have $[u_1, [x_1, x_1y]] = 0, \forall\, x_1, y \in \mathscr{S}$. Thus,

$$\begin{aligned} 0 &= u_1(x_1x_1y + x_1y'x_1) + (x_1x_1y + x_1y'x_1)u_1' \\ &= u_1x_1(x_1y + y'x_1) + x_1(x_1y + y'x_1)u_1', \ \forall\, x_1, y \in \mathscr{S}. \end{aligned}$$

By equation (1), we obtain that $u_1x_1(x_1y + y'x_1) + x_1u_1'(x_1y + y'x_1) = 0, \forall\, x_1, y \in \mathscr{S}$ which is equivalent to

$$(u_1x_1 + x_1u_1')(x_1y + y'x_1) = 0, \ \forall\, x_1, y \in \mathscr{S}. \tag{2}$$

Changing y with yu_1 in equation (2) and applying A_2-condition, we have

$$\begin{aligned} 0 &= (u_1x_1 + x_1u_1')(x_1yu_1 + yu_1'x_1) = (u_1x_1 + x_1u_1')(x_1yu_1 + x_1y_\circ u_1 + yu_1'x_1) \\ &= (u_1x_1 + x_1u_1')(x_1yu_1 + y_\circ x_1u_1 + yu_1'x_1) = (u_1x_1 + x_1u_1')((x_1y + y'x_1)u_1 + y(x_1u_1 + u_1'x_1)). \end{aligned}$$

Moreover, by equation (2), we obtain $(u_1x_1 + x_1u_1')y(x_1u_1 + u_1'x_1) = 0, \forall\, x_1, y \in \mathscr{S}$. Equivalently,

$$(u_1x_1 + x_1u_1')\mathscr{S}(x_1u_1 + u_1'x_1) = (0), \ \forall\, x_1 \in \mathscr{S}$$

and in that case $(\mathscr{S}(x_1u_1 + u_1'x_1))^2 = (0)$, which is a contradiction, as $\mathscr{S}$ does not have any non-zero nilpotent ideal. Thus, $u_1x_1 + x_1u_1' = 0, \forall\, x_1 \in \mathscr{S}$, that is, $u_1x_1 = x_1u_1, \forall x_1 \in \mathscr{S}$. Hence $u_1 \in \mathscr{Z}(\mathscr{S})$. ■

Remark 3.11. *Let $\mathscr{Z}(\mathscr{S})$ be the center of $\mathscr{S}$. We define the extended centroid of $\mathscr{L}$ by the set $\mathscr{Z}_{\mathscr{S}(\mathscr{L})} = \{s \in \mathscr{S} : sl = ls, \forall\, l \in \mathscr{L}\}$. Also, one can easily check that $\mathscr{Z}(\mathscr{S}) \subseteq \mathscr{Z}_{\mathscr{S}(\mathscr{L})}$.*

Theorem 3.12. *Let char $\mathscr{S} \neq 3$ and $\mathcal{P}$ be an additive submonoid of $\mathscr{S}$. If $[[p, [p, \mathcal{P}]], s] = (0), \forall\, p \in \mathcal{P}, s \in \mathscr{S}$, then $[\mathcal{P}, [\mathcal{P}, \mathcal{P}]] \subseteq \mathscr{Z}(\mathscr{S})$.*

Proof. The given hypothesis infers that

$$[[k,[k,m_1]],y]=0,\ \forall\, k,m_1\in\mathcal{P},y\in\mathscr{S}. \tag{1}$$

Again by hypothesis, $[[l_1+k,[l_1+k,m_1]],y]=0$, for any $l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}$. Then by Lemma 2.4, we obtain

$$0=[[l_1+k,[l_1,m_1]+[k,m_1]],y]=[[l_1+k,[l_1,m_1]],y]+[[l_1+k,[k,m_1]],y]$$

for any $l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}$. Further, by Lemma 2.4 and equation (1), we get that

$$[[k,[l_1,m_1]],y]+[[l_1,[k,m_1]],y]=0,\ \forall\, l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}. \tag{2}$$

By using Jacobi identity, we can substitute $[[l_1,[k,m_1]]+[[l_1,k],m_1]$ in place of $[[k,[l_1,m_1]]$ in equation (2) and thus we have

$$2[[l_1,[k,m_1]],y]+[[l_1,k],m_1],y]=0,\ \forall\, l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}. \tag{3}$$

By interchanging l_1 and m_1 in equation (3), we have

$$2[[m_1,[k,l_1]],y]+[[m_1,k],l_1],y]=0,\forall\, l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}$$

which is equivalent to

$$2[[[l_1,k],m_1],y]+[[l_1,[k,m_1]],y]=0,\ \forall\, l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}. \tag{4}$$

Further, by adding equations (3) and (4), we get that

$$3([[[l_1,k],m_1],y]+[[l_1,[k,m_1]],y])=0,\ \forall\, l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}.$$

Again using Jacobi identity, we have $3[[[l_1,m_1],k],y]=0,\forall\, l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}$. As *char* $\mathscr{S}\neq 3$, so we are left with $[[[l_1,m_1],k],y]=0,\forall\, l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}$. Therefore, $[\mathcal{P},[\mathcal{P},\mathcal{P}]]\subseteq\mathscr{Z}(\mathscr{S})$. ■

The next corollary is an important outcome of the previous result.

Corollary 3.13. *If char $\mathscr{S}\neq 3$ and $[[l_1,[l_1,\mathscr{L}]],\mathscr{L}]=(0)$, for any $l_1\in\mathscr{L}$, then $[\mathscr{L},[\mathscr{L},\mathscr{L}]]\subseteq\mathscr{Z}_{\mathscr{S}(\mathscr{L})}$.*

Theorem 3.14. *If char $\mathscr{S}\neq 2$ and $[[\mathscr{L},[\mathscr{L},\mathscr{L}]],\mathscr{L}]=(0)$, then $[\mathscr{L},[\mathscr{L},\mathscr{L}]]\subseteq\mathscr{Z}(\mathscr{S})$.*

Proof. Since $\mathscr{L}$ is a Lie ideal of $\mathscr{S}$, therefore $[\mathscr{L},[\mathscr{L},\mathscr{L}]]$ is also a Lie ideal of $\mathscr{S}$. Thus, $[l_1,s_1]\in[\mathscr{L},[\mathscr{L},\mathscr{L}]],\ \forall\ l_1\in[\mathscr{L},[\mathscr{L},\mathscr{L}]]$ and $s_1\in\mathscr{S}$. Moreover, by hypothesis $[l_1,[l_1,s_1]]=0,\ \forall\ l_1\in[\mathscr{L},[\mathscr{L},\mathscr{L}]],s_1\in\mathscr{S}$ and hence Lemma 3.2, concludes that $[\mathscr{L},[\mathscr{L},\mathscr{L}]]\subseteq\mathscr{Z}(\mathscr{S})$. ■

4. Lie structure of $\mathscr{S}$

The idea behind the results proved in this section was first brought to the author's attention during the study of the Lie structure of rings given by Herstein [7–9]. Throughout this section, $\mathscr{L}$ denotes a 2-Lie ideal (that is, a Lie ideal having property $2l_1m_1\in\mathscr{L},\forall\, l_1,m_1\in\mathscr{L}$) of $\mathscr{S}$. We now begin this section with an example:

Example 4.1. *Consider $\mathscr{S}=\mathbb{Z}\times\mathbb{Z}^+=\{(u_1,r_1):u_1\in\mathbb{Z},r_1\in\mathbb{Z}^+\}$, where $\mathbb{Z}^+$ is the set of all positive integers with binary operations $\oplus$ and $\odot$ by $(u_1,r_1)\oplus(v,s)=(u_1+v,lcm(r_1,s))$ and $(u_1,r_1)\odot(v,s)=(u_1v,gcd(r_1,s)),\ \forall\ (u_1,r_1),(v,s)\in\mathscr{S}$. Further, define the pseudo inverse of an element (u_1,r_1) of $\mathscr{S}$ by $(u_1,r_1)'=(-u_1,r_1)$. Then clearly, $\mathscr{S}$ is an additively regular semiring with A_2-condition. Indeed, the set $\mathscr{L}=\{(0,s):s\in\mathbb{Z}^+\}$ is a 2-Lie ideal of $\mathscr{S}$.*

We now introduce a more general result which is a generalization of [9, Lemma 1.3].

Theorem 4.2. *If char $\mathscr{S} \neq 2$, then either $\mathscr{L}$ is contained in $\mathscr{Z}(\mathscr{S})$ or $\mathscr{L}$ contains a non-zero ideal of $\mathscr{S}$.*

Proof. In case $[\mathscr{L},\mathscr{L}] = (0)$, then by Proposition 3.3, we have $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$. But, if $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$, then again by Proposition 3.3, $[\mathscr{L},\mathscr{L}] \neq (0)$, so we shall prove the containment of the ideal $2\mathscr{S}[\mathscr{L},\mathscr{L}]\mathscr{S}$ of $\mathscr{S}$ in $\mathscr{L}$. We claim that $2\mathscr{S}[\mathscr{L},\mathscr{L}] \subseteq \mathscr{L}$. For this, let $2s[l_1, m_1] \in 2\mathscr{S}[\mathscr{L},\mathscr{L}]$, for any $l_1, m_1 \in \mathscr{L}, s \in \mathscr{S}$. Then

$$\begin{aligned}
2s[l_1, m_1] &= 2(sl_1m_1 + sm_1'l_1) \\
&= 2(sl_1m_1 + s_\circ l_1m_1 + sm_1'l_1), \text{by Lemma 2.1} \\
&= 2(sl_1m_1 + l_1s_\circ m_1 + sm_1'l_1), \text{by } A_2\text{−condition} \\
&= 2(sl_1m_1 + l_1sm_1 + l_1s'm_1 + s'm_1l_1), \text{by Lemma 2.1} \\
&= 2((l_1sm_1 + s'm_1l_1) + (l_1s'm_1 + sl_1m_1)) = 2[l_1, sm_1] + 2[l_1, s']m_1 \in \mathscr{L}.
\end{aligned}$$

Hence, $2\mathscr{S}[\mathscr{L},\mathscr{L}] \subseteq \mathscr{L}$. This infers that

$$[2r[l_1, m_1], s] \in \mathscr{L}, \forall\, l_1, m_1 \in \mathscr{L}, r, s \in \mathscr{S}$$

which is equivalent to

$$2r[l_1, m_1]s + 2s'r[l_1, m_1] \in \mathscr{L}. \tag{1}$$

Also, since $2sr[l_1, m_1] \in \mathscr{L}$, therefore equation (1) gives

$$2r[l_1, m_1]s + 2s'r[l_1, m_1] + 2sr[l_1, m_1] \in \mathscr{L}.$$

Equivalently, $2r[l_1, m_1]s + 2s_\circ r[l_1, m_1] \in \mathscr{L}$. Thus, we obtain

$$2r(l_1m_1 + m_1'l_1)s + 2(s + s')r(l_1m_1 + m_1'l_1) \in \mathscr{L}.$$

In other words, $2r(l_1m_1 + m_1'l_1)s + 2(s_\circ rl_1m_1 + s_\circ rm_1'l_1) \in \mathscr{L}$. Then, A_2-condition yields

$$2r(l_1m_1 + m_1'l_1)s + 2r_\circ l_1m_1s + 2r_\circ m_1'l_1s \in \mathscr{L}.$$

Therefore, Lemma 2.1 concludes that $2r(l_1m_1 + m_1'l_1)s \in \mathscr{L}$, $\forall\, l_1, m_1 \in \mathscr{L}, r, s \in \mathscr{S}$ which gives $2\mathscr{S}[\mathscr{L},\mathscr{L}]\mathscr{S} \subseteq \mathscr{L}$. The theorem is thereby established. ■

Definition 4.3. *[2] A semiring $\mathscr{S}$ is called ideal-simple (id-simple for short), if $\mathscr{S}$ is non-trivial and $\mathscr{I} = \mathscr{S}$, whenever $\mathscr{I}$ is a non-zero ideal of $\mathscr{S}$ such that $\mathscr{I}$ contains atleast two elements respectively.*

The above result immediately implies the following theorem which is a generalization of [9, Theorem 1.2]

Theorem 4.4. *If $\mathscr{S}$ is an id-simple semiring with char $\mathscr{S} \neq 2$, then either $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$ or $\mathscr{L}$ coincides with $\mathscr{S}$.*

Lemma 4.5. *If char $\mathscr{S} \neq 2$ and $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$, then there exists an ideal $\mathscr{I}$ of $\mathscr{S}$ such that $[\mathscr{I},\mathscr{S}] \subseteq \mathscr{L}$.*

Proof. As proved in the Theorem 4.2, the non-zero ideal $2\mathscr{S}[\mathscr{L},\mathscr{L}]\mathscr{S}$ of $\mathscr{S}$ is contained in $\mathscr{L}$, then it follows easily that $[2\mathscr{S}[\mathscr{L},\mathscr{L}]\mathscr{S},\mathscr{S}] \subseteq \mathscr{L}$. Hence proved. ■

In the rest of this section, $\mathscr{S}$ denotes a prime semiring with *char* $\mathscr{S} \neq 2$. It is easy to observe that every id-simple semiring is a prime semiring. Therefore, all the forthcoming results of this section are also true for an id-simple semiring.

Lemma 4.6. *If $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$ and $u_1, v \in \mathscr{S}$ with $u_1\mathscr{L}v = (0)$, then either $u_1 = 0$ or $v = 0$.*

Proof. By the above lemma, there exists an ideal $\mathscr{J}$ of $\mathscr{S}$ with $[\mathscr{J},\mathscr{S}] \subseteq \mathscr{L}$. Let $l_1 \in \mathscr{L}, i \in \mathscr{J}, s \in \mathscr{S}$. Then $[iu_1l_1, s] \in [\mathscr{J},\mathscr{S}] \subseteq \mathscr{L}$. Henceforth,

$$\begin{aligned}0 &= u_1[iu_1l_1, s]v = u_1[iu_1, s]l_1v + u_1iu_1[l_1, s]v = u_1[iu_1, s]l_1v, \text{ as } u_1\mathscr{L}v = (0)\\ &= u_1(iu_1s + s'iu_1)l_1v = u_1iu_1sl_1v, \text{ as } u_1\mathscr{L}v = (0).\end{aligned}$$

This shows that $u_1\mathscr{J}u_1\mathscr{S}\mathscr{L}v = (0)$. If $u_1 \neq 0$ and by using the fact that $\mathscr{S}$ is prime, we obtain $\mathscr{L}v = (0)$, then by Lemma 3.5, $v = 0$. ■

Theorem 4.7. *If* $[u_1, [\mathscr{L},\mathscr{L}]] = (0)$, *for any* $u_1 \in \mathscr{S}$, *then either* $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$ *or* $[u_1,\mathscr{L}] = (0)$.

Proof. Assume that $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$. For any $l_1, m_1 \in \mathscr{L}$, hypothesis gives that $[u_1, [l_1, m_1]] = 0$ which leads to

$$u_1(l_1m_1 + m_1'l_1) + (l_1m_1 + m_1'l_1)u_1' + (l_1m_1 + m_1'l_1)u_1 = (l_1m_1 + m_1'l_1)u_1.$$

This infers that

$$u_1(l_1m_1 + m_1'l_1) = (l_1m_1 + m_1'l_1)u_1. \tag{1}$$

Again by hypothesis, we have

$$[u_1, [l_1, 2l_1m_1]] = 0, \forall\, l_1, m_1 \in \mathscr{L}.$$

Since $char\ \mathscr{S} \neq 2$, therefore $[u_1, [l_1, l_1m_1]] = 0, \forall\, l_1, m_1 \in \mathscr{L}$ which is equivalent to

$$u_1l_1(l_1m_1 + m_1'l_1) + l_1(l_1m_1 + m_1'l_1)u_1' = 0, \forall\, l_1, m_1 \in \mathscr{L}.$$

Then by using equation (1), we get

$$u_1l_1(l_1m_1 + m_1'l_1) + l_1u_1'(l_1m_1 + m_1'l_1) = 0$$

which concludes that

$$(u_1l_1 + l_1u_1')(l_1m_1 + m_1'l_1) = 0,\ \forall\, l_1, m_1 \in \mathscr{L}. \tag{2}$$

The replacement of m_1 with $2m_1n$, where $n \in \mathscr{L}$, in equation (2) gives

$$2(u_1l_1 + l_1u_1')(l_1m_1n + m_1'nl_1) = 0.$$

But $char\ \mathscr{S} \neq 2$ gives

$$\begin{aligned}0 &= (u_1l_1 + l_1u_1')(l_1m_1n + m_1'nl_1) = (u_1l_1 + l_1u_1')(l_1m_1n + m_{1\circ}nl_1 + m_1n'l_1)\\ &= (u_1l_1 + l_1u_1')((l_1m_1 + m_1'l_1)n + m_1(l_1n + n'l_1))\\ &= (u_1l_1 + l_1u_1')(l_1m_1 + m_1'l_1)n + (u_1l_1 + l_1u_1')m_1(l_1n + n'l_1),\end{aligned}$$

then equation (2) yields $(u_1l_1 + l_1u_1')m_1(l_1n + n'l_1) = 0, \forall\, l_1, m_1, n \in \mathscr{L}$. In other words,

$$(u_1l_1 + l_1u_1')\mathscr{L}(l_1n + n'l_1) = (0),\ \forall\, l_1, n \in \mathscr{L}.$$

By the above lemma, for any $l_1 \in \mathscr{L}$, we obtain either $u_1l_1 + l_1u_1' = 0$ or $l_1n + n'l_1 = 0, \forall\, n \in \mathscr{L}$. Now, if $[l_1,\mathscr{L}] = (0), \forall\, l_1 \in \mathscr{L}$, then $[\mathscr{L},\mathscr{L}] = (0)$ and hence Proposition 3.3 yields $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$, which is absurd.

Therefore, there exists some $k \in \mathscr{L}$ with $[k,\mathscr{L}] \neq (0)$, then $[u_1, k] = 0$. We now claim that $[u_1,\mathscr{L}] = (0)$. For this, if possible, let $j(\neq k) \in \mathscr{L}$ with $[u_1, j] \neq 0$. Thus, $[j,\mathscr{L}] = (0)$. This infers that $[j + k,\mathscr{L}] \neq (0)$ and $[u_1, j + k] \neq 0$ hold simultaneously, which is not true. Hence, $[u_1,\mathscr{L}] = (0)$. ■

An application of the above theorem is as follows:

Corollary 4.8. *If $u_1 \in \mathscr{S}$ satisfies $[u_1, [\mathscr{L}, \mathscr{L}]] = (0)$, then either $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$ or u_1 commutes with every element of $\mathscr{L}$.*

The upcoming theorem is a partial extension of [8, Theorem 1].

Theorem 4.9. *If $[u_1, [u_1, \mathscr{L}]] = (0)$, for any $u_1 \in \mathscr{S}$, then either $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$ or $[u_1, \mathscr{L}] = (0)$.*

Proof. Let $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$. By given hypothesis, $[u_1, [u_1, x_1]] = 0, \forall\, x_1 \in \mathscr{L}$. This infers that $u_1[u_1, x_1] + [u_1, x_1]u_1' + [u_1, x_1]u_1 = [u_1, x_1]u_1, \forall\, x_1 \in \mathscr{L}$. Further, A_2-condition implies that

$$u_1(u_1x_1 + x_1'u_1) = (u_1x_1 + x_1'u_1)u_1,\ \forall\, x_1 \in \mathscr{L}. \tag{1}$$

Again by using hypothesis, we obtain $[u_1, [u_1, 2l_1m_1]] = 0$, for any $l_1, m_1 \in \mathscr{L}$. Then $char\ \mathscr{S} \neq 2$ gives that $[u_1, [u_1, l_1m_1]] = 0$, for any $l_1, m_1 \in \mathscr{L}$. Thus,

$$\begin{aligned} 0 &= u_1(u_1l_1m_1 + l_1'm_1u_1) + (u_1l_1m_1 + l_1'm_1u_1)u_1' \\ &= u_1(u_1l_1m_1 + l_{1\circ}m_1u_1 + l_1m_1'u_1) + (u_1l_1m_1 + l_{1\circ}m_1u_1 + l_1'm_1u_1)u_1' \\ &= u_1(u_1l_1m_1 + l_{1\circ}u_1m_1 + l_1m_1'u_1) + (u_1l_1m_1 + l_{1\circ}u_1m_1 + l_1'm_1u_1)u_1' \end{aligned}$$

which gives that

$$u_1((u_1l_1 + l_1'u_1)m_1 + l_1(u_1m_1 + m_1'u_1)) + ((u_1l_1 + l_1'u_1)m_1 + l_1(u_1m_1 + m_1'u_1))u_1' = 0, \text{ for any } l_1, m_1 \in \mathscr{L}.$$

By using equation (1), we are left with $2(u_1l_1 + l_1'u_1)(u_1m_1 + m_1'u_1) = 0, \forall\, l_1, m_1 \in \mathscr{L}$. Since $char\ \mathscr{S} \neq 2$, so we obtain

$$(u_1l_1 + l_1'u_1)(u_1m_1 + m_1'u_1) = 0,\ \forall\, l_1, m_1 \in \mathscr{L}$$

or

$$[u_1, l_1][u_1, m_1] = 0,\ \forall\, l_1, m_1 \in \mathscr{L}. \tag{2}$$

Further, by replacing l_1 by $2l_1w$, for any $w \in \mathscr{L}$ in the above equation, we have

$$2[u_1, l_1w][u_1, m_1] = 0,\ \forall\, l_1, m_1 \in \mathscr{L}.$$

Again, since $char\ \mathscr{S} \neq 2$, so we left with $[u_1, l_1w][u_1, m_1] = 0, \forall\, l_1, m_1 \in \mathscr{L}$. By Lemma 2.4 and equation (2), we deduce that $[u_1, l_1]w[u_1, m_1] = 0, \forall\, l_1, m_1, w \in \mathscr{L}$. Equivalently, $[u_1, l_1]\mathscr{L}[u_1, m_1] = (0), \forall\, l_1, m_1 \in \mathscr{L}$. Lemma 4.6, infers that either $[u_1, l_1] = 0$ or $[u_1, m_1] = 0, \forall\, l_1, m_1 \in \mathscr{L}$. Both cases implies that $[u_1, \mathscr{L}] = (0)$. ■

Corollary 4.10. *Let $[u_1, [u_1, \mathscr{L}]] = (0)$, for any $u_1 \in \mathscr{S}$. Then either $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$ or u_1 commutes with every element of $\mathscr{L}$.*

Now, we divert our attention to the study of the Lie structure of higher commutators as a Lie ideal of $\mathscr{S}$ and hence as a Lie subsemiring of $\mathscr{S}$.

Theorem 4.11. *If $a \in \mathscr{S}$ satisfies $[a, [a, (\mathcal{H})]] = (0)$, where $(\mathcal{H})$ is an ideal generated by $\mathcal{H}$, for some higher commutator $\mathcal{H}$ of $\mathscr{S}$, then either $(\mathcal{H}) \subseteq \mathscr{Z}(\mathscr{S})$ or $a \in \mathscr{Z}(\mathscr{S})$.*

Proof. Let $\mathcal{H}$ be a higher commutator of $\mathscr{S}$. Then clearly $(\mathcal{H})$ is a Lie ideal of $\mathscr{S}$. Suppose that $(\mathcal{H}) \nsubseteq \mathscr{Z}(\mathscr{S})$. Thus, by Theorem 4.9, we have

$$[a, (\mathcal{H})] = (0). \tag{1}$$

By Corollary 3.8, $(\mathcal{H}) \supseteq \mathscr{S}^{(k)}$, for some k. Thus, equation (1) implies that $[a, \mathscr{S}^{(k)}] = (0)$, that is, $[a, [\mathscr{S}^{(k-1)}, \mathscr{S}^{(k-1)}]] = (0)$. By Theorem 4.7 and the same argument as above we can say that $[a, \mathscr{S}^{(k-1)}] = (0)$. Now, repeating the same process and using Theorem 3.10, we end up with $a \in \mathscr{Z}(\mathscr{S})$. ■

5. Derivations in semirings

Throughout this section, $\mathscr{S}$ represents a prime semiring with $char\ \mathscr{S} \neq 2$.

Definition 5.1. *[4] An additive map $d : \mathscr{S} \to \mathscr{S}$ is called a derivation of $\mathscr{S}$, if $(xy)^d = x^d y + xy^d$, $\forall\, x, y \in \mathscr{S}$.*

Definition 5.2. *Let $\mathcal{T}$ be any arbitrary subset of $\mathscr{S}$. Then the centralizer of $\mathcal{T}$ in $\mathscr{S}$ is $\mathcal{C}_{\mathscr{S}}(\mathcal{T}) = \{x \in \mathscr{S} : [x, \mathcal{T}] = (0)\}$.*

Lemma 5.3. *$\mathcal{C}_{\mathscr{S}}(\mathscr{L})$ is a Lie ideal and subsemiring of $\mathscr{S}$.*

Proof. Let $t_1 \in \mathcal{C}_{\mathscr{S}}(\mathscr{L})$. For any $s \in \mathscr{S}$, $[[t_1, s], l_1] = [t_1, [s, l_1]] + [s, [l_1, t_1]]$, by Jacobi identity. This gives that $[[t_1, s], l_1] = 0$, as $[t_1, \mathscr{L}] = (0)$. Thus, $[t_1, s] \in \mathcal{C}_{\mathscr{S}}(\mathscr{L})$. This concludes that $\mathcal{C}_{\mathscr{S}}(\mathscr{L})$ is a Lie ideal of $\mathscr{S}$. Now, let $t_1, t_2 \in \mathcal{C}_{\mathscr{S}}(\mathscr{L})$. Then $[t_1 t_2, l_1] = t_1[t_2, l_1] + [t_1, l_1]t_2 = 0$, $\forall\, l_1 \in \mathscr{L}$ which yields $t_1 t_2 \in \mathcal{C}_{\mathscr{S}}(\mathscr{L})$. This proves the lemma. ■

Observe that the centralizer of a Lie ideal of $\mathscr{S}$ is a 2-Lie ideal of $\mathscr{S}$.

Theorem 5.4. *If $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$, then $\mathcal{C}_{\mathscr{S}}(\mathscr{L}) \subseteq \mathscr{Z}(\mathscr{S})$.*

Proof. By the above lemma, $\mathcal{C}_{\mathscr{S}}(\mathscr{L})$ is a Lie ideal and subsemiring of $\mathscr{S}$. We now claim that $\mathcal{C}_{\mathscr{S}}(\mathscr{L})$ can not contain a non-zero ideal of $\mathscr{S}$. On contrary, let $\mathscr{I}$ be a non-zero ideal of $\mathscr{S}$ such that $\mathscr{I} \subseteq \mathcal{C}_{\mathscr{S}}(\mathscr{L})$, i.e., $[\mathscr{I}, \mathscr{L}] = (0)$. This concludes that $[\mathscr{S}\mathscr{I}, \mathscr{L}] = (0)$ which implies that $[si, l_1] = 0$, $\forall\, s \in \mathscr{S}, i \in \mathscr{I}, l_1 \in \mathscr{L}$. Thus, $s[i, l_1] + [s, l_1]i = 0$ leading to $[s, l_1]i = 0$, $\forall\, s \in \mathscr{S}, i \in \mathscr{I}, l_1 \in \mathscr{L}$. Hence, $[s, l_1]\mathscr{I} = (0)$, $\forall\, s \in \mathscr{S}, l_1 \in \mathscr{L}$. This deduces that $[s, l_1]\mathscr{S}\mathscr{I} = (0)$, $\forall\, s \in \mathscr{S}, l_1 \in \mathscr{L}$. Primeness of $\mathscr{S}$ yields $[s, l_1] = 0$, $\forall\, s \in \mathscr{S}, l_1 \in \mathscr{L}$. Therefore, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$, which is absurd. This concludes that $\mathcal{C}_{\mathscr{S}}(\mathscr{L})$ does not contain any non-zero ideal of $\mathscr{S}$. By Theorem 4.2, we get that $\mathcal{C}_{\mathscr{S}}(\mathscr{L}) \subseteq \mathscr{Z}(\mathscr{S})$. ■

The next result can be directly deduced as an outcome of Theorem 4.7.

Theorem 5.5. *If $\mathscr{L}$ is a 2-Lie ideal of $\mathscr{S}$ such that $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$, then $\mathcal{C}_{\mathscr{S}}([\mathscr{L}, \mathscr{L}]) = \mathcal{C}_{\mathscr{S}}(\mathscr{L})$.*

Theorem 5.6. *If d is a derivation of $\mathscr{S}$ such that $\mathscr{L}^d = (0)$, then either $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$ or $d = 0$.*

Proof. As $\mathscr{L}^d = (0)$, therefore $0 = [l_1, s]^d = (l_1 s + s' l_1)^d = l_1^d s + l_1 s^d + (s')^d l_1 + s' l_1^d$, $\forall\, l_1 \in \mathscr{L}, s \in \mathscr{S}$. This infers

$$l_1 s^d + (s')^d l_1 = 0,\ \forall\, l_1 \in \mathscr{L}, s \in \mathscr{S}. \tag{1}$$

In equation (1), the replacement of s with sm_1, where $m_1 \in \mathscr{L}$, gives $l_1(s^d m_1 + s m_1^d) + ((s')^d m_1 + s' m_1^d) l_1 = 0$, $\forall\, l_1 \in \mathscr{L}, s \in \mathscr{S}$. The given hypothesis concludes that

$$l_1 s^d m_1 + (s')^d m_1 l_1 = 0,\ \forall\, l_1, m_1 \in \mathscr{L}, s \in \mathscr{S}. \tag{2}$$

By applying Lemma 2.1 on equation (1), we get $l_1 s^d = s^d l_1$ and then using this in equation (2), we have $s^d l_1 m_1 + (s')^d m_1 l_1 = 0$, $\forall\, l_1, m_1 \in \mathscr{L}, s \in \mathscr{S}$. In other words,

$$s^d [l_1, m_1] = 0,\ \forall\, l_1, m_1 \in \mathscr{L}, s \in \mathscr{S}. \tag{3}$$

By putting st in place of s, with $t \in \mathscr{S}$, we get that $(s^d t + s t^d)[l_1, m_1] = 0$. Then equation (3) yields $s^d \mathscr{S}[l_1, m_1] = (0)$. By the primeness of $\mathscr{S}$, either $d = 0$ or $[l_1, m_1] = 0$, $\forall\, l_1, m_1 \in \mathscr{L}$. By Proposition 3.3, we get the desired conclusion. ■

Proposition 5.7. *If $\mathscr{L}$ is a 2-Lie ideal of $\mathscr{S}$ and d is a non-zero derivation of $\mathscr{S}$ with $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$. Suppose that $a\mathscr{L}^d = (0)$ or $\mathscr{L}^d a = (0)$, then $a = 0$.*

Proof. As $d \neq 0$ and $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$, so by the above theorem, $\mathscr{L}^d \neq (0)$. Since $[l_1,s]l_1 = l_1(sl_1) + (s'l_1)l_1 \in [\mathscr{L},\mathscr{S}] \subseteq \mathscr{L}$, for any $l_1 \in \mathscr{L}, s \in \mathscr{S}$, therefore $0 = a([l_1,s]l_1)^d = a([l_1,s]^d l_1 + [l_1,s]l_1^d)$. The given hypothesis infers that $a[l_1,s]l_1^d = 0, \forall\, l_1 \in \mathscr{L}, s \in \mathscr{S}$. Replacing s by $m_1^d r$, with $m_1 \in \mathscr{L}, r \in \mathscr{S}$, we get that $0 = a[l_1, m_1^d r]l_1^d = am_1^d[l_1,r]l_1^d + a[l_1,m_1^d]rl_1^d, \forall\, l_1, m_1 \in \mathscr{L}, r \in \mathscr{S}$. This concludes that $a(l_1 m_1^d + m_1^d l_1')rl_1^d = 0, \forall\, l_1, m_1 \in \mathscr{L}, r \in \mathscr{S}$, as $am_1^d = 0$. This implies that $al_1 m_1^d r l_1^d = 0, \forall\, l_1, m_1 \in \mathscr{L}, r \in \mathscr{S}$. Equivalently, $al_1 \mathscr{L}^d \mathscr{S} l_1^d = (0), \forall\, l_1 \in \mathscr{L}$. Primeness of $\mathscr{S}$ gives that for any $l_1 \in \mathscr{L}$, either $al_1\mathscr{L}^d = (0)$ or $l_1^d = 0$. But $\mathscr{L}^d \neq (0)$, so there exists some $n \in \mathscr{L}$ such that $n^d \neq 0$ and $an\mathscr{L}^d = (0)$. We further claim that $ap\mathscr{L}^d = (0), \forall\, p \in \mathscr{L}$. If possible, let $p(\neq n) \in \mathscr{L}$ with $ap\mathscr{L}^d \neq (0)$. This deduces that $p^d = 0$. Thus, $a(p+n)\mathscr{L}^d = ap\mathscr{L}^d + an\mathscr{L}^d \neq (0)$ and $(p+n)^d = n^d \neq 0$ hold simultaneously and it leads to a contradiction. Hence, $ap\mathscr{L}^d = (0), \forall\, p \in \mathscr{L}$, equivalently $a\mathscr{L}\mathscr{L}^d = (0)$. In view of Lemma 4.6, we obtain $a = 0$. ■

Finally, we give an extension of [3, Theorem 1].

Theorem 5.8. *If d is a non-zero derivation of $\mathscr{S}$ and $\mathscr{L}$ is a 2-Lie ideal of $\mathscr{S}$ with $\mathscr{L}^{d^2} = (0)$, then $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$.*

Proof. As we have proved earlier in Theorem 4.2, the ideal $2\mathscr{S}[\mathscr{L},\mathscr{L}]\mathscr{S} \subseteq \mathscr{L}$, therefore

$$(2s[l_1,m_1]n)^{d^2} = 0, \ \forall\ l_1, m_1, n \in \mathscr{L}, s \in \mathscr{S}.$$

This gives that

$$0 = ((2s[l_1,m_1])^d n + 2s[l_1,m_1]n^d)^d = (2s[l_1,m_1])^{d^2} n + (2s[l_1,m_1])^d n^d + (2s[l_1,m_1])^d n^d + 2s[l_1,m_1]n^{d^2}.$$

Since $2\mathscr{S}[\mathscr{L},\mathscr{L}] \subseteq \mathscr{L}$, $n \in \mathscr{L}$ and $char\ \mathscr{S} \neq 2$, therefore given hypothesis leads to

$$(s[l_1,m_1])^d n^d = 0, \ \forall\ l_1, m_1, n \in \mathscr{L}, s \in \mathscr{S}.$$

This infers that

$$s^d[l_1,m_1]n^d + s[l_1,m_1]^d n^d = 0, \ \forall\, l_1, m_1, n \in \mathscr{L}, s \in \mathscr{S}. \quad (1)$$

Replacing s by sr, with $r \in \mathscr{S}$ and obtain

$$s^d r[l_1,m_1]n^d + s(r^d[l_1,m_1]n^d + r[l_1,m_1]^d n^d) = 0.$$

Then equation (1) implies that $s^d r[l_1,m_1]n^d = 0$ which is equivalent to $s^d\mathscr{S}[l_1,m_1]n^d = (0), \forall\, l_1, m_1, n \in \mathscr{L}, s \in \mathscr{S}$. By primeness of $\mathscr{S}$, we get that

$$[l_1,m_1]n^d = 0, \ \forall\, l_1, m_1, n \in \mathscr{L}. \quad (2)$$

Further, replacing m_1 by $2m_1 t$, with $t \in \mathscr{L}$, we have $2([l_1,m_1]tn^d + m_1[l_1,t]n^d) = 0$. By using $char\ \mathscr{S} \neq 2$ and equation (2), we are left with $[l_1,m_1]\mathscr{L}n^d = (0), \forall\, l_1, m_1, n \in \mathscr{L}$. Then Lemma 4.6 gives either $[l_1,m_1] = 0$ or $n^d = 0, \forall\, l_1, m_1, n \in \mathscr{L}$. Then Proposition 3.3 and Theorem 5.6 concludes that $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$. ■

6. Conclusions

This paper characterized the Lie structure of semirings and action of derivations on Lie ideals of semirings. It is observed that for a prime semiring $\mathscr{S}$, with $char\ \mathscr{S} \neq 2$ and $[a,[\mathscr{L},\mathscr{L}]] = (0)$, for any Lie ideal $\mathscr{L}$ of $\mathscr{S}$ and $a \in \mathscr{S}$, either $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$ or $[a,\mathscr{L}] = (0)$ and thereby partially generalized Herstein's theorems in the framework of additively regular semirings and their higher commutators. Moreover, an extension to Herstein's result: "For a ring $\mathcal{R}$ with $char\ \mathcal{R} \neq 2$, any Lie ideal $\mathscr{L}$ is either contained in the center of $\mathcal{R}$ or contains a non-zero ideal of $\mathcal{R}$" is established which also enable us to extend Bergen's theorem for derivations.

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Some coefficient properties of a certain family of regular functions associated with lemniscate of Bernoulli and Opoola differential operator

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Abstract. In this exploration, we introduce a certain family of regular (or analytic) functions in association with the right-half of the Lemniscate of Bernoulli and the well-known Opoola differential operator. For the regular function f studied in this work, some estimates for the early coefficients, Fekete-Szegö functionals and second and third Hankel determinants are established. Another established result is the sharp upper estimate of the third Hankel determinant for the inverse function f^{-1} of f.

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Keywords: Regular function, Lemniscate of Bernoulli, Fekete-Szegö functional, inverse function, coefficient bounds, Hankel determinant, Opoola differential operator.

Contents

1. Introductory Statements

Firstly, we represent by $\mathcal{A}$, the family of *normalized and regular functions* whose form is of the Taylor's series

$$f(z) = z + \sum_{x=2}^{\infty} a_x z^x, \quad f(0) = f'(0) - 1 = 0 \tag{1.1}$$

and $z \in \Sigma := \{z \in \mathbb{C}, \text{ such that } |z| < 1\}$. Also, represented by $\mathcal{S}$ is the family of functions $f \in \mathcal{A}$ that are also univalent in Σ. A famous subfamily of $\mathcal{S}$ is the family $\mathcal{ST}$ of starlike functions. A function $f \in \mathcal{S}$ is said to be in $\mathcal{ST}$ if the condition $\mathcal{R}e(z(f'/f)) > 0$ holds. For function class $\mathcal{S}$, the Koebe one-quarter theorem, see [10],

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is a famous theorem that affirms that the range of every function $f \in \mathcal{S}$ includes the disk $\{w : |w| < 0.25\}$. For this purpose, $f \in \mathcal{S}$ has the inverse function f^{-1} where

$$f^{-1}(f(z)) = z, \quad z \in \Sigma,$$

$$f(f^{-1}(w)) = w, \quad |w| < r_0(f),\ r_0(f) \geq 0.25,$$

and some computations show that

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

We represent the family of regular functions of the form

$$\wp(z) = 1 + \sum_{x=1}^{\infty} p_x z^x, \quad z \in \Sigma \tag{1.3}$$

by $\mathcal{P}$ where $\mathcal{P}$ is called the family of functions with positive real parts in Σ. A generalization of (1.3) is the function

$$\wp_\sigma(z) = 1 + (1-\sigma)\sum_{x=1}^{\infty} p_x z^x, \quad z \in \Sigma,\ 0 \leq \sigma < 1, \tag{1.4}$$

known as the function with positive real parts of order σ. Let $\mathcal{P}(\sigma)$ represent the family of functions $\wp_\sigma(z)$.

Let "$\prec$" represent subordination. Then for $f, F \in \mathcal{A}$, $f(z) \prec F(z)$ if there exists a Schwarz function

$$s(z) = \sum_{x=1}^{\infty} s_x z^x, \quad z \in \Sigma$$

such that $s(0) = 0$, $|s(z)| = |z| < 1$, and $f(z) = F(s(z))$. Suppose $F(z)$ is univalent in Σ, then

$$f(z) \prec F(z) \text{ if and only if } f(0) = F(0) \text{ and } f(\Sigma) \subset F(\Sigma).$$

Recently, the direction of research in theory of geometric functions shows that the study of some prescribed domains $\wp(\Sigma)$ is inexhaustible. In fact, special cases of functions $\wp(z)$ have greatly motivated many researchers to study various kinds of natural image domains of $\wp(\Sigma)$. Some of these domains can be found in [7, 9, 12, 13, 15, 16, 18, 21, 25–27, 29, 31] and the citations therein. *Precisely*, Sokól and Stankiewicz [32] reported the subfamily $\mathcal{SL}(\ell b) \subset \mathcal{ST}$ satisfying the condition

$$\varphi(z) = z(f'/f) \prec \ell b(z) = \sqrt{1+z}, \quad z \in \Sigma \tag{1.5}$$

such that function φ lies in the domain bounded by the *right half of the lemniscate of Bernoulli* which is geometrically represented by $|\varphi^2 - 1| < 1$, $\forall z \in \Sigma$. One can find some descriptive diagrams and more properties of domain $|\varphi^2 - 1| < 1$ in [32]. The work of Lockwood [20] is a treatise of curves available for further research.

The differential operator $\mathcal{D}^{n,\beta}_{\tau,\mu} : \mathcal{A} \longrightarrow \mathcal{A}$ was announced by Opoola [23], see also [4, 17, 27]. For $f \in \mathcal{A}$ of the form (1.1),

$$\begin{aligned}
&\mathcal{D}^{0,\beta}_{\tau,\mu} f(z) = f(z)\\
&\mathcal{D}^{1,\beta}_{\tau,\mu} f(z) = (1 + (\beta - \mu - 1)\tau) f(z) - z\tau(\beta - \mu) + z\tau f'(z) = \mathcal{J}_\tau(f(z))\\
&\mathcal{D}^{2,\beta}_{\tau,\mu} f(z) = \mathcal{J}_\tau(\mathcal{D}^{1,\beta}_{\tau,\mu} f(z))\\
&\mathcal{D}^{3,\beta}_{\tau,\mu} f(z) = \mathcal{J}_\tau(\mathcal{D}^{2,\beta}_{\tau,\mu} f(z))
\end{aligned}$$

and

$$\mathcal{D}^{n,\beta}_{\tau,\mu} f(z) = \mathcal{J}_\tau(\mathcal{D}^{n-1,\beta}_{\tau,\mu} f(z))$$

which can be simplified as

$$\mathcal{D}^{n,\beta}_{\tau,\mu} f(z) = z + \sum_{x=2}^{\infty} (1 + (x + \beta - \mu - 1)\tau)^n a_x z^x, \quad z \in \Sigma \tag{1.6}$$

for parameters in (2.2). It is clear that from (1.6),

1. $\mathcal{D}^{0,\beta}_{\tau,\mu} f(z) = \mathcal{D}^{n,\beta}_{0,\mu} f(z) = \mathcal{D}^{0,\beta}_{0,\mu} f(z) = f(z)$.
2. $\mathcal{D}^{n,\beta}_{1,\beta} f(z) = \mathcal{D}^{n,\mu}_{1,\mu} f(z) = \mathcal{D}^n f(z)$ is the famous Sălăgean differential operator, see [3, 30].
3. $\mathcal{D}^{n,\beta}_{\tau,\beta} f(z) = \mathcal{D}^{n,\mu}_{\tau,\mu} f(z) = \mathcal{D}^n_\tau f(z)$ is the famous Al-Oboudi differential operator, see [2].

2. A New Family of Regular Functions

The function f in $\mathcal{A}$ is in the family $\mathcal{B}^{n,\beta}_{\tau,\mu}(\delta, \gamma, \ell b)$ if it satisfies the condition

$$(1 - e^{-2i\delta}\gamma^2 z^2) \frac{\mathcal{D}^{n+1,\beta}_{\tau,\mu} f(z)}{z} \prec \ell b(z) \tag{2.1}$$

for

$$n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \ 0 \le \mu \le \beta; \ \beta, \tau \ge 0, \ \delta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \ 0 \le \gamma \le 1, \ z \in \Sigma, \tag{2.2}$$

$\ell b(z)$ and $\mathcal{D}^{n+1,\beta}_{\tau,\mu} f(z)$ are functions declared in (1.5) and (1.6), respectively. We however demonstrate that the following are special cases of $\mathcal{B}^{n,\beta}_{\tau,\mu}(\delta, \gamma, \ell b)$. Let $\widetilde{\wp}_0(z) = (1+z)/(1-z)$ and $\widetilde{\wp}_\sigma(z) = (1+(1-2\sigma)z)/(1-z)$ be the extremal functions, respectively in $\mathcal{P}$ and $\mathcal{P}(\sigma)$, then

1. $\mathcal{B}^{0,\beta}_{\tau,\mu}(0, 0, \widetilde{\wp}_0) = R$, the family of bounded turning functions presented in [1].
2. $\mathcal{B}^{0,\beta}_{\tau,\mu}(0, 0, \widetilde{\wp}_\sigma) = R(\sigma)$, the family of bounded functions of order σ presented in [33] and
3. $\mathcal{B}^{0,\beta}_{\tau,\mu}(0, 1, \widetilde{\wp}) = H$, the family of functions presented in [11].

In this investigation, a new subfamily of regular functions is defined and some estimates for early coefficients, Fekete-Szegö functional (for both real and complex parameters), and the second, and third Hankel determinants for the functions $f \in \mathcal{A}$ are established. We also established the upper estimate for the third Hankel determinant for the inverse function f^{-1} of $f \in \mathcal{A}$. We are inspired by the works in [18].

3. Lemmas

The lemmas that follow shall be needed.

Lemma 3.1 ([6]). *If $\wp(z) \in \mathcal{P}$ and $\alpha \in \mathbb{R}$, then*

$$\left| p_2 - \alpha \frac{p_1^2}{2} \right| \le \begin{cases} 2(1-\alpha) & when \quad \alpha \le 0, \\ 2 & when \ 0 \le \alpha \le 2, \\ 2(\alpha - 1) & when \quad \alpha \ge 2. \end{cases}$$

Lemma 3.2 ([6]). *If $\wp(z) \in \mathcal{P}$ and $\beta \in \mathbb{C}$, then*

$$\left| p_2 - \beta \frac{p_1^2}{2} \right| \le 2 \max\{1, |1 - \beta|\}.$$

Lemma 3.3 ([14]). *If $\wp(z) \in \mathcal{P}$, $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{N}$, then*

$$|p_{x+y} - \alpha p_x p_y| \le \begin{cases} 2 & when \quad 0 \le \alpha \le 1, \\ 2|2\alpha - 1| & elsewhere. \end{cases}$$

Lemma 3.4 ([10]). *If $\wp(z) \in \mathcal{P}$, then $|p_x| \le 2$ and $x \in \mathbb{N}$.*

4. Main Results

Henceforth, it is assumed that all parameters are as declared in (2.2) unless otherwise stated. Our results are therefore as follows.

4.1. Coefficient Estimates

Theorem 4.1. *If* $f \in \mathcal{B}^{n,\beta}_{\tau,\mu}(\delta, \gamma, \ell b)$, *then*

$$|a_2| \leq \frac{1}{2\phi_2} \tag{4.1}$$

$$|a_3| \leq \frac{13+8\gamma^2}{8\phi_3} \tag{4.2}$$

$$|a_4| \leq \frac{25+8\gamma^2}{16\phi_4} \tag{4.3}$$

$$|a_5| \leq \frac{1603+832\gamma^2+512\gamma^4}{512\phi_5} \tag{4.4}$$

where

$$\phi_x = (1+(x+\beta-\mu-1)\tau)^{n+1}. \tag{4.5}$$

Proof. Let $f \in \mathcal{B}^{n,\beta}_{\tau,\mu}(\delta, \gamma, \ell b)$, then the definition of subordination permits us to represent (2.1) as

$$(1-e^{-2i\delta}\gamma^2 z^2)\frac{\mathcal{D}^{n+1,\beta}_{\tau,\mu} f(z)}{z} = \ell b(s(z))$$

or

$$(1-e^{-2i\delta}\gamma^2 z^2)(\mathcal{D}^{n+1,\beta}_{\tau,\mu} f(z)) = z[1+s(z)]^{1/2}. \tag{4.6}$$

For brevity, we use ϕ_x in (4.5) so that simple computation shows that (4.6) expands as

$$\begin{aligned} z + \phi_2 a_2 z^2 + (\phi_3 a_3 - e^{-2i\delta}\gamma^2)z^3 + (\phi_4 a_4 - e^{-2i\delta}\gamma^2\phi_2 a_2)z^4 + (\phi_5 a_5 - e^{-2i\delta}\gamma^2\phi_3 a_3)z^5 + \cdots \\ = z + \frac{1}{4}p_1 z^2 + \frac{1}{4}\left(p_2 - \frac{17}{8}p_1^2\right)z^3 + \frac{1}{4}\left(\frac{13}{32}p_1^3 - \frac{5}{4}p_1p_2 + p_3\right)z^4 \\ + \frac{1}{4}\left(\frac{419}{2048}p_1^4 + \frac{105}{96}p_1^2p_2 - \frac{5}{4}p_1p_3 - \frac{5}{8}p_2^2 + p_4\right)z^5 + \cdots \end{aligned} \tag{4.7}$$

so that the comparison of the coefficients yields

$$a_2 = \frac{p_1}{4\phi_2} \tag{4.8}$$

$$a_3 = \frac{(p_2 - \frac{17}{8}p_1^2) + 4e^{-2i\delta}\gamma^2}{4\phi_3} \tag{4.9}$$

$$a_4 = \frac{(p_3 - \frac{5}{4}p_1p_2 + \frac{13}{32}p_1^3) + e^{-2i\delta}\gamma^2 p_1}{4\phi_4} \tag{4.10}$$

and

$$a_5 = \frac{(p_4 - \frac{5}{4}p_1p_3 - \frac{5}{8}p_2^2 + \frac{35}{32}p_1^2p_2 + \frac{419}{2048}p_1^4) + (p_2 - \frac{17}{8}p_1^2)e^{-2i\delta}\gamma^2 + 4e^{-4i\delta}\gamma^4}{4\phi_5}. \tag{4.11}$$

Application of triangle inequality and Lemma 3.4 in (4.8) yields our result in (4.1). Also, from (4.9),

$$|a_3| \le \frac{|p_2 - \frac{17}{8}p_1^2| + 4|e^{-2i\delta}|\gamma^2}{4\phi_3}$$

and the application of Lemma 3.1 yields the result in (4.2). From (4.10), we have the presentation

$$|a_4| \le \frac{|p_3 - \frac{5}{4}p_1p_2| + \frac{13}{32}|p_1^3| + |e^{-2i\delta}|\gamma^2|p_1|}{4\phi_4}$$

which by the application of Lemmas 3.1 and 3.4 yields our result in (4.3). To obtain estimate for a_5 we have from (4.11) that

$$a_5 = \frac{(p_4 - \frac{5}{4}p_1p_3) - \frac{5}{8}p_2(p_2 - \frac{7}{2}\frac{p_1^2}{2}) + \frac{419}{2048}p_1^4 + (p_2 - \frac{17}{4}\frac{p_1^2}{2})e^{-2i\delta}\gamma^2 + 4e^{-4i\delta}\gamma^4}{4\phi_5}.$$

and

$$|a_5| \le \frac{|p_4 - \frac{5}{4}p_1p_3| + \frac{5}{8}|p_2||p_2 - \frac{7}{2}\frac{p_1^2}{2}| + \frac{419}{2048}|p_1^4| + |p_2 - \frac{17}{4}\frac{p_1^2}{2}||e^{-2i\delta}|\gamma^2 + 4|e^{-4i\delta}|\gamma^4}{4\phi_5}$$

which by the application of Lemmas 3.1, 3.3 and 3.4 yields our result in (4.4). ■

4.2. Estimates for Fekete-Szegö Functional

Another commonly studied property of the coefficient problems of $f \in \mathcal{A}$ is the Fekete-Szegö functional

$$\mathcal{FS}(\varepsilon, f) = \left|a_3 - \varepsilon a_2^2\right|, \quad \varepsilon \in \mathbb{R} \tag{4.12}$$

announced in [8]. Interested reader may see [4, 5, 17–19, 24] and the citations therein for more properties, applications, and background details.

Theorem 4.2. *If $f \in \mathcal{B}^{n,\beta}_{\tau,\mu}(\delta, \gamma, \ell b)$, then for real parameter ε,*

$$\left|a_3 - \varepsilon a_2^2\right| \le \begin{cases} \frac{1-\alpha+2\gamma^2}{2\phi_3} & when \quad \varepsilon \le -\frac{17\phi_2^2}{2\phi_3} \\ \frac{1+2\gamma^2}{2\phi_3} & when \quad -\frac{17\phi_2^2}{2\phi_3} \le \varepsilon \le -\frac{9\phi_2^2}{2\phi_3} \\ \frac{\alpha-1+2\gamma^2}{2\phi_3} & when \quad \varepsilon \ge -\frac{9\phi_2^2}{2\phi_3} \end{cases} \tag{4.13}$$

where

$$\alpha = \frac{17\phi_2^2 + 2\varepsilon\phi_3}{4\phi_2^2}. \tag{4.14}$$

Proof. Let $\varepsilon \in \mathbb{R}$. If we substitute (4.8) and (4.9) into (4.12) we will arrive at

$$|a_3 - \varepsilon a_2^2| = \left|\frac{(p_2 - \frac{17}{8}p_1^2) + 4e^{-2i\delta}\gamma^2}{4\phi_3} - \frac{\varepsilon p_1^2}{16\phi_2^2}\right|$$

so that

$$|a_3 - \varepsilon a_2^2| \le \frac{1}{4\phi_3}\left|p_2 - \left(\frac{17\phi_2^2 + 2\varepsilon\phi_3}{4\phi_2^2}\right)\frac{p_1^2}{2}\right| + \left|\frac{e^{-2i\delta}\gamma^2}{\phi_3}\right|$$

or

$$|a_3 - \varepsilon a_2^2| \le \frac{1}{4\phi_3}\left|p_2 - \alpha\frac{p_1^2}{2}\right| + \frac{\gamma^2}{\phi_3}$$

where α is defined in (4.14). The application of Lemma 3.1 means that for α that satisfies conditions $\alpha \le 0$, $0 \le \alpha \le 2$ and $\alpha \ge 2$, we have the results in (4.13). ■

Theorem 4.3. *If $f \in \mathcal{B}_{\tau,\mu}^{n,\beta}(\delta, \gamma, \ell b)$, then for complex parameter ξ,*

$$\left|a_3 - \xi a_2^2\right| \leq \frac{1}{2\phi_3} \max\{1, |1-\beta|\} + \frac{\gamma^2}{\phi_3} \tag{4.15}$$

where

$$\beta = \frac{17\phi_2^2 + 2\xi\phi_3}{4\phi_2^2} \tag{4.16}$$

Proof. Let $\xi \in \mathbb{C}$. If we substitute (4.8) and (4.9) into (4.12) we will arrive at the inequality

$$|a_3 - \xi a_2^2| \leq \frac{1}{4\phi_3} \left| p_2 - \left(\frac{17\phi_2^2 + 2\xi\phi_3}{4\phi_2^2}\right) \frac{p_1^2}{2} \right| + \left| \frac{e^{-2i\delta}\gamma^2}{\phi_3} \right|$$

or

$$|a_3 - \xi a_2^2| \leq \frac{1}{4\phi_3} \left| p_2 - \beta \frac{p_1^2}{2} \right| + \frac{\gamma^2}{\phi_3}$$

where β is defined in (4.16). The application of Lemma 3.2 produce the result in (4.15). ■

4.3. Estimates for some Hankel Determinants

The yth-Hankel determinant

$$\mathcal{HD}_{y,x}(f) = \begin{vmatrix} 1 & a_{x+1} & a_{x+2} & \cdots & a_{x+y-1} \\ a_{x+1} & a_{x+2} & \cdots & \cdots & a_{x+y} \\ a_{x+2} & a_{x+3} & \cdots & \cdots & a_{x+y+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{x+y-1} & a_{x+y} & \cdots & \cdots & a_{x+2(y-1)} \end{vmatrix} \tag{4.17}$$

($x, y \in \mathbb{N}$) was introduced by Pommerenke [28]. (4.17) has its elements from the coefficients of f in (1.1). Observe that from (4.17), we can establish that

$$|\mathcal{HD}_{2,1}(f)| = |a_3 - a_2^2|, \tag{4.18}$$

$$|\mathcal{HD}_{2,2}(f)| = |a_2 a_4 - a_3^2|, \tag{4.19}$$

$$\mathcal{HD}_{3,1}(f) = a_3(a_2 a_4 - a_3^2) + a_4(a_2 a_3 - a_4) + a_5(a_3 - a_2^2) \tag{4.20}$$

hence,

$$|\mathcal{HD}_{3,1}(f)| \leq |a_3||\mathcal{HD}_{2,2}(f)| + |a_4||\mathcal{G}_2(f)| + |a_5||\mathcal{HD}_{2,1}(f)|. \tag{4.21}$$

where

$$|\mathcal{G}_x(f)| = |a_x a_{x+1} - a_{x+2}|, \quad x = \{2, 3, 4, \ldots\}. \tag{4.22}$$

Even though the functionals in (4.12) and (4.18) have different historical background, yet it can be observed that the functionals are related since $|\mathcal{HD}_{2,1}(f)| = \mathcal{FS}(1, f)$.

For the inverse functions f^{-1} in (1.2), Obradovic and Tuneski [22] established that

$$|\mathcal{HD}_{3,1}(f^{-1})| = |\mathcal{HD}_{3,1}(f) - (a_3 - a_2^2)^3| \tag{4.23}$$

and obtained some estimates for some subfamilies of $\mathcal{S}$. Interested reader may see [4, 5, 17–19] and the citations therein for some properties and applications; and more background details on Hankel determinants.

Theorem 4.4. *If $f \in \mathcal{B}_{\tau,\mu}^{n,\beta}(\delta, \gamma, \ell b)$, then*

$$|\mathcal{HD}_{2,1}(f)| \leq \frac{1 + 2\gamma^2}{2\phi_3} \tag{4.24}$$

Proof. Substituting $\xi = 1$ in (4.15) yields (4.24). ■

Theorem 4.5. *If $f \in \mathcal{B}^{n,\beta}_{\tau,\mu}(\delta,\gamma,\ell b)$, then*

$$|\mathcal{HD}_{2,2}(f)| \leq -4A + 8B - 2C + 8D - E + 4F + \frac{\gamma^4}{\phi_2^2} \tag{4.25}$$

where

$$\left.\begin{array}{c} A = \frac{1}{16\phi_2\phi_4}, \quad B = \frac{289\phi_2\phi_4 - 26\phi_3^2}{1024\phi_2\phi_3^2\phi_4}, \quad C = \frac{1}{16\phi_3^2}, \\ D = \frac{17\phi_4 - 5\phi_3^2}{64\phi_2\phi_3^2\phi_4}, \quad E = \frac{1}{2\phi_3^2}\gamma^2 \quad \text{and} \quad F = \frac{17\phi_2\phi_4 + \phi_3^2}{16\phi_2\phi_3^2\phi_4}\gamma^2 \end{array}\right\} \tag{4.26}$$

Proof. Substituting (4.8), (4.9) and (4.10) into (4.19) simplifies to

$$\begin{aligned} \mathcal{HD}_{2,2}(f) =& \frac{1}{16\phi_2\phi_4}p_1p_3 - \frac{289\phi_2\phi_4 - 26\phi_3^2}{1024\phi_2\phi_3^2\phi_4}p_1^4 - \frac{1}{16\phi_3^2}p_2^2 + \frac{17\phi_4 - 5\phi_3^2}{64\phi_2\phi_3^2\phi_4}p_1^2p_2 \\ & - \frac{1}{2\phi_3^2}e^{-2i\delta}\gamma^2p_2 + \frac{17\phi_2\phi_4 + \phi_3^2}{16\phi_2\phi_3^2\phi_4}e^{-2i\delta}\gamma^2p_1^2 - \frac{e^{-4i\delta}\gamma^4}{\phi_2^2} \end{aligned}$$

and for brevity we get

$$\mathcal{HD}_{2,2}(f) = Ap_1p_3 - Bp_1^4 - Cp_2^2 + Dp_1^2p_2 - Ee^{-2i\delta}p_2 + Fe^{-2i\delta}p_1^2 - \frac{e^{-4i\delta}\gamma^4}{\phi_2^2}$$

for A, B, C, D, E and F in (4.26). Now some rearrangement and simplifications yield

$$|\mathcal{HD}_{2,2}(f)| = \left| Ap_1\left(p_3 - \frac{B}{A}p_1^3\right) - Cp_2\left(p_2 - \frac{2D}{C}\frac{p_1^2}{2}\right) - Ee^{-2i\delta}\left(p_2 - \frac{2F}{E}\frac{p_1^2}{2}\right) - \frac{e^{-4i\delta}\gamma^4}{\phi_2^2}\right|$$

so that

$$|\mathcal{HD}_{2,2}(f)| \leq |Ap_1|\left|p_3 - \frac{B}{A}p_1^3\right| + |Cp_2|\left|p_2 - \frac{2D}{C}\frac{p_1^2}{2}\right| + |Ee^{-2i\delta}|\left|p_2 - \frac{2F}{E}\frac{p_1^2}{2}\right| + \left|\frac{e^{-4i\delta}\gamma^4}{\phi_2^2}\right|$$

and the appropriate application of Lemmas 3.1, 3.3 and 3.4 yields (4.25). ■

Theorem 4.6. *If $f \in \mathcal{B}^{n,\beta}_{\tau,\mu}(\delta,\gamma,\ell b)$, then*

$$|\mathcal{G}_2(f)| \leq -2G + 4H + 8I + 2J \tag{4.27}$$

where

$$G = \frac{1}{4\phi_4}, \quad H = \frac{\phi_4 + 5\phi_2\phi_3}{16\phi_2\phi_3\phi_4}, \quad I = \frac{17\phi_4 + 13\phi_2\phi_3}{128\phi_2\phi_3\phi_4}, \quad \text{and} \quad J = \frac{\phi_2\phi_3 - \phi_4}{4\phi_2\phi_3\phi_4}\gamma^2. \tag{4.28}$$

Proof. Substituting (4.8), (4.9) and (4.10) into (4.22) simplifies to

$$\mathcal{G}_2(f) = a_2a_3 - a_4 = -\frac{1}{4\phi_4}p_3 + \frac{\phi_4 + 5\phi_2\phi_3}{16\phi_2\phi_3\phi_4}p_1p_2 - \frac{17\phi_4 + 13\phi_2\phi_3}{128\phi_2\phi_3\phi_4}p_1^3 - \frac{\phi_2\phi_3 - \phi_4}{4\phi_2\phi_3\phi_4}e^{-2i\delta}\gamma^2p_1$$

and for brevity we get

$$\mathcal{G}_2(f) = -Gp_3 + Hp_1p_2 - Ip_1^3 - Je^{-2i\delta}p_1$$

for G, H, I and J in (4.28). Now some rearrangement and simplifications yield

$$|\mathcal{G}_2(f)| = \left|-G\left(p_3 - \frac{H}{G}p_1p_2\right) - Ip_1^3 - Je^{-2i\delta}p_1\right|$$

so that

$$|\mathcal{G}_2(f)| \leq |-G|\left|p_3 - \frac{H}{G}p_1p_2\right| + |Ip_1^3| + |Je^{-2i\delta}p_1|$$

and the appropriate application of Lemmas 3.3 and 3.4 yields (4.27). ■

Theorem 4.7. *If $f \in \mathcal{B}^{n,\beta}_{\tau,\mu}(\delta,\gamma,\ell b)$, then*

$$|\mathcal{HD}_{3,1}(f)| \le \left(\frac{13+8\gamma^2}{8\phi_3}\right)\left[-4A+8B-2C+8D-E+4F+\frac{\gamma^4}{\phi_2^2}\right] + \left(\frac{25+8\gamma^2}{16\phi_4}\right)\left[-2G+4H+8I+2J\right] + \left(\frac{1603+832\gamma^2+512\gamma^4}{512\phi_5}\right)\left[\frac{1+2\gamma^2}{2\phi_3}\right] \tag{4.29}$$

where $A, B, C, \ldots, J$ are defined in (4.26) *and* (4.28).

Proof. Substitute (4.2), (4.3), (4.4), (4.24), (4.25) and (4.27) into (4.21) yields (4.29). ■

Theorem 4.8. *If $f \in \mathcal{B}^{n,\beta}_{\tau,\mu}(\delta,\gamma,\ell b)$, then*

$$|\mathcal{HD}_{3,1}(f^{-1})| \le \frac{13+8\gamma^2}{4\phi_3}\left[-4A+8B-2C+8D-E+4F+\frac{\gamma^2}{\phi_2^2}\right] + \frac{1+2\gamma^2}{2\phi_3}\left[4L-2K+4N-2M+8P-4R+16Q+\frac{\gamma^4}{\phi_5}\right] + \left[\frac{25+8\gamma^2}{16\phi_4}\right]^2 + \left[\frac{1}{2\phi_2}\right]^6 \tag{4.30}$$

where $A, B, C, \ldots, J$ are defined in (4.26) *and* (4.28)*, and*

$$\left.\begin{aligned} K = \tfrac{1}{4\phi_5},\ L = \tfrac{5}{16\phi_5},\ M = \tfrac{1}{4\phi_5}\gamma^2,\ N = \tfrac{17\phi_2^2\phi_3-6\phi_5}{32\phi_2^2\phi_3\phi_5}\gamma^2, \\ R = \tfrac{5}{32\phi_5},\ P = \tfrac{6\phi_5+35\phi_2^2\phi_3}{128\phi_2^2\phi_3\phi_5},\ Q = \tfrac{816\phi_5-419\phi_2^2\phi_3}{8192\phi_2^2\phi_3\phi_5}. \end{aligned}\right\} \tag{4.31}$$

Proof. Substituting (4.20) into (4.23) yields

$$\begin{aligned} \mathcal{HD}_{3,1}(f^{-1}) &= \Big(a_3(a_2a_4-a_3^2)+a_4(a_2a_3-a_4)+a_5(a_3-a_2^2)\Big) - \Big(a_3-a_2^2\Big)^3 \\ &= 2a_2a_3a_4 - 2a_3^3 - a_4^2 + a_3a_5 - a_2^2a_5 + 3a_2^2a_3^2 - 3a_2^4a_3 + a_2^6 \\ &= 2a_3(a_2a_4-a_3^2) + a_5(a_3-a_2^2) + 3a_2^2a_3(a_3-a_2^2) - a_4^2 + a_2^6 \\ &= 2a_3(a_2a_4-a_3^2) + (a_3-a_2^2)(3a_2^2a_3+a_5) - a_4^2 + a_2^6 \end{aligned}$$

so that

$$|\mathcal{HD}_{3,1}(f^{-1})| \le 2|a_3||a_2a_4-a_3^2| + |a_3-a_2^2||3a_2^2a_3+a_5| + |a_4|^2 + |a_2|^6 \tag{4.32}$$

or

$$|\mathcal{HD}_{3,1}(f^{-1})| \le 2|a_3||\mathcal{HD}_{2,2}(f)| + |\mathcal{HD}_{2,1}(f)||3a_2^2a_3+a_5| + |a_4|^2 + |a_2|^6. \tag{4.33}$$

Observe that by using (4.8), (4.9) and (4.11),

$$3a_2^2a_3+a_5 = \frac{1}{4\phi_5}p_4 - \frac{5}{16\phi_5}p_1p_3 + \frac{1}{4\phi_5}e^{-2i\delta}\gamma^2p_2 - \frac{17\phi_2^2\phi_3-6\phi_5}{32\phi_2^2\phi_3\phi_5}e^{-2i\delta}\gamma^2p_1^2 - \frac{5}{32\phi_5}p_2^2 + \frac{6\phi_5+35\phi_2^2\phi_3}{128\phi_2^2\phi_3\phi_5}p_1^2p_2 + \frac{816\phi_5-419\phi_2^2\phi_3}{8192\phi_2^2\phi_3\phi_5}p_1^4 + \frac{e^{-4i\delta}\gamma^4}{\phi_5}$$

so that for brevity,

$$3a_2^2a_3+a_5 = Kp_4 - Lp_1p_3 + Me^{-2i\delta}p_2 - Ne^{-2i\delta}p_1^2 - Rp_2^2 + Pp_1^2p_2 + Qp_1^4 + \frac{e^{-4i\delta}\gamma^4}{\phi_5}$$

for K, L, M, N, R, P and Q in (4.31). Now some rearrangement and simplifications yield

$$|3a_2^2a_3 + a_5| = \left| K\left(p_4 - \frac{L}{K}p_1p_3\right) + Me^{-2i\delta}\left(p_2 - \frac{2N}{M}\frac{p_1^2}{2}\right) - Rp_2\left(p_2 - \frac{2P}{R}\frac{p_1^2}{2}\right) + Qp_1^4 + \frac{e^{-4i\delta}\gamma^4}{\phi_5}\right|$$

so that

$$|3a_2^2a_3 + a_5| = |K|\left|p_4 - \frac{L}{K}p_1p_3\right| + |Me^{-2i\delta}|\left|p_2 - \frac{2N}{M}\frac{p_1^2}{2}\right| + |Rp_2|\left|p_2 - \frac{2P}{R}\frac{p_1^2}{2}\right| + |Qp_1^4| + \left|\frac{e^{-4i\delta}\gamma^4}{\phi_5}\right|$$

and the appropriate application of Lemmas 3.4, 3.1 and 3.3 yields

$$|3a_2^2a_3 + a_5| \leq 4L - 2K + 4N - 2M + 8P - 4R + 16Q + \frac{\gamma^4}{\phi_5}. \tag{4.34}$$

Now substituting (4.1), (4.2), (4.3), (4.24), (4.25) and (4.34) into (4.33) yields (4.30). ■

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The outer-independent edge-vertex domination in trees

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Abstract. Let $G = (V, E)$ be a finite simple graph. A vertex $v \in V$ is edge-vertex dominated by an edge $e \in E$ if e is incident with v or e is incident with a vertex adjacent to v. An edge-vertex dominating set of G is a subset $D \subseteq E$ such that every vertex of G is edge-vertex dominated by an edge of D. A subset $D \subseteq E$ is called an *outer-independent edge-vertex dominating set* of G if D is an edge-vertex dominating set of G and the set $V(G) \setminus I(D)$ is independent, where $I(D)$ is the set of vertices incident to an edge of D. The *outer-independent edge-vertex domination number* of G, denoted by $\gamma_{ev}^{oi}(G)$, is the smallest cardinality of an outer-connected edge-vertex dominating set of G. In this paper, we study outer-independent edge-vertex domination numbers. In particular, we prove that $\frac{n-l+1}{3} \leq \gamma_{ev}^{oi}(T) \leq \frac{2n-s-2}{3}$ for every tree T of order $n \geq 3$ with l leaves and s support vertices. We also characterize the trees attaining the bounds.

AMS Subject Classifications: 05C69.

Keywords: edge-vertex dominating set, outer independent edge-vertex dominating set.

Contents

1. Introduction and Terminology

Let $G = (V, E)$ be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. The cardinality of V is called the *order* of G. The set $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ is called the *open neighborhood* of $v \in V(G)$. The *degree* of $v \in V(G)$ is the cardinality of $N(v)$. We denote it by $deg_G(v)$. The *distance* between two distinct vertices in G is the length of a shortest path between them. The *diameter* of G is denoted by $diam(G)$. A *diametral path* of G is a path with the length which equals $diam(G)$.

Let T be a tree. A vertex v of T is called *leaf* if $deg_T(v) = 1$. A *support vertex* is a vertex adjacent to a leaf. A *weak support vertex* is a support vertex that is adjacent to exactly one leaf. A *rooted tree* T differentiates one vertex r called the root. For a vertex $v(\neq r) \in V(T)$, the parent of v is the neighbor of v placed on the unique (r, v)-path, while a child of v is any other neighbor of v. We denote the set of children of v by $C(v)$. A descendant of v is a vertex $w \neq v$ such that v is contained in the unique (r, w)-path. In particular, every child of v is also a descendant of v. We denote the set of descendants of v by $D(v)$. The subtree induced by $D(v) \cup \{v\}$ is denoted by T_v. The *star* is a complete bipartite graph $K_{1,t}$. The *double star* is the graph obtained by joining the centers of two stars $K_{1,p}$ and $K_{1,q}$. Subdividing an edge e is to delete e, add a new vertex x, and join x to the ends of e. A *healthy spider* $S_{t,t}$ is the graph obtained from a star $K_{1,t}$ by subdividing each edges of $K_{1,t}$. For a subset $S \subseteq V(G)$, $G - S$ denotes the subgraph of G induced by $V(G) \setminus S$.

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A vertex $v \in V(G)$ is edge-vertex dominated by an edge $e \in E(G)$ if e is incident with v or e is incident with a vertex adjacent to v (See [2]). An edge-vertex dominating set of G is a subset $D \subseteq E(G)$ such that every vertex of G is edge-vertex dominated by an edge of D (See [2]). A subset $D \subseteq E(G)$ is called an *outer-independent edge-vertex dominating set* (OIEVDS) of G if D is an edge-vertex dominating set of G and the set $V(G) \setminus I(D)$ is independent, where $I(D)$ is the set of vertices incident to an edge of D. The *outer-independent edge-vertex domination number* of G, denoted by $\gamma_{ev}^{oi}(G)$, is the smallest cardinality of an outer-connected edge-vertex dominating set of G. A $\gamma_{ev}^{oi}(G)$-set is an OIEVDS of G with the cardinality $\gamma_{ev}^{oi}(G)$.

Edge-vertex domination in graphs was introduced and studied in [2, 4]. Recently, variations of outer-independent and edge-vertex domination were given in [1, 5, 6]. In this paper, we study outer-independent edge-vertex domination numbers. We prove that $\frac{n-l+1}{3} \leq \gamma_{ev}^{oi}(T) \leq \frac{2n-s-2}{3}$ for every tree T of order $n \geq 3$ with l leaves and s support vertices. We also characterize the trees attaining the bounds.

Finally, we give a lemma whose proof follows from straightforward observation.

Lemma 1.1. *The following holds.*

1. *Every support vertex of T is incident to an edge of every $\gamma_{ev}^{oi}(T)$-set.*
2. *For every tree T with diameter at least three, there exists a $\gamma_{ev}^{oi}(T)$-set whose elements are not incident to any leaf.*

2. Main Result 1

In this section, we prove that if T is a tree of order $n \geq 3$ with l leaves, then $\frac{n-l+1}{3} \leq \gamma_{ev}^{oi}(T)$. We also give a characterization of all trees with $\frac{n-l+1}{3} = \gamma_{ev}^{oi}(T)$.

First of all, we introduce a family $\mathcal{T}$ of trees that be obtained from $T_1, \ldots, T_m$ $(m \geq 1)$ of trees such that T_1 is a path P_4 with two support vertices u, v, and let $S(T_1) = \{uv\}$. If $m \geq 2$, then T_{i+1} be obtained recursively from T_i by one of the following two operations for $1 \leq i \leq m-1$.

Operation $\mathcal{O}_1$:
(i) Attach a vertex by joining it to a vertex incident to edges of $S(T_i)$.
(ii) Set $S(T_{i+1}) = S(T_i)$.

Operation $\mathcal{O}_2$:
(i) Attach a path $P_3 := uvw$ by joining u to a leaf of T_i.
(ii) Set $S(T_{i+1}) = S(T_i) \cup \{uv\}$.

Proposition 2.1. *If a tree T belongs to $\mathcal{T}$, then $\gamma_{ev}^{oi}(T) = \frac{n-l+1}{3}$.*

Proof. We use the induction on the number of operations performed to construct the tree T. If $T = T_1 \cong P_4$, then $\gamma_{ev}^{oi}(T) = 1$. Let m be a positive integer. Suppose that every tree T' constructed by $m-1$ operations satisfies $\gamma_{ev}^{oi}(T') = \frac{n'-l'+1}{3}$. Let $T = T_{m+1}$ be a tree constructed by m operations.

First, we assume that T is obtained from T' by Operation $\mathcal{O}_1$. Then $n = n' + 1$ and $l = l' + 1$. It is easy to see that $S(T') = S(T)$ is an OIEVDS of T. Thus, $\gamma_{ev}^{oi}(T) = \gamma_{ev}^{oi}(T') = \frac{n'-l'+1}{3} = \frac{n-1-(l-1)+1}{3} = \frac{n-l+1}{3}$.

Second, we assume that T is obtained from T' by Operation $\mathcal{O}_2$. Then $n = n' + 3$ and $l = l'$. It is easy to see that $S(T) = S(T') \cup \{uv\}$ is an OIEVDS of T and $\gamma_{ev}^{oi}(T) = \gamma_{ev}^{oi}(T') + 1$ Thus, $\gamma_{ev}^{oi}(T) = \gamma_{ev}^{oi}(T') + 1 = \frac{n'-l'+1}{3} + 1 = \frac{n-3-l+1}{3} + 1 = \frac{n-l+1}{3}$. ■

Theorem 2.2. *Let T be a tree of order $n \geq 3$ with l leaves. Then $\frac{n-l+1}{3} \leq \gamma_{ev}^{oi}(T)$ with equality if and only if $T \in \mathcal{T}$.*

Proof. If $T = P_3$, then clearly $\frac{n-l+1}{3} = \frac{2}{3} < \gamma_{ev}^{oi}(T) = 1$. Assume that the order of T is at least 4. If T is a star, then $\frac{n-l+1}{3} = \frac{2}{3} < \gamma_{ev}^{oi}(T) = 1$. If T is a double star, then $\frac{n-l+1}{3} = 1 = \gamma_{ev}^{oi}(T)$. By using Operation $\mathcal{O}_1$ repeatably, we have $T \in \mathcal{T}$.

Now assume that $diam(T) \geq 4$. We use the induction on the order of T. Suppose that every tree T' of order $n'(< n)$ satisfies $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$ with equality only if $T' \in \mathcal{T}$.

Among all of diametrical paths in T, we choose $x_0x_1 \ldots x_d$ so that it maximizes $deg_T(x_{d-1})$. Root T at x_0. We divide our consideration into four cases.

Case 1. $deg_T(x_{d-1}) \geq 3$.

Let u ($\neq x_d$) be a leaf adjacent to x_{d-1}. Let $T' = T - \{u\}$. Then $n = n' + 1$ and $l = l' + 1$. It is easy to see that any $\gamma_{ev}^{oi}(T')$-set D is an OIEVDS of T. Applying the induction hypothesis to T', we have $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$. Thus, $\frac{n-l+1}{3} = \frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T)$. If $\frac{n-l+1}{3} = \gamma_{ev}^{oi}(T)$, then $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$ and $T' \in \mathcal{T}$. By Operation $\mathcal{O}_1$, we have $T \in \mathcal{T}$.

Case 2. $deg_T(x_{d-1}) = 2$ and $deg_T(x_{d-2}) \geq 3$.

Assume that there exists a support vertex $v \in C(x_{d-2}) \setminus \{x_{d-1}\}$. Let $T' = T - T_v$. Then $n = n' + 2$ and $l = l' + 1$. It is easy to see that $\gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T) - 1$. Applying the induction hypothesis to T', we have $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$. Thus, $\frac{n-2-(l-1)+1}{3} = \frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T) - 1$ and so $\frac{n-l+1}{3} < \gamma_{ev}^{oi}(T)$.

Assume that there exists a leaf $u \in C(x_{d-2})$. Let $T' = T - \{u\}$. Then $n = n' + 1$ and $l = l' + 1$. It is easy to see that $\gamma_{ev}^{oi}(T') = \gamma_{ev}^{oi}(T)$. Applying the induction hypothesis to T', we have $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$. Thus, $\frac{n-1-(l-1)+1}{3} = \frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T)$ and so $\frac{n-l+1}{3} \leq \gamma_{ev}^{oi}(T)$. If $\frac{n-l+1}{3} = \gamma_{ev}^{oi}(T)$, then $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$ and $T' \in \mathcal{T}$. By Operation $\mathcal{O}_1$, we have $T \in \mathcal{T}$.

Case 3. $deg_T(x_{d-1}) = 2$, $deg_T(x_{d-2}) = 2$ and $deg_T(x_{d-3}) \geq 3$.

Let $T' = T - T_{x_{d-2}}$. Then $n = n' + 3$ and $l = l' + 1$. It is easy to see that $\gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T) - 1$. Applying the induction hypothesis to T', we have $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$. Thus, $\frac{n-3-(l-1)+1}{3} = \frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T) - 1$ and so $\frac{n-l+1}{3} < \gamma_{ev}^{oi}(T)$.

Case 4. $deg_T(x_{d-1}) = 2$, $deg_T(x_{d-2}) = 2$ and $deg_T(x_{d-3}) = 2$.

Let $T' = T - T_{x_{d-2}}$. Then $n = n' + 3$ and $l = l'$. It is easy to see that $\gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T) - 1$. Applying the induction hypothesis to T', we have $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$. Thus, $\frac{n-3-l+1}{3} = \frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T) - 1$ and so $\frac{n-l+1}{3} \leq \gamma_{ev}^{oi}(T)$. If $\frac{n-l+1}{3} = \gamma_{ev}^{oi}(T)$, then $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$ and $T' \in \mathcal{T}$. By Operation $\mathcal{O}_2$, we have $T \in \mathcal{T}$. ■

3. Main Result 2

In this section, we prove that if T is a tree of order $n \geq 3$ with s support vertices, then $\gamma_{ev}^{oi}(T) \leq \frac{2n-s-2}{3}$. We also give a characterization of all trees with $\gamma_{ev}^{oi}(T) = \frac{2n-s-2}{3}$.

Theorem 3.1. *Let T be a tree of order $n \geq 3$ with s support vertices. Then $\gamma_{ev}^{oi}(T) \leq \frac{2n-s-2}{3}$ with equality if and only if T is a healthy spider.*

Proof. If $T = P_3$, then clearly $\gamma_{ev}^{oi}(T) = 1$ and T is a healthy spider. Assume that the order of T is at least 4. If T is a star, then $\gamma_{ev}^{oi}(T) = 1 < \frac{2n-s-2}{3}$. If T is a double star, then $\gamma_{ev}^{oi}(T) = 1 < \frac{2n-s-2}{3} = \frac{2n-4}{3}$.

Now assume that $diam(T) \geq 4$. We use the induction on the order of T. Suppose that every tree T' of order $n'(< n)$ satisfies $\gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3}$ with equality only if T' is a healthy spider. Among all of diametrical paths in T, we choose $x_0x_1 \ldots x_d$ so that it maximizes $deg_T(x_{d-1})$. Root T at x_0. We divide our consideration into three cases.

Case 1. $deg_T(x_{d-1}) \geq 3$.

Let u ($\neq x_d$) be a leaf adjacent to x_{d-1}. Let $T' = T - \{u\}$. Then $n = n' + 1$ and $s = s'$. It is easy to see that any $\gamma_{ev}^{oi}(T')$-set is an OIEVDS of T. So, $\gamma_{ev}^{oi}(T) \leq \gamma_{ev}^{oi}(T')$. Applying the induction hypothesis to T', we have $\gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3}$. Thus, $\gamma_{ev}^{oi}(T) \leq \gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3} < \frac{2n-s-2}{3}$.

Case 2. $deg_T(x_{d-1}) = 2$ and $deg_T(x_{d-2}) = 2$.

Let $T' = T - \{x_{d-2}, x_{d-1}, x_d\}$. It is easy to see that $\gamma_{ev}^{oi}(T) = \gamma_{ev}^{oi}(T') + 1$, $s - 1 \leq s' \leq s$ and $n = n' + 3$. Applying the induction hypothesis to T', we have $\gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3}$. Thus, $\gamma_{ev}^{oi}(T) - 1 = \gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3} \leq \frac{2n-6-s+1-2}{3}$ and so $\gamma_{ev}^{oi}(T) < \frac{2n-s-2}{3}$.

Case 3. $deg_T(x_{d-1}) = 2$ and $deg_T(x_{d-2}) \geq 3$.

Assume that there exists a leaf $c \in C(x_{d-2})$. Let $T' = T - \{v\}$. By the argument as in Case 1, we have $\gamma_{ev}^{oi}(T) < \frac{2n-s-2}{3}$.

Assume that there exists a support vertex $c \in C(x_{d-2}) \setminus \{x_{d-1}\}$. By the assumption, c is weak and has a child w. Let $T' = T - T_c$. It is easy to see that $\gamma_{ev}^{oi}(T) = \gamma_{ev}^{oi}(T') + 1$, $s = s' + 1$ and $n = n' + 2$. Applying the induction hypothesis to T', we have $\gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3}$. Thus, $\gamma_{ev}^{oi}(T) - 1 = \gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3} = \frac{2n-4-s+1-2}{3}$ and so $\gamma_{ev}^{oi}(T) \leq \frac{2n-s-2}{3}$.

Now we assume that $T_{x_{d-2}}$ is a healthy spider $S_{t,t}$. Let $T' = T - V(T_{x_{d-2}})$. It is easy to see that $\gamma_{ev}^{oi}(T) \leq \gamma_{ev}^{oi}(T') + t$, $s - t \leq s'$ and $n = n' + 2t + 1$. If $|V(T')| \geq 3$, then by the induction hypothesis on T', we have $\gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3}$. Thus, $\gamma_{ev}^{oi}(T) - t \leq \gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3} = \frac{2n-4t-2-s+t-2}{3}$ and so $\gamma_{ev}^{oi}(T) < \frac{2n-s-2}{3}$. If $|V(T')| = 2$, then clearly $\gamma_{ev}^{oi}(T) = \frac{2n-s-2}{3}$ and T is a healthy spider. ■

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