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Approximation of time separating stochastic processes by neural networks revisited

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Abstract. Here we study the univariate quantitative approximation of time separating stochastic process over the whole real line by the normalized bell and squashing type neural network operators. Activation functions here are of compact support. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged stochastic function or its high order derivative. The approximations are pointwise and with respect to the Lp norm. The feed-forward neural networks are with one hidden layer. We finish with a great variety of special applications.

AMS Subject Classifications: 40A05, 40A99, 46A70, 46A99.

Keywords: Time separating stochastic process, neural network approximation, modulus of continuity, activation functions of compact support, squashing functions.

Contents

1. Introduction

The first author in [2] and [3], was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" activation functions are assumed to be of compact support. The functions under approximation were from the whole $\mathbb{R}$ into $\mathbb{R}$. Here we perform quantitative approximations of time separating stochastic processes by these neural network operators. We follow the above-described pattern

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and produce pointwise and L_p quantitative estimates. This article is a continuation of [4], where the activation functions had been over the whole real line.
We give several interesting applications. Specific motivations came by:

1. Stationary Gaussian processes with an explicit representation such as

$$X_t = \cos(\alpha t)\,\xi_1 + \sin(\alpha t)\,\xi_2, \alpha \in \mathbb{R},$$

where ξ_1, ξ_2 are independent random variables with the standard normal distribution, see [6].

2. By the "Fourier model" of a stationary process, see [7].

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x\rangle + b_j\right),\ x \in \mathbb{R}^s, s \in \mathbb{N},$$

where for $0 \le j \le n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x\rangle$ is the inner product of a_j and x, and σ is the activation function of the network.

2. About Neural Networks Approximation

In this section we follow [3].

Definition 2.1. *(see [5]) A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero. The function $b(x)$ may have jump discontinuities. In this work we consider only centered bell-shaped functions of compact support $[-T, T]$, $T > 0$.*

Example 2.2. (1) *$b(x)$ can be the characteristic function over $[-1, 1]$.*
(2) *$b(x)$ can be the hat function over $[-1, 1]$, i.e.,*

$$b(x) = \begin{cases} 1 + x, & -1 \le x \le 0, \\ 1 - x, & 0 < x \le 1 \\ 0, & \textit{elsewhere.} \end{cases}$$

Here we consider functions $f : \mathbb{R} \to \mathbb{R}$ that are either continuous and bounded, or uniformly continuous.

In the article we follow we study the pointwise convergence with rates over the real line, to the unit operator, of the "normalized bell type neural network operators",

$$(H_n(f))(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}, \tag{1}$$

where $0 < \alpha < 1$ and $x \in \mathbb{R}$, $n \in \mathbb{N}$. The terms in the ratio of sums (1) can be nonzero iff

$$\left| n^{1-\alpha}\left(x - \frac{k}{n}\right)\right| \le T, \text{ i.e. } \left|x - \frac{k}{n}\right| \le \frac{T}{n^{1-\alpha}}$$

iff

$$nx - Tn^{\alpha} \le k \le nx + Tn^{\alpha}. \tag{2}$$

In order to have the desired order of numbers

$$-n^2 \leq nx - Tn^{\alpha} \leq nx + Tn^{\alpha} \leq n^2, \tag{3}$$

it is sufficient enough to assume that

$$n \geq T + |x|. \tag{4}$$

When $x \in [-T, T]$ it is enough to assume $n \geq 2T$ which implies (3).

Proposition 2.3. *Let $a \leq b$, $a, b \in \mathbb{R}$. Let $card(k)$ (≥ 0) be the maximum number of integers contained in $[a, b]$. Then*

$$\max(0, (b-a)-1) \leq card(k) \leq (b-a)+1.$$

Note 2.4. *We would like to establish a lower bound on $card(k)$ over the interval $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$. From Proposition 2.3 we get that*

$$card(k) \geq \max(2Tn^{\alpha} - 1, 0).$$

We obtain $card(k) \geq 1$, if

$$2Tn^{\alpha} - 1 \geq 1 \text{ iff } n \geq T^{-\frac{1}{\alpha}}.$$

So to have the desired order (3) and $card(k) \geq 1$ over $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$, we need to consider

$$n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right). \tag{5}$$

Also notice that $card(k) \to +\infty$, as $n \to +\infty$.

Denote by $[\cdot]$ the integral part of a number and by $\lceil \cdot \rceil$ its ceiling. Here comes the first result we use.

Theorem 2.5. *([3], Ch.1) Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then*

$$|(H_n(f))(x) - f(x)| \leq \omega_1\left(f, \frac{T}{n^{1-\alpha}}\right), \tag{6}$$

where ω_1 is the first modulus of continuity of f.

The second result we use follows.

Theorem 2.6. *([3], Ch.1) Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Let $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, such that $f^{(N)}$ is a uniformly continuous function or $f^{(N)}$ is continuous and bounded. Then*

$$|(H_n(f))(x) - f(x)| \leq \left(\sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right| T^j}{n^{j(1-\alpha)} j!}\right) + \tag{7}$$

$$\omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

Notice that as $n \to \infty$ we have that R.H.S.(7)$\to 0$, therefore L.H.S.(7)$\to 0$, i.e., (7) gives us with rates the pointwise convergence of $(H_n(f))(x) \to f(x)$, as $n \to +\infty$, $x \in \mathbb{R}$.

Corollary 2.7. *([3], Ch.1) Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T, T]$. Let $x \in [-T^*, T^*]$, $T^* > 0$, and $n \in \mathbb{N}$ be such that $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$. Consider $p \geq 1$. Then*

$$\|H_n(f) - f\|_{p,[-T^*,T^*]} \leq \omega_1\left(f, \frac{T}{n^{1-\alpha}}\right) \cdot 2^{\frac{1}{p}} \cdot T^{*\frac{1}{p}}. \tag{8}$$

From (8) we get the L_p convergence of $H_n(f)$ to f with rates.

Corollary 2.8. *([3], Ch.1) Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T,T]$. Let $x \in [-T^*, T^*]$, $T^* > 0$, and $n \in \mathbb{N}$ be such that $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$. Consider $p \geq 1$. Then*

$$\|H_n(f) - f\|_{p,[-T^*,T^*]} \leq \tag{9}$$

$$\left(\sum_{j=1}^{N} \frac{T^j \cdot \left\|f^{(j)}\right\|_{p,[-T^*,T^*]}}{n^{j(1-\alpha)} j!}\right) + \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \frac{2^{\frac{1}{p}} T^N T^{*\frac{1}{p}}}{N! n^{N(1-\alpha)}},$$

where $N \geq 1$.

Here from (9) we get again the L_p convergence of $H_n(f)$ to f with rates.

2.1. The "Normalized Squashing Type Operators" and their Convergence to the Unit with Rates

We need

Definition 2.9. *Let the nonnegative function $S : \mathbb{R} \to \mathbb{R}$, S has compact support $[-T,T]$, $T > 0$, and is nondecreasing there and it can be continuous only on either $(-\infty, T]$ or $[-T,T]$. S can have jump discontinuities. We call S the "squashing function" (see also [5]).*

Let $f : \mathbb{R} \to \mathbb{R}$ be either uniformly continuous or continuous and bounded.
For $x \in \mathbb{R}$ we define the "normalized squashing type operator"

$$(K_n(f))(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right) \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)}, \tag{10}$$

$0 < \alpha < 1$ and $n \in \mathbb{N} : n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. It is clear that

$$(K_n(f))(x) = \frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} f\left(\frac{k}{n}\right) \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)}{W(x)}, \tag{11}$$

where

$$W(x) := \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right).$$

Here we give the pointwise convergence with rates of $(K_n f)(x) \to f(x)$, as $n \to +\infty$, $x \in \mathbb{R}$.

Theorem 2.10. *([3], Ch.1) Under the above terms and assumptions we obtain*

$$|K_n(f)(x) - f(x)| \leq \omega_1\left(f, \frac{T}{n^{1-\alpha}}\right). \tag{12}$$

We also give

Theorem 2.11. *([3], Ch.1) Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Let $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, such that $f^{(N)}$ is a uniformly continuous function or $f^{(N)}$ is continuous and bounded. Then*

$$|(K_n(f))(x) - f(x)| \leq \left(\sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right| T^j}{j! n^{j(1-\alpha)}}\right) + \tag{13}$$

$$\omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

So we obtain the pointwise convergence of $K_n(f)$ to f with rates.

Note 2.12. *The maps H_n, K_n are positive linear operators reproducing constants, in particular*

$$H_n(1) = K_n(1) = 1. \tag{14}$$

3. Time Seperating Stochastic Processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\omega \in \Omega; Y_1, Y_2, \ldots, Y_m, m \in \mathbb{N}$, be real-valued random variables on Ω with finite expectations, and $h_1(t), h_2(t), \ldots h_m(t) : \mathbb{R} \to \mathbb{R}$, such that $h_i(t)$, $i = 1, 2, \ldots \ldots m$ are all uniformly continuous or $h_i(t)$ $i = 1, 2, \ldots \ldots m$ are all continuous and bounded for every $i = 1, 2, \ldots \ldots m$.
Clearly, then

$$Y(t, \omega) := \sum_{i=1}^{m} h_i(t) Y_i(\omega), t \in \mathbb{R}, \tag{15}$$

is a quite common stochastic process separating time.
We can assume that $h_i \in C^r(\mathbb{R}), i = 1, 2, ..., m; r \in \mathbb{N}$. Consequently, we have that the expectation

$$(EY)(t) = \sum_{i=1}^{m} h_i(t) EY_i \in C(\mathbb{R}) \text{ or } C^r(\mathbb{R}). \tag{16}$$

A classical example of a stochastic process separating time is

$$(\sin t) Y_1(\omega) + (\cos t) Y_2(\omega), t \in \mathbb{R}.$$

Notice that $|\sin t| \leq 1$ and $|\cos t| \leq 1$.
Another typical example is

$$\sinh(t) Y_1(\omega) + \cosh(t) Y_2(\omega), t \in \mathbb{R}. \tag{17}$$

In this article we will apply the results of section 2, to $f(t) = (EY)(t)$. We will finish with several applications. See the related [6], [7].

4. Main Results

We present the following general approximation of the seperating stochastic processes by neural network operators.

Theorem 4.1. *Let $(EY)(t)$ as in (16), Let also $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then*

$$|(H_n(E(Y)))(t) - (E(Y))(t)| \leq \omega_1\left((E(Y)), \frac{T}{n^{1-\alpha}}\right), \tag{18}$$

where ω_1 is the first modulus of continuity of $E(Y)$.

Proof. $E(Y)$ are uniformly continuous or continuous and bounded in $\mathbb{R}$, Thus, the conclusion comes from Theorem 2.5. ∎

Our second main result follows.

Theorem 4.2. *Let $(EY)(t)$ as in (16), Let also $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then*

$$|(H_n(E(Y)))(t) - (E(Y))(t)| \leq \left(\sum_{j=1}^{N} \frac{\left|(E(Y))^{(j)}(t)\right| T^j}{n^{j(1-\alpha)} j!}\right) + \tag{19}$$

$$\omega_1\left((E(Y))^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

Notice that as $n \to \infty$ we have that R.H.S.(19)$\to 0$, therefore L.H.S.(19)$\to 0$, i.e., (19) gives us with rates the pointwise convergence of $(H_n(E(Y)))(t) \to (E(Y))(t)$, as $n \to +\infty$, $x \in \mathbb{R}$.

Proof. Notice that Let $(E(Y)) \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, such that $(E(Y))^{(N)}$ is a uniformly continuous function or $(E(Y))^{(N)}$ is continuous and bounded. Thus, the conclusion comes from Theorem 2.5. ■

We continue with,

Corollary 4.3. *Let $(EY)(t)$ as in (16). Let also $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T, T]$. Let $t \in [-T^*, T^*]$, $T^* > 0$, and $n \in \mathbb{N}$ be such that $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$. Consider $p \geq 1$. Then*

$$\|H_n(E(Y)) - E(Y)\|_{p,[-T^*,T^*]} \leq \omega_1\left(E(Y)\, \frac{T}{n^{1-\alpha}}\right) \cdot 2^{\frac{1}{p}} \cdot T^{*\frac{1}{p}}. \tag{20}$$

From (20) we get the L_p convergence of $H_n(E(Y))$ to $E(Y)$ with rates.

Corollary 4.4. *Let $(EY)(t)$ as in (16). Let also $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T, T]$. Let $t \in [-T^*, T^*]$, $T^* > 0$, and $n \in \mathbb{N}$ be such that $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$. Consider $p \geq 1$. Then*

$$\|H_n(E(Y)) - E(Y)\|_{p,[-T^*,T^*]} \leq \tag{21}$$

$$\left(\sum_{j=1}^{N} \frac{T^j \cdot \left\|(E(Y))^{(j)}\right\|_{p,[-T^*,T^*]}}{n^{j(1-\alpha)} j!}\right) + \omega_1\left((E(Y))^{(N)}, \frac{T}{n^{1-\alpha}}\right) \frac{2^{\frac{1}{p}} T^N T^{*\frac{1}{p}}}{N! n^{N(1-\alpha)}},$$

where $N \geq 1$.

We also give the next

Theorem 4.5. *Let $t \in \mathbb{R}$ and $(EY)(t)$ as in (16). Under the terms and assumptions of Definition 2.9 and the "normalized squashing type operator" as defined in (10). We obtain*

$$|K_n(EY)(t) - (EY)(t)| \leq \omega_1\left(EY, \frac{T}{n^{1-\alpha}}\right). \tag{22}$$

Proof. From Theorem 2.10. ■

We also give

Theorem 4.6. *Let $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Let also $(EY)(t)$ as in (16). Then*

$$|(K_n(EY))(t) - (EY)(t)| \leq \left(\sum_{j=1}^{N} \frac{\left|(EY)^{(j)}(t)\right| T^j}{j! n^{j(1-\alpha)}}\right) + \tag{23}$$

$$\omega_1\left((EY)^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

So we obtain the pointwise convergence of $K_n(EY)$ to (EY) with rates.

Proof. $(EY) \in C^N(\mathbb{R})$. Further more $(EY)^{(N)}$ is a uniformly continuous function or $(EY)^{(N)}$ is continuous and bounded. Hence the conclusion comes from Theorem 2.11. ■

5. Applications

For the next applications we consider (Ω, F, P) be a probability space and Y_1, Y_2 be real valued random variables on Ω with finite expectations. We consider the stochastic processes $Z_i(t, \omega)$ for $i = 1, 2, 3, 4$ where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$Z_1(t, \omega) = \sin(\xi t) Y_1(\omega) + \cos(\xi t) Y_2(\omega), \tag{24}$$

where $\xi > 0$ is fixed;

$$Z_2(t, \omega) = sech(\mu t) Y_1(\omega) + \tanh(\mu t) Y_2(\omega), \tag{25}$$

where $\mu > 0$ is fixed.
Here $sechx := \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, x \in \mathbb{R}$.

$$Z_3(t, \omega) = \frac{1}{1 + e^{-\ell_1 t}} Y_1(\omega) + \frac{1}{1 + e^{-\ell_2 t}} Y_2(\omega), \tag{26}$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$Z_4(t, \omega) = e^{-e^{-\mu_1 t}} Y_1(\omega) + e^{-e^{-\mu_2 t}} Y_2(\omega), \tag{27}$$

where $\mu_1, \mu_2 > 0$ are fixed;

The expectations of $Z_i, i = 1, 2, 3, 4$ are

$$(EZ_1)(t) = \sin(\xi t) E(Y_1) + \cos(\xi t) E(Y_2), \tag{28}$$

$$(EZ_2)(t) = sech(\mu t) E(Y_1) + \tanh(\mu t) E(Y_2), \tag{29}$$

$$(EZ_3)(t) = \frac{1}{1 + e^{-\ell_1 t}} E(Y_1) + \frac{1}{1 + e^{-\ell_2 t}} E(Y_2), \tag{30}$$

$$(EZ_4)(t) = e^{-e^{-\mu_1 t}} E(Y_1) + e^{-e^{-\mu_2 t}} E(Y_2). \tag{31}$$

For the next $(EZ_i)(t), i = 1, 2, 3, 4$ are as defined in relations between (28) and (31) respectively.
We present the following result.

Proposition 5.1. *Let $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then for $i = 1, 2, 3, 4$*

$$|(H_n(EZ_i))(t) - (EZ_i)(t)| \leq \omega_1\left((EZ_i), \frac{T}{n^{1-\alpha}}\right), \tag{32}$$

where ω_1 is the first modulus of continuity of (EZ_i).

Proof. From Theorem 4.1. ■

We also give

Proposition 5.2. *Let $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then for $i = 1, 2, 3, 4$*

$$|(H_n(EZ_i))(t) - (EZ_i)(t)| \leq \left(\sum_{j=1}^{N} \frac{\left|(EZ_i)^{(j)}(t)\right| T^j}{n^{j(1-\alpha)} j!}\right) + \tag{33}$$

$$\omega_1\left((EZ_i)^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

Notice that as $n \to \infty$ we have that R.H.S.(33)$\to 0$, therefore L.H.S.(33)$\to 0$, i.e., (33) gives us with rates the pointwise convergence of $(H_n(EZ_i))(t) \to (EZ_i)(t)$, as $n \to +\infty$, $x \in \mathbb{R}$.

Proof. From Theorem 4.2. ■

We continue with,

Corollary 5.3. *Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T, T]$. Let $t \in [-T^*, T^*]$, $T^* > 0$, and $n \in \mathbb{N}$ be such that $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$. Consider $p \geq 1$. Then for $i = 1, 2, 3, 4$*

$$\|H_n(EZ_i) - (EZ_i)\|_{p,[-T^*,T^*]} \leq \omega_1\left((EZ_i)\ \frac{T}{n^{1-\alpha}}\right) \cdot 2^{\frac{1}{p}} \cdot T^{*\frac{1}{p}}. \tag{34}$$

From (34) we get the L_p convergence of $H_n((EZ_i))$ to (EZ_i) with rates.

Corollary 5.4. *Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T, T]$. Let $t \in [-T^*, T^*]$, $T^* > 0$, and $n \in \mathbb{N}$ be such that $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$. Consider $p \geq 1$. Then for $i = 1, 2, 3, 4$*

$$\|H_n((EZ_i)) - (EZ_i)\|_{p,[-T^*,T^*]} \leq \tag{35}$$

$$\left(\sum_{j=1}^{N} \frac{T^j \cdot \left\|(EZ_i)^{(j)}\right\|_{p,[-T^*,T^*]}}{n^{j(1-\alpha)} j!}\right) + \omega_1\left((EZ_i)^{(N)}, \frac{T}{n^{1-\alpha}}\right) \frac{2^{\frac{1}{p}} T^N T^{*\frac{1}{p}}}{N! n^{N(1-\alpha)}},$$

where $N \geq 1$.

Proposition 5.5. *Under the terms and assumptions of Definition 2.9 and the "normalized squashing type operator" as defined in (10). Then for $i = 1, 2, 3, 4$ we obtain*

$$|K_n(EZ_i)(t) - (EZ_i)(t)| \leq \omega_1\left((EZ_i), \frac{T}{n^{1-\alpha}}\right). \tag{36}$$

Proof. From Theorem 4.5. ■

We also give

Proposition 5.6. *Let $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then for $i = 1, 2, 3, 4$*

$$|(K_n(EZ_i))(t) - (EZ_i)(t)| \leq \left(\sum_{j=1}^{N} \frac{\left|(EZ_i)^{(j)}(t)\right| T^j}{j! n^{j(1-\alpha)}}\right) + \tag{37}$$

$$\omega_1\left((EZ_i)^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

So we obtain the pointwise convergence of $K_n(EZ_i)$ to (EZ_i) with rates.

Proof. From Theorem 4.6. ■

6. Specific Applications

Let $(\Omega, \mathcal{F}, P)$, where Ω is the set of non-negative integers, be a probability space, $Y_{1,1}, Y_{2,1}$ be real-valued random variables on Ω following Poisson distributions with parameters $\lambda_1, \lambda_2 \in (0, \infty)$ respectively.
We consider the stochastic processes $Z_{i,1}(t,\omega)$ for $i = 1,2,3,4$, where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$Z_{1,1}(t,\omega) = \sin(\xi t)\, Y_{1,1}(\omega) + \cos(\xi t)\, Y_{2,1}(\omega), \tag{38}$$

where $\xi > 0$ is fixed;

$$Z_{2,1}(t,\omega) = sech\,(\mu t)\, Y_{1,1}(\omega) + \tanh(\mu t)\, Y_{2,1}(\omega), \tag{39}$$

where $\mu > 0$ is fixed.

$$Z_{3,1}(t,\omega) = \frac{1}{1+e^{-\ell_1 t}} Y_{1,1}(\omega) + \frac{1}{1+e^{-\ell_2 t}} Y_{2,1}(\omega), \tag{40}$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$Z_{4,1}(t,\omega) = e^{-e^{-\mu_1 t}} Y_{1,1}(\omega) + e^{-e^{-\mu_2 t}} Y_{2,1}(\omega), \tag{41}$$

where $\mu_1, \mu_2 > 0$ are fixed;
Since $E(Y_{1,1}) = \lambda_1$ and $E(Y_{2,1}) = \lambda_2$, the expectations of $Z_{i,1}, i = 1,2,3,4,$ are

$$(EZ_{1,1})(t) = \lambda_1 \sin(\xi t) + \lambda_2 cos\,(\xi t), \tag{42}$$

$$(EZ_{2,1})(t) = \lambda_1 sech\,(\mu t) + \lambda_2 \tanh(\mu t), \tag{43}$$

$$(EZ_{3,1})(t) = \frac{\lambda_1}{1+e^{-\ell_1 t}} + \frac{\lambda_2}{1+e^{-\ell_2 t}}, \tag{44}$$

$$(EZ_{4,1})(t) = \lambda_1 e^{-e^{-\mu_1 t}} + \lambda_2 e^{-e^{-\mu_2 t}}. \tag{45}$$

For the next we consider $(\Omega, \mathcal{F}, P)$, where $\Omega = \mathbb{R}$, be a probability space, $Y_{1,2}, Y_{2,2}$ be real-valued random variables on Ω following Gaussian distributions with expectations $\hat{\mu}_1, \hat{\mu}_2 \in \mathbb{R}$ respectively.
We consider the stochastic processes $Z_{i,2}(t,\omega)$ for $i = 1,2,3,4$, where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$Z_{1,2}(t,\omega) = \sin(\xi t)\, Y_{1,2}(\omega) + \cos(\xi t)\, Y_{2,2}(\omega), \tag{46}$$

where $\xi > 0$ is fixed;

$$Z_{2,2}(t,\omega) = sech\,(\mu t)\, Y_{1,2}(\omega) + \tanh(\mu t)\, Y_{2,2}(\omega), \tag{47}$$

where $\mu > 0$ is fixed.

$$Z_{3,2}(t,\omega) = \frac{1}{1+e^{-\ell_1 t}} Y_{1,2}(\omega) + \frac{1}{1+e^{-\ell_2 t}} Y_{2,2}(\omega), \tag{48}$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$Z_{4,2}(t,\omega) = e^{-e^{-\mu_1 t}} Y_{1,2}(\omega) + e^{-e^{-\mu_2 t}} Y_{2,2}(\omega), \tag{49}$$

where $\mu_1, \mu_2 > 0$ are fixed;
Since $E(Y_{1,2}) = \hat{\mu}_1$ and $E(Y_{2,2}) = \hat{\mu}_2$, The expectations of $Z_{i,2}, i = 1,2,3,5$ are

$$(EZ_{1,2})(t) = \hat{\mu}_1 \sin(\xi t) + \hat{\mu}_2 cos\,(\xi t), \tag{50}$$

$$(EZ_{2,2})(t) = \hat{\mu}_1 sech\,(\mu t) + \hat{\mu}_2 \tanh(\mu t), \tag{51}$$

$$(EZ_{3,2})(t) = \frac{\hat{\mu}_1}{1+e^{-\ell_1 t}} + \frac{\hat{\mu}_2}{1+e^{-\ell_2 t}}, \tag{52}$$

$$(EZ_{4,2})(t) = \hat{\mu}_1 e^{-e^{-\mu_1 t}} + \hat{\mu}_2 e^{-e^{-\mu_2 t}}. \tag{53}$$

Furthermore, we consider $(\Omega, \mathcal{F}, P)$, where $\Omega = [0, \infty)$, be a probability space, $Y_{1,3}, Y_{2,3}$ be real-valued random variables on Ω following Weibull distributions with scale parameters 1 and shape parameters $\gamma_1, \gamma_2 \in (0, \infty)$ respectively.
We consider the stochastic processes $Z_{i,3}(t, \omega)$ for $i = 1, 2, 3, 4$, where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$Z_{1,3}(t, \omega) = \sin(\xi t)\, Y_{1,3}(\omega) + \cos(\xi t)\, Y_{2,3}(\omega), \tag{54}$$

where $\xi > 0$ is fixed;

$$Z_{2,3}(t, \omega) = sech\,(\mu t)\, Y_{1,3}(\omega) + \tanh(\mu t)\, Y_{2,3}(\omega), \tag{55}$$

where $\mu > 0$ is fixed.

$$Z_{3,3}(t, \omega) = \frac{1}{1 + e^{-\ell_1 t}} Y_{1,3}(\omega) + \frac{1}{1 + e^{-\ell_2 t}} Y_{2,3}(\omega), \tag{56}$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$Z_{4,3}(t, \omega) = e^{-e^{-\mu_1 t}} Y_{1,3}(\omega) + e^{-e^{-\mu_2 t}} Y_{2,3}(\omega), \tag{57}$$

where $\mu_1, \mu_2 > 0$ are fixed;
Since $E(Y_{1,3}) = \Gamma\left(1 + \frac{1}{\gamma_1}\right)$ and $E(Y_{2,3}) = \Gamma\left(1 + \frac{1}{\gamma_2}\right)$, where $\Gamma(\cdot)$ is the Gamma function, The expectations of $Z_{i,3}, i = 1, 2, 3, 4$, are

$$(EZ_{1,3})(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right) \sin(\xi t) + \Gamma\left(1 + \frac{1}{\gamma_2}\right) cos\,(\xi t), \tag{58}$$

$$(EZ_{2,3})(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right) sech\,(\mu t) + \Gamma\left(1 + \frac{1}{\gamma_2}\right) \tanh(\mu t), \tag{59}$$

$$(EZ_{3,3})(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right) \frac{1}{1 + e^{-\ell_1 t}} + \Gamma\left(1 + \frac{1}{\gamma_2}\right) \frac{1}{1 + e^{-\ell_2 t}}, \tag{60}$$

$$(EZ_{4,3})(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right) e^{-e^{-\mu_1 t}} + \Gamma\left(1 + \frac{1}{\gamma_2}\right) e^{-e^{-\mu_2 t}}. \tag{61}$$

We present the following result.

Proposition 6.1. *Let $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then for $i = 1, 2, 3, 4$ and $k = 1, 2, 3$*

$$|(H_n(EZ_{i,k}))(t) - (EZ_{i,k})(t)| \leq \omega_1\left((EZ_{i,k}), \frac{T}{n^{1-\alpha}}\right), \tag{62}$$

where ω_1 is the first modulus of continuity of $(EZ_{i,k})$.

Proof. From Proposition 5.1. ■

We also give

Proposition 6.2. *Let $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then for $i = 1, 2, 3, 4$ and $k = 1, 2, 3$*

$$|(H_n(EZ_{i,k}))(t) - (EZ_{i,k})(t)| \leq \left(\sum_{j=1}^{N} \frac{\left|(EZ_{i,k})^{(j)}(t)\right| T^j}{n^{j(1-\alpha)} j!}\right) + \tag{63}$$

$$\omega_1\left((EZ_{i,k})^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

Notice that as $n \to \infty$ we have that R.H.S.(63)$\to 0$, therefore L.H.S.(63)$\to 0$, i.e., (63) gives us with rates the pointwise convergence of $(H_n(EZ_{i,k}))(t) \to (EZ_{i,k})(t)$, as $n \to +\infty$, $x \in \mathbb{R}$.

Proof. From Proposition 5.2. ∎

We continue with,

Corollary 6.3. *Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T, T]$. Let $t \in [-T^*, T^*]$, $T^* > 0$, and $n \in \mathbb{N}$ be such that $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$. Consider $p \geq 1$. Then for $i = 1, 2, 3, 4$ and $k = 1, 2, 3$*

$$\left\| H_n\left(EZ_{i,k}\right) - \left(EZ_{i,k}\right)\right\|_{p,[-T^*,T^*]} \leq \omega_1\left(\left(EZ_{i,k}\right) \frac{T}{n^{1-\alpha}}\right) \cdot 2^{\frac{1}{p}} \cdot T^{*\frac{1}{p}}. \tag{64}$$

From (64) we get the L_p convergence of $H_n\left(\left(EZ_i\right)\right)$ to $\left(EZ_{i,k}\right)$ with rates.

Corollary 6.4. *Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T, T]$. Let $t \in [-T^*, T^*]$, $T^* > 0$, and $n \in \mathbb{N}$ be such that $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$. Consider $p \geq 1$. Then for $i = 1, 2, 3, 4$ and $k = 1, 2, 3$*

$$\left\| H_n\left(\left(EZ_{i,k}\right)\right) - \left(EZ_{i,k}\right)\right\|_{p,[-T^*,T^*]} \leq \tag{65}$$

$$\left(\sum_{j=1}^{N} \frac{T^j \cdot \left\|\left(EZ_{i,k}\right)^{(j)}\right\|_{p,[-T^*,T^*]}}{n^{j(1-\alpha)} j!}\right) + \omega_1\left(\left(EZ_{i,k}\right)^{(N)}, \frac{T}{n^{1-\alpha}}\right) \frac{2^{\frac{1}{p}} T^N T^{*\frac{1}{p}}}{N! n^{N(1-\alpha)}},$$

where $N \geq 1$.

Proposition 6.5. *Under the terms and assumptions of Definition 2.9 and the "normalized squashing type operator" as defined in (10), for $i = 1, 2, 3, 4$ and $k = 1, 2, 3$ we obtain*

$$\left|K_n\left(EZ_{i,k}\right)(t) - \left(EZ_{i,k}\right)(t)\right| \leq \omega_1\left(\left(EZ_{i,k}\right), \frac{T}{n^{1-\alpha}}\right). \tag{66}$$

Proof. From Proposition 5.5. ∎

We also give

Proposition 6.6. *Let $t \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then for $i = 1, 2, 3, 4$ and $k = 1, 2, 3$*

$$\left|\left(K_n\left(EZ_{i,k}\right)\right)(t) - \left(EZ_{i,k}\right)(t)\right| \leq \left(\sum_{j=1}^{N} \frac{\left|\left(EZ_{i,k}\right)^{(j)}(t)\right| T^j}{j! n^{j(1-\alpha)}}\right) + \tag{67}$$

$$\omega_1\left(\left(EZ_{i,k}\right)^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

So we obtain the pointwise convergence of $K_n\left(EZ_{i,k}\right)$ to $\left(EZ_{i,k}\right)$ with rates.

Proof. From Proposition 5.6. ∎

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On existence of extremal integrable solutions and integral inequalities for nonlinear Volterra type integral equations

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Abstract. We prove the existence of maximal and minimal integrable solutions of nonlinear Volterra type integral equations. Two basic integral inequalities are obtained in the form of extremal integrable solutions which are further exploited for proving the boundedness and uniqueness of the integrable solutions of the considered integral equation.
AMS Subject Classifications: 47H10, 35A35.

Keywords: Volterra integral equation; Tarski fixed point principle; Extremal integrable solutions; integral inequalities.

Contents

1. Introduction

The integral inequalities is an important topic discussed in the theory of differential and integral equations because they have nice applications for proving the boundedness and uniqueness of the solutions of such nonlinear equations. There exists a good amount of literature on integral inequalities and applications, see for example, Lakshmikantham and Leela [14] and references therein. The existence of maximal and minimal solutions of the integral equations play a significant role in the theory of integral inequalities and applications related to the integral equations. The most of the integral inequalities are about the continuous solutions of the continuous integral equations, but the study for integrable solutions for discontinuous integral equations is very rare. Therefore, it is interesting to prove some basic integral inequalities related to Volterra integral equations involving integrable solutions which is the main motivation of the present paper. The present paper deals with the extremal integrable solutions and integral inequalities involving integrable solutions related to nonlinear discontinuous Volterra type integral equations.

Given a closed and bounded interval $J = [0, T]$ in the real line $\mathbb{R}$, consider the nonlinear Volterra type integral equation (in short VIE),

$$x(t) = q(t) + \lambda \int_0^t k(t,s) f(s, x(s))\, ds, \quad t \in J, \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $\lambda > 0$, the functions $q : J \to \mathbb{R}$ and $f : J \times \mathbb{R} \to \mathbb{R}$ satisfy certain integrability and Chandrabhan type conditions to be specified later.

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Definition 1.1. *By an integrable solution of the VIE (1.1) we mean a function $x \in L^1(J,\mathbb{R})$ that satisfies the equation* (1.1) *on J, where $L^1(J,\mathbb{R})$ is the space of Lebesgue integrable real-valued functions defined on J.*

The VIE (1.1) is quite known and discussed sufficiently in the literature for different aspects of the solutions. The existence of an integrable solution has been discussed in Emmanule [11] and Banas [1] via measure of weak noncompactness whereas existence is discussed in Banas and El-Sayed [2] via Schauder fixed point principle. However, to the best of authors knowledge, no result is so far proved for maximal and minimal integrable solutions. Here we prove the existence of the extremal integrable solutions and integral inequalities for the VIE (1.1) under certain monotonicity condition along with applications. In the following section we present some preliminaries and notations needed in what follows.

2. Preliminaries

We place the problem of VIE (1.1) in the function space $L^1(J,\mathbb{R})$ of Lebesgue integrable real-valued functions defined on J. We define a standard norm $\|\cdot\|_{L^1}$ in $L^1(J,\mathbb{R})$ by

$$\|x\|_{L^1} = \int_0^T |x(t)|\,dt. \tag{2.1}$$

Clearly, $L^1(J,\mathbb{R})$ becomes a Banach space w.r.t. the norm $\|\cdot\|_{L^1}$ defined above. Next, we introduce an order relation $\preceq$ in $L^1(J,\mathbb{R})$ by the cone K given by

$$K = \{x \in L^1(J,\mathbb{R}) \mid x(t) \geq 0 \ \text{ a.e. } \ t \in J\}. \tag{2.2}$$

Thus,

$$x \preceq y \iff y - x \in K,$$

or equivalently,

$$x \preceq y \iff x(t) \leq y(t) \ \text{ a.e. } \ t \in J. \tag{2.3}$$

Lemma 2.1. *The partially ordered set $\big(L^1(J,\mathbb{R}),\preceq\big)$ is a Banach lattice.*

Proof. Let K be a order cone in $L^1(J,\mathbb{R})$ and let $x, y \in K$ be such that $x \preceq y$, Then, we have

$$\|x\|_{L^1} = \int_0^T |x(t)|\,dt = \int_0^T x(t)\,dt \leq \int_0^T y(t)\,dt = \int_0^T |y(t)\,dt = \|y\|_{L^1}.$$

Hence $\big(L^1(J,\mathbb{R}),\preceq\big)$ is a Banach lattice. □

Lemma 2.2. *The partially ordered set $\big(L^1(J,\mathbb{R}),\preceq\big)$ is a complete lattice.*

Proof. To finish, it is enough to show that $\big(L^1(J,\mathbb{R}),\preceq\big)$ is an abstract L-space. Let $x, y \in \big(L^1(J,\mathbb{R}),\preceq\big)$ be such that $x \succeq 0$ and $y \succeq 0$. Then by definition of the norm $\|\cdot\|_{L^1}$, we have

$$\begin{aligned}
\|x+y\|_{L^1} &= \int_0^T |x(t)+y(t)|\,dt \\
&= \int_0^T \big[x(t)+y(t)\big]\,dt \\
&= \int_0^T x(t)\,dt + \int_0^T y(t)\,dt \\
&= \int_0^T |x(t)|\,dt + \int_0^T |y(t)|\,dt \\
&= \|x\|_{L^1} + \|y\|_{L^1}.
\end{aligned} \tag{2.4}$$

This shows that $\big(L^1(J,\mathbb{R}),\preceq\big)$ is an abstract L-space. Hence by a theorem of uniformly monotone Banach lattice (Birkhoff [3, page 373]), $\big(L^1(J,\mathbb{R}),\preceq\big)$ is a complete lattice. $\square$

As a consequence of Lemmas 2.1 and 2.2, we obtain the following useful result. See also Dhage [7] and Dhage and Patil [10] and references therein.

Lemma 2.3 (Birkhoff [3])**.** *A non-empty closed and bounded subset of the complete Banach lattice $\big(L^1(J,\mathbb{R}),\preceq\big)$ is a complete lattice.*

Now, the basic tool used in this paper is the algebraic fixed point theorem of Tarski [15]. Before stating this result, we mention a useful concept of isotone mapping on a lattice L into itself.

Definition 2.4. *A mapping on a lattice $(L,\preceq)$ is called isotone increasing if preserve the order relation $\preceq$, that is, if $x,y\in L$ with $x\preceq y$, then $\mathcal{T}x\preceq\mathcal{T}y$.*

Theorem 2.5 (Tarski [15])**.** *Let $(L,\preceq)$ be a partially ordered set and let $T:L\to L$ be a mapping. Suppose that*

(a) $\mathcal{T}$ is isotone increasing,

(b) $(L,\preceq)$ is a complete lattice, and

(c) $F_{\mathcal{T}}=\{u\in L\mid\mathcal{T}u=u\}$.

Then $F_{\mathcal{T}}\neq\emptyset$ and $\big(F_{\mathcal{T}},\preceq\big)$ is a complete lattice.

In the following section we prove the main results of this paper under suitable conditions.

3. Existence of Extremal Integrable Solutions

We consider the following definitions appeared in Dhage [6, 7] whcich is useful for dealing with the discontinuous differential and integral equations.

Definition 3.1. *A function $f:J\times\mathbb{R}\to\mathbb{R}$ is said to be **Chandrabhan** if*

(i) the map $t\mapsto f(t,x)$ is measurable for each $x\in\mathbb{R}$, and

(ii) the map $x\mapsto f(t,x)$ is nondecreasing for almost every $t\in J$.

*Furthermore, a Chandrabhan function $f(t,x)$ is called $L^1_{\mathbb{R}}$-**Chandrabhan** if*

(iii) there exists a function $h\in L^1(J,\mathbb{R})$ such that

$$|f(t,x)|\leq h(t)\ \text{ a. e. }\ t\in J,$$

for all $x\in\mathbb{R}$.

Similarly, we have

Definition 3.2. *A function $k:J\times J\to\mathbb{R}$ is said to satisfy **integrability condition** if*

(i) the map $(t,s)\mapsto k(t,s)$ is jointly measurable, and

(iii) there exists a function $\gamma_k\in L^1(J,\mathbb{R})$ such that

$$|k(t,s)|\leq\gamma_k(s)\ \text{ a. e. } t,s\in J.$$

Lemma 3.3 (Dhage [6, 7])**.** *If $f(t,x)$ is Chandrabhan, then the function $t \mapsto f(t,x(t))$ is measurable. Moreover, if $f(t,x)$ is $L^1_{\mathbb{R}}$-Chandrabhan, then $f(\cdot,x(\cdot))$ is Lebesgue integrable on J for each $x \in L^1(J,,\mathbb{R})$.*

Proof. The proof is similar to an analogous result for Caratheódory functions $f(t,x)$ given in Granas and Dugundji [13]. We omit the details. □

Definition 3.4. *An integrable solution $x_M \in L^1(J,\mathbb{R})$ of the VIE* (1.1) *is said to be maximal if x is any other integrable solution, then $x(t) \leq x_M(t)$ for almost every $t \in J$. Similarly, a minimal integrable solution x_m of the VIE* (1.1) *is defined on J.*

We need the following hypotheses in what follows.

(H_1) The function $q : J \to \mathbb{R}$ is Lebesgue integrable.

(H_2) The function k is nonnegative and satisfy integrability condition on $J \times J$.

(H_3) The function f is $L^1_{\mathbb{R}}$-Chandrabhan on $J \times \mathbb{R}$.

Theorem 3.5. *Assume that hypotheses (H_1) through (H_3) hold. Then the VIE* (1.1) *has a maximal and a minimal integrable solution defined on J.*

Proof. Define a subset S of the complete lattice $\big(L^1(J,\mathbb{R}),\preceq\big)$ by

$$S = \big\{x \in \big(L^1(J,\mathbb{R}),\preceq\big) \mid \|x\|_{L^1} \leq r\big\} \tag{3.1}$$

where, $r = \|q\|_{L^1} + \lambda\,\|H\|_{L^1}T$ and $H(t) = \gamma(t)h(t)$, $t \in J$.

By Lemma 2.3, $(S,\preceq)$ is a complete lattice. Define an operator $\mathcal{T}$ on S by

$$\mathcal{T}x(t) = q(t) + \lambda \int_0^t k(t,s)f(s,x(s))\,ds, \;\; t \in J. \tag{3.2}$$

Then the VIE (1.1) is transformd into an operator equation

$$\mathcal{T}x(t) = x(t), \;\; t \in J. \tag{3.3}$$

We show that $\mathcal{T}$ defines a mapping $\mathcal{T} : S \to S$. Since the functions k and f are L^1_J-Caratheódory and $L^1_{\mathbb{R}}$-Chandrabhan on $J \times J$ and $J \times \mathbb{R}$ respectively, the integral on right hand of the equation (3.2) exists. Moreover, the integral is continuous and hence Lebesgue integrable. Again the sum of two Lebesgue integrable functions is again Lebesgue integrable on J. Hence $\mathcal{T}x \in L^1(J,\mathbb{R})$. Moreover, we have

$$\begin{aligned} |\mathcal{T}x(t)| &\leq |q(t)| + \lambda \int_0^t k(t,s)|f(s,x(s))|\,ds \\ &\leq |q(t)| + \lambda \int_0^t \gamma(s)h(s)\,ds \\ &\leq |q(t)| + \lambda \int_0^t H(s)\,ds \\ &\leq |q(t)| + \lambda\,\|H\|_{L^1}. \end{aligned} \tag{3.4}$$

Therefore, taking the integral on both sides from 0 to T, we obtain

$$\begin{aligned} \|\mathcal{T}x\|_{L^1} &= \int_0^T |\mathcal{T}x(t)|\,dt \\ &\leq \int_0^T |q(t)|\,dt + \lambda \int_0^T \|H\|_{L^1}\,dt \\ &= \|q\|_{L^1} + \lambda\,\|H\|_{L^1}T, \end{aligned}$$

which implies that $\mathcal{T}$ maps S into itself. Next we show that $\mathcal{T}$is isotone increasing operator on S into itself. Let $x, y \in S$ be such that $x \preceq y$. Then, in view of hypotheses (H_2) and (H_3),

$$\begin{aligned} \mathcal{T}x(t) &= q(t) + \lambda \int_0^t k(t,s) f(s, x(s))\, ds \\ &\leq q(t) + \lambda \int_0^t k(t,s) f(s, y(s))\, ds \\ &= \mathcal{T}y(t) \end{aligned} \tag{3.5}$$

for almost every $t \in J$. This shows that $\mathcal{T}x \preceq \mathcal{T}y$ almost everywhere on J. As a result, $\mathcal{T}$ is isotone increasing operator on S. Now by application of Theorem 2.5 implies that $\mathcal{T}$ has a fixed point and the set $F_{\mathcal{T}}$ of all fixed points is a complete lattice. Thus, $F_{\mathcal{T}} \neq \emptyset$ and $\big(F_{\mathcal{T}}, \preceq\big)$ is a complete lattice. Consequently $x_m = \wedge F_{\mathcal{T}}$ and $x_M = \vee F_{\mathcal{T}}$ both exist and are respectively the minimal and maximal integrable solutions of the VIE (1.1) on J. This complete the proof. □

Example 3.6. *Let $J = [0,1] \subset \mathbb{R}$ and consider he nnlinear Volterra inegral equation,*

$$x(t) = t^2 + \int_0^t (t-s) \tanh x(s)\, ds, \ \ t \in [0,1]. \tag{3.6}$$

Here, $q(t) = t^2$, $k(t,s) = t - s$ and $f(t,x) = \tanh x$ for $t \in [0,1]$ and $x \in \mathbb{R}$. Thus the above functions satisfy all the conditions of Theorem 3.5, whence the VIE (3.6) *has maximal and minimal integrable solutions defined on* $[0,1]$.

4. Integral Inequalities

Next, we prove two basic integral inequalities involving the integrable solutions related to the VIE (1.1) on J.

Theorem 4.1. *Assume that the hypotheses (H_1)- (H_3) hold. If there exists an element $u \in S$ such that*

$$u(t) \leq q(t) + \lambda \int_0^t k(t,s) f(s, u(s))\, ds, \tag{4.1}$$

for every $t \in J$, then there exists a maximal integrable solution x_M of the VIE (1.1) *such that*

$$u(t) \leq x_M(t) \ \text{ a. e. t} \in \text{J}. \tag{4.2}$$

Proof. Let $P = \sup S$ which does exist since $(S, \preceq)$ is a complete lattice. Now consider the lattice interval $[u, P]$ which is a closed set and hence a complete lattice. Define an operator $\mathcal{T}$ on $[u, P]$ by (3.2). Then from (4.1) we get $u \preceq \mathcal{T}u$ everywhere on J. Since $\mathcal{T}$ is isotone increasing, it maps the lattice integral $[u, P]$ into itself. Now, by an application of Theorem 2.5, $\mathcal{T}$ has a maximal fixed point x_M in $[u, P]$ which corresponds to the maximal integrable solution of the VIE (1.1) in $[u, P]$. By nature of x_M we have, $u(t) \leq x_M(t)$ a. e. t ∈ J.. This completes proof. □

Theorem 4.2. *Assume that the hypotheses (H_1)- (H_3) hold. If there exists an element $v \in S$ such that*

$$v(t) \geq q(t) + \lambda \int_0^t k(t,s) f(s, v(s))\, ds, \tag{4.3}$$

for every $t \in J$, then there exists a minimal integrable solution x_m of the VIE (1.1) *such that*

$$v(t) \geq x_m(t) \ \text{ a. e. t} \in \text{J}. \tag{4.4}$$

Proof. The proof is similar to Theorem 4.1 with appropriate modifications. We omit the details. □

Next, we apply the integral inequality stated in Theorem 4.1 to the VIE (1.1) for proving the boundedness and uniqueness of the integrable solutions on J. Now, consider the scaler VIE

$$r(t) = p(t) + \lambda \int_0^t k(t,s)F(s, r(s))\,ds, \;\; t \in J, \tag{4.5}$$

where $p : J \to \mathbb{R}_+$ is Lebesgue integrable and $F : J \times \mathbb{R}_+ \to \mathbb{R}_+$ is a Chandrabhan function.

Theorem 4.3. *Assume that all hypotheses of Theorem 3.5 are satisfied with q and f replaced by p and F given in* (1.1) *and* (4.5) *respectively. Further suppose that the functions q, f and p, F satisfy the inequalities*

$$\left.\begin{aligned} |q(t)| &\le p(t) \;\; \text{a.e. t} \in \text{J}, \\ |f(t,x)| &\le F(t,|x|) \;\; \text{a.e. t} \in \text{J}, \end{aligned}\right\} \tag{4.6}$$

Then for any integrable solution u of the VIE (1.1)*, we obtain*

$$|u(t)| \le r_M(t) \;\; \text{a.e. t} \in \text{J}, \tag{4.7}$$

where r_M is a maximal integrable solution of the VIE (4.6) *on J.*

Proof. By Theorem 3.5, the scalar VIE (4.5) has a maximal integrable solution r_M on J. Let $u \in L^1(J, \mathbb{R})$ be any integrable solution of the VIE (1.1) on J. Then we have

$$u(t) = q(t) + \lambda \int_0^t k(t,s)f(s, u(s))\,ds, \;\; t \in J.$$

Therefore, by Theorem 4.1,

$$\begin{aligned} |u(t)| &\le |q(t)| + \lambda \int_0^t k(t,s)|f(s, u(s))|\,ds \\ &\le p(t) + \lambda \int_0^t k(t,s)F(s, |u(s)|)\,ds \\ &\le r_M(t) \end{aligned}$$

for almost every $t \in J$. This completes the proof. □

Finally, we prove uniqueness result for the integrable solution of the VIE (1.1) on J.

Theorem 4.4. *Assume that all hypotheses of Theorem 3.5 are satisfied with q and f replaced by p and F given in* (1.1) *and* (4.5) *respectively. Further suppose that the functions f and F satisfy the inequality*

$$|f(t,x) - f(t,y)| \le F(t, |x-y|) \; \text{a.e. t} \in \text{J}, \tag{4.8}$$

for all $x, y \in \mathbb{R}$. Further, if identically zero function is the only solution of the VIE (4.5) *with $p \equiv 0$ on J, then the VIE* (1.1) *has a unique integrable solution on J.*

Proof. Suppose that u and v are two integrable solutions of the VIE (1.1) on J. Then we have

$$u(t) = q(t) + \lambda \int_0^t k(t,s)f(s, u(s))\,ds, \;\; t \in J,$$

and

$$v(t) = q(t) + \lambda \int_0^t k(t,s)f(s, v(s))\,ds, \;\; t \in J.$$

Therefore, by inequality (4.8), we obtain

$$\begin{aligned}|u(t)-v(t)| &\leq \lambda \int_0^t k(t,s)|f(s,u(s)-f(s,v(s)|\,ds\\ &\leq \lambda \int_0^t k(t,s)F(s,|u(s)-v(s)|)\,ds,\end{aligned}$$

for almost every $t \in J$. Now, applying integral inequality given in Theorem 4.1 yields that $u(t) = v(t)$ a. e. t $\in$ J. This completes the proof. □

Remark 4.5. *Under the hypotheses (H_1)- (H_3), the results of this paper may be extended to nonlinear integral equation of Fredholm type*

$$x(t) = q(t) + \lambda \int_0^T k(t,s)f(s,x(s))\,ds, \quad t \in J, \tag{4.9}$$

using the similar arguments with appropriate modifications.

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Study of the inverse continuous Bernoulli distribution

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Abstract. The continuous Bernoulli distribution, a one-parameter probability distribution defined over the interval $[0, 1]$, has recently received increased attention in applied statistics. Numerous studies have highlighted both its merits and limitations, and proposed extended variants. In this article, we present an innovative modification of the continuous Bernoulli distribution through an inverse transformation, introducing the inverse continuous Bernoulli distribution. The main feature of this distribution is that it transfers the properties of the continuous distribution to the interval $[1, +\infty)$ without the need for additional parameters. The first part of this article elucidates the mathematical properties of this novel inverse distribution, including essential probability functions and quantiles. Inference for the associated model is performed using the famous maximum likelihood estimation. A comprehensive simulation study is carried out to evaluate the effectiveness of the estimated model. Its performance is then evaluated in a practical context using data sets from a variety of sources. In particular, our results demonstrate its superior performance to a wide range of analogous models defined over the support interval $[1, +\infty)$, even outperforming the well-established Pareto model.

AMS Subject Classifications: 60E05, 62E99.

Keywords: Continuous Bernoulli distribution, inverse distribution, quantiles, data fitting.

Contents

1. Introduction

In order to explain the mathematical basis of this investigation, we will first examine the continuous Bernoulli (CB) distribution, which was originally introduced in the work of [10]. The definition of this distribution can be succinctly stated as follows:

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Definition 1.1. *The cumulative distribution function (cdf) below defines the CB distribution with the parameter* $\omega \in (0,1)$*:*

$$F_{CB}(x;\omega) = \begin{cases} 0, & x < 0, \\ x, & \omega = \dfrac{1}{2} \text{ and } x \in [0,1], \\ \dfrac{\omega^x (1-\omega)^{1-x} + \omega - 1}{2\omega - 1}, & \omega \in (0,1)/\left\{\dfrac{1}{2}\right\} \text{ and } x \in [0,1], \\ 1, & x > 1. \end{cases} \tag{1.1}$$

Thus, the CB distribution has a support restricted to the interval $[0,1]$, with a single parameter similar to the conventional power distribution. This distribution finds applications in a wide variety of fields, with particular emphasis on machine learning, probability theory and statistics. In the context of variational autoencoders, it proves to be highly effective in replicating the pixel intensities of real-world images. For a more in-depth exploration of these topics, see the following references: [10], [8], [12] and [9].

Recently, there have been notable developments in the scientific literature in the area of CB distribution extensions. Of particular importance is the power CB (PCB) distribution, as elucidated by the authors in [2]. The PCB distribution introduces a notable extension to the cdf, as shown in Equation (1.1). This extension is achieved by the inclusion of a shape parameter, which serves to increase the modeling versatility of the CB distribution. In [2], the authors proposed a statistical methodology to explore the fundamental mathematical properties of the PCB distribution. Maximum likelihood estimation was also used to explore the nuances of parameter estimation. In order to illustrate the practical utility of the PCB distribution, an in-depth investigation was carried out using two different data sets. These data sets include a collection of trade share data and a comprehensive set of polyester fiber tensile strength data. Through these real-world data fitting exercises, the flexibility and applicability of the PCB distribution was rigorously assessed. The accuracy of the PCB distribution is underlined by the consideration of fair competitors. Conventional statistical standards show that the PCB distribution provides superior results. In addition, the transmuted CB (TCB) distribution introduced in [3] stands out as a highly effective extension. A notable feature of the CB distribution is its parameter, which orchestrates a linear trade-off between the minimum and maximum values of two continuous random variables. Using a statistical approach, the authors derive the fundamental mathematical properties of the TCB distribution. To illustrate the suitability of the model, they examined three proportional data sets: the time to infection of kidney dialysis patients, records of flood peaks, and waiting times for service in a bank. The empirical results highlight the superior fit of the TCB distribution to these data sets compared to well-established competitors. In a related context, [12, Chapter 9] introduced a two-dimensional CB distribution along with some of its key attributes. In addition, the authors in [9] elaborated an exponentiated variant of the CB distribution to construct a fractile (quantile) regression model for responses in the range $[0,1]$. More recently, the authors in [4] used the CB distribution to give rise to the Op family, considering its cdf as a distribution generator. In particular, the OpTL distribution, rooted in the Topp-Leone distribution, emerged as a novel two-parameter distribution with support in the interval $[0,1]$. Notably, the OpTL distribution demonstrated superior fit performance compared to contemporary models, including the CB distribution itself. In summary, it is evident that research around the CB distribution will continue to flourish in future, both from a theoretical and practical perspective.

In this article, we use the CB distribution to formulate a novel one-parameter distribution defined over the interval $[1,+\infty)$. To achieve this goal, we focus on the conventional inverse scheme, a well-established methodology known for its invaluable contributions to the modeling and analysis of data in various domains. The inverse scheme not only facilitates the exploration of reciprocal relationships, but also provides valuable insights into the characteristics of skewed and heavy-tailed distributions. Moreover, its practical utility extends to important domains such as finance, reliability and extreme value analysis. In the specific context of this study, we briefly present our approach: Starting with a random variable X following the CB distribution, we introduce the concept of an inverse CB (ICB) distribution, characterized by the distribution of the inverse random variable

$Y = 1/X$. The cdf of the ICB distribution, denoted as $F_{ICB}(x;\omega)$, is expressed as a function of $F_{CB}(x;\omega)$, represented as $F_{ICB}(x;\omega) = 1 - F_{CB}(1/x;\omega)$. The exact definition is explained below.

Definition 1.2. *The cdf given below defines the ICB distribution with parameter $\omega \in (0,1)$:*

$$F_{ICB}(x;\omega) = \begin{cases} 0, & x < 1, \\ 1 - \dfrac{1}{x}, & \omega = \dfrac{1}{2} \text{ and } x \geq 1, \\ 1 - \dfrac{\omega^{1/x}(1-\omega)^{1-1/x} + \omega - 1}{2\omega - 1}, & \omega \in (0,1)/\left\{\dfrac{1}{2}\right\} \text{ and } x \geq 1. \end{cases}$$

This cdf is at the core of the theory and inference of the ICB distribution, which will be explained in detail in this article. In particular, the support of the ICB distribution lies within the interval $[1,+\infty)$, making it a direct rival to the Pareto distribution. The creation of such distributions, i.e., with support $[1,+\infty)$, is crucial for modeling various real-world phenomena, such as reliability analysis and survival data, where non-negative outcomes beyond a lower bound are of primary interest. Such distributions provide a robust framework for handling situations where values cannot be less than one, thus ensuring a more accurate representation of the underlying processes. In our research, we empirically demonstrate that the ICB distribution can provide a more robust and accurate fit to real-world data compared to the Pareto distribution, thus supporting its importance in statistical modeling.

The subsequent organization of the article unfolds as follows: Section 2 deals with the basic probability functions that govern the ICB distribution. Section 3 is dedicated to the parameter estimation, simulations and real-world applications. Finally, our study ends in Section 4, where we present our final results.

2. Probability Functions

This section deals with the analysis of the probability density function (pdf), hazard rate function (hrf) and quantile function (qf) associated with the ICB distribution.

First, the pdf relevant to the ICB distribution is obtained by differentiating $F_{ICB}(x;\omega)$ as follows:

$$f_{ICB}(x;\omega) = \begin{cases} 0, & x < 1, \\ \dfrac{1}{x^2}, & \omega = \dfrac{1}{2} \text{ and } x \geq 1, \\ c_\omega \dfrac{1}{x^2}\omega^{1/x}(1-\omega)^{1-1/x}, & \omega \in (0,1)/\left\{\dfrac{1}{2}\right\} \text{ and } x \geq 1, \end{cases} \tag{2.1}$$

where

$$c_\omega = \frac{2\,\mathrm{arctanh}(1-2\omega)}{1-2\omega} \quad \left(\text{or, equivalently, } c_\omega = \frac{\ln(1-\omega) - \ln(\omega)}{1-2\omega}\right).$$

Such a pdf with support $[1,+\infty)$ is essential for modeling and analyzing random variables that are restricted to positive values, such as durations, lifetimes and various natural phenomena. It provides insight into the probability of observing certain outcomes within this constrained range, making it a valuable tool in fields such as reliability analysis and survival studies.

By manipulating $f_{ICB}(x;\omega)$ and $F_{ICB}(x;\omega)$, we obtain the hrf corresponding to the ICB distribution. More precisely, we have

$$h_{ICB}(x;\omega) = \frac{f_{ICB}(x;\omega)}{1-F_{ICB}(x;\omega)}$$
$$= \begin{cases} 0, & x<1, \\ \dfrac{1}{x}, & \omega = \dfrac{1}{2} \text{ and } x \geq 1, \\ c_\omega^* \dfrac{\omega^{1/x}(1-\omega)^{1-1/x}}{x^2\left[1-\omega-\omega^{1/x}(1-\omega)^{1-1/x}\right]}, & \omega \in (0,1)/\left\{\dfrac{1}{2}\right\} \text{ and } x \geq 1, \end{cases}$$

where

$$c_\omega^* = 2\operatorname{arctanh}(1-2\omega) \quad (\text{or, equivalently, } c_\omega^* = \ln(1-\omega)-\ln(\omega)).$$

The qf corresponding to the ICB distribution is obtained as

$$Q_{ICB}(x;\omega) = F_{ICB}^{-1}(x;\omega)$$
$$= \begin{cases} \dfrac{1}{1-x}, & \omega = \dfrac{1}{2} \text{ and } x \in [0,1], \\ \dfrac{c_\omega^*}{\ln(1-\omega)-\ln[(2\omega-1)(1-x)+1-\omega]}, & \omega \in (0,1)/\left\{\dfrac{1}{2}\right\} \text{ and } x \in [0,1]. \end{cases} \tag{2.2}$$

Having an analytical expression for the qf in distributions with support $[1,+\infty)$ is important for efficient risk assessment and decision making, especially in scenarios involving reliability and tail risk analysis where accurate modeling of extreme events is paramount.

In addition, the qf in Equation (2.2) is used to generate random samples of size n from the ICB distribution. Analytically, for a fixed value of the parameter ω, we obtain the median, lower, and upper quartiles of the ICB distribution when x takes the values $1/2$, $1/4$, and $3/4$, respectively, in this qf.
Furthermore, the Galton skewness and Moor kurtosis as proposed by [7] and [11], respectively, can be obtained by utilizing Equation (2.2) as follows:

$$S_G = \frac{Q(6/8;\omega)-2Q(4/8;\omega)+Q(2/8;\omega)}{Q(6/8;\omega)-Q(2/8;\omega)}$$

and

$$K_M = \frac{Q(7/8;\omega)-Q(5/8;\omega)+Q(3/8;\omega)-Q(1/8;\omega)}{Q(6/8;\omega)-Q(2/8;\omega)}.$$

Table 1 shows the summary statistics of the ICB distribution for different choice of the parameter value ω.

Table 1: Summary statistics of the ICB distribution for varying values of ω

ω	$Q(1/2;\omega)$	$Q(1/4;\omega)$	$Q(3/4;\omega)$	S_G	K_M
0.1	3.7381	2.0000	8.7429	0.4845	2.1481
0.2	2.9495	1.6769	6.6765	0.4909	2.1558
0.3	2.5182	1.5141	5.4966	0.4957	2.1633
0.4	2.2239	1.4094	4.6599	0.4989	2.1690
0.51	1.9802	1.3267	3.9405	0.4999	2.1714
0.6	1.8171	1.2732	3.4425	0.4986	2.1678
0.7	1.6587	1.2224	2.9453	0.4935	2.1527
0.8	1.5129	1.1762	2.4772	0.4823	2.1139
0.9	1.3652	1.1292	2.0000	0.4579	2.0148
0.99	1.1746	1.0659	1.4188	0.3841	1.6875

This table shows that the median, lower and upper quartiles of the ICB distribution are monotonically decreasing functions of the parameter ω, while the skewness and kurtosis are monotonically increasing functions for $\omega \in [0, 0.5)$ and decreasing for $\omega \in (0.5, 1)$.

3. Parameter Estimation, Simulations and Applications

3.1. Parameter Estimation

Here, we adapt the maximum likelihood estimation method in estimating the parameter ω of the ICB distribution, which is supposed to be unknown. Let $x_1, x_2, \ldots, x_n$ be n independent observations of size n from a random variable X following the ICB distribution. Then, the likelihood function associated to Equation (2.1) is specified by

$$L(\omega; x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{ICB}(x_i; \omega)$$
$$= \begin{cases} \left[\prod_{i=1}^{n} \frac{1}{x_i^2}\right], & \omega = \frac{1}{2}, \\ c_\omega^n \left[\prod_{i=1}^{n} \frac{1}{x_i^2}\right] \omega^{\sum_{i=1}^{n} \frac{1}{x_i}} (1-\omega)^{n-\sum_{i=1}^{n} \frac{1}{x_i}}, & \omega \in (0,1)/\left\{\frac{1}{2}\right\}, \end{cases} \tag{3.1}$$

and $\min(x_1, \ldots, x_n) \geq 1$.

By taking the natural logarithm of Equation (3.1), the corresponding log-likelihood function is obtained as

$$\ell(\omega; x_1, \ldots, x_n) = \sum_{i=1}^{n} \ln[f_{ICB}(x_i; \omega)]$$
$$= \begin{cases} -2\sum_{i=1}^{n} \ln(x_i), & \omega = \frac{1}{2}, \\ n\ln(c_\omega) - 2\sum_{i=1}^{n} \ln(x_i) + \ln(\omega)\sum_{i=1}^{n} \frac{1}{x_i} + \ln(1-\omega)\left(n - \sum_{i=1}^{n} \frac{1}{x_i}\right), & \omega \in (0,1)/\left\{\frac{1}{2}\right\}, \end{cases} \tag{3.2}$$

and $\min(x_1, \ldots, x_n) \geq 1$. The maximum likelihood estimate (MLE) of ω, say $\hat{\omega}$, can be obtained by maximizing $\ell(\omega; x_1, \ldots, x_n)$ with respect to ω. In our case, this can be achieved by taking the first derivative of Equation (3.2) with respect to ω and equating the corresponding expression to zero, i.e., $\frac{\partial \ell(\omega; x_1, \ldots, x_n)}{\partial \omega} = 0$.

3.2. Monte Carlo Simulation Study

One of the most important aspects of any statistical model is the performance of its parameter estimate(s). Here we perform a Monte Carlo simulation study to examine the asymptotic behavior of the MLE of ω from the ICB distribution. To achieve this, random samples were generated from the ICB distribution using Equation (2.2). The simulation is repeated 1000 times for different sample sizes ($n = 30,\ 50,\ 100,\ 200$ and 500) and different choices of the parameter value ($\omega = 0.1,\ 0.4,\ 0.6$ and 0.8). The performance of the MLE $\hat{\omega}$ is examined in terms of mean estimate, average bias, mean square error (MSE), and coverage probability. The numerical computation of these quantities is displayed in Tables 2, 3, 4 and 5, respectively.

Table 2: Mean estimate of the MLE $\hat{\omega}$

n	$\omega = 0.1$	$\omega = 0.4$	$\omega = 0.6$	$\omega = 0.8$
30	0.1116	0.4023	0.5984	0.7801
50	0.1084	0.4021	0.5977	0.7908
100	0.1042	0.4014	0.5968	0.7940
200	0.1037	0.4007	0.5946	0.7960
500	0.1013	0.4006	0.6002	0.7999

Table 3: Average bias of the MLE $\hat{\omega}$

n	$\omega = 0.1$	$\omega = 0.4$	$\omega = 0.6$	$\omega = 0.8$
30	0.0116	0.0023	-0.0016	-0.0198
50	0.0084	0.0021	-0.0023	-0.0091
100	0.0042	0.0014	-0.0032	-0.0059
200	0.0037	0.0007	-0.0054	-0.0040
500	0.0013	0.0006	0.0002	-0.0006

Table 4: MSE of the MLE $\hat{\omega}$

n	$\omega = 0.1$	$\omega = 0.4$	$\omega = 0.6$	$\omega = 0.8$
30	0.0045	0.0208	0.0202	0.0119
50	0.0028	0.0126	0.0137	0.0063
100	0.0012	0.0069	0.0069	0.0034
200	0.0007	0.0035	0.0034	0.0017
500	0.0002	0.0013	0.0013	0.0006

Table 5: Coverage probability of the $100(1-\alpha)\%$ confidence interval of the MLE $\hat{\omega}$

$1-\alpha$	n	$\omega = 0.1$	$\omega = 0.4$	$\omega = 0.6$	$\omega = 0.8$
0.95	30	0.910	0.896	0.898	0.903
	50	0.916	0.920	0.909	0.921
	100	0.932	0.933	0.931	0.943
	200	0.947	0.942	0.946	0.951
	500	0.945	0.954	0.955	0.952
0.90	30	0.879	0.856	0.843	0.864
	50	0.870	0.876	0.858	0.884
	100	0.896	0.875	0.871	0.883
	200	0.903	0.885	0.892	0.904
	500	0.904	0.901	0.906	0.906

Remarks:

i.) In Table 2, the data reveal a converging trend as n increases, with the mean estimate $\hat{\omega}$ approaching the true parameter value.

ii.) The findings in Table 3 shed light on the relationship between sample size and bias. As n increases, we observe a corresponding decrease (increase) in average bias. In addition, this table illustrates the existence of both negative and positive biases for the MLE $\hat{\omega}$.

iii.) The results in Table 4 show a notable trend as n increases, with the MSE steadily approaching zero.

iv.) Table 5 provides insights into the coverage probability of two different confidence intervals for the estimator $\hat{\omega}$. As n increases, our observations indicate a convergence of the coverage probability towards the nominal levels associated with the 95% and 90% confidence intervals, respectively.

3.3. Applications

In this part, we assess the suitability of the ICB distribution in the context of a real-life scenario, employing two distinct data sets. Specifically, we examine the fits of the Pareto and New Pareto (NP) distributions. The NP distribution is derived from [1]. Both of them share the same support interval of $[1, +\infty)$ with the ICB distribution. These distributions are evaluated for their capacity to model the data sets alongside the ICB distribution.

Data sets:

Data set I comprises 31 recorded flood peak exceedances (measured in m^3/s) for the Wheaton River in the vicinity of Carcross, located within the Yukon Territory, Canada. The data set spans the years from 1958 to 1984. For this data set, the authors in [5] conducted an investigation employing this specific data set to assess the suitability of the generalized Pareto distribution. In a separate study, the authors in [6] harnessed this same data set to elucidate the adaptability of a generalized Lindley distribution. The data set is organized and presented as follows: 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.

On the other hand, Data set II consists of the time-to-failure (103 h) of turbocharger of one type of engine given in [13]. The data set is shown as follows: 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0.

To facilitate model comparison, we employ the following information criteria: the maximized log-likelihood (ℓ^*), the Akaike information criterion (AIC), the corrected Akaike information criterion ($AICc$), the Bayesian information criterion (BIC), and the Hannan-Quinn information criterion ($HQIC$). These criteria are rigorously defined as follows:

$$AIC = -2\ell^* + 2k, \qquad AICc = AIC + \frac{2k\,(k+1)}{n-k-1},$$

$$BIC = -2\ell^* + k\ln(n), \qquad HQIC = -2\ell^* + 2k\ln[\ln(n)],$$

where n is the sample size and k is the number of parameter(s) in the considered model.

A smaller value of AIC, $AICc$, BIC and $HQIC$ indicates a better fit of the respective distributions to the analyzed data set. Table 6 provides a comprehensive summary of the goodness of fit results for the ICB, Pareto and NP models applied to the two data sets considered.

Table 6: Summary results for Data sets I and II

Models	MLE	ℓ^*	AIC	$AICc$	BIC	$HQIC$
Data set I						
ICB	$\hat{\omega} = 0.004$	-120.8408	243.6816	243.6897	247.8962	245.3354
NP	$\hat{\alpha} = 0.6305$	-125.1094	252.2187	252.2268	256.4333	253.8725
Pareto	$\hat{\theta} = 0.4054$	-135.4429	272.8858	272.8939	277.1005	274.5397
Data set II						
ICB	$\hat{\omega} = 0.0058$	-114.3882	230.7765	230.7845	234.9911	232.4303
NP	$\hat{\alpha} = 0.8611$	-126.3683	254.7366	254.7447	258.9512	256.3904
Pareto	$\hat{\theta} = 0.5165$	-143.8719	289.7438	289.7518	293.9384	291.3976

From this table, it can be seen that the AIC, $AICc$, BIC and $HQIC$ values associated with the ICB distribution show a significant reduction compared to those of the Pareto and NP distributions. As a result, the ICB distribution consistently outperforms its competitors in analising the two data sets considered.

4. Conclusion

In conclusion, the introduction of the ICB distribution represents an advance in the field of probability distributions. This considered modification has allowed us to extend the properties of the CB distribution to the interval $[1, +\infty)$ without the need for additional parameters. Through a study of its mathematical properties, including quantiles, we have laid the foundation for understanding and using this novel distribution.

Our article has also shown that the ICB distribution can be effectively used in statistical modeling. The use of maximum likelihood estimation for inference proved to be a robust and practical approach. Through a thorough simulation study, we provided empirical evidence of the model's performance, which was consistently superior to both the Pareto and new Pareto models when applied to diverse real-world data sets. These results show the potential of the ICB distribution as a valuable tool for modeling data in various domains. This article provides the foundations for further exploration and adoption of the ICB distribution in statistical and data analysis contexts.

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On β-γ-connectedness and $\beta_{(\gamma,\delta)}$-continuous functions

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Abstract. The purpose of this work is to present the idea of β-γ-separated sets, examine their characteristics in topological spaces and then define the notation for β-γ-connected and β-γ-disconnectedness. In addition, the study of topological qualities that involves for β-γ-connected spaces via β-γ-separated sets. An analysis is conducted on the properties of β-γ-connected spaces and how they behave under $\beta_{(\gamma,\delta)}$-continuous functions. We also provide the ideas of β-γ-components in a space X and β-γ-locally linked spaces.
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Keywords: Operation, β-γ-open, β-γ-closed, β-γ-closure, β-γ-connected and $\beta_{(\gamma,\delta)}$-continuous functions.

Contents

1. Introduction

One of the most significant, practical and basic notations in general topology and other high level mathematical discipline now a days is connectedness. The notation of connectedness is fruitful in computing, topology, algebraic topology and advanced calculus. Many researchers across the globe have investigated properties of connectedness ([2], [3], [4], [5], [6]) and obtained new and interesting results.

The idea of β-open set in topological spaces was first proposed by M.E. Abd El-Monsef, S.N. El-Deeb and R.A. Mahmoud in 1983. Their proof was that the set of all β-open sets in (X, τ) is finer topology on X then τ. The researchers worked on two related topologies that were tested on the same underlying structure to determine if they share the same topological properties. The basic properties of β-connectedness were obtained by Jafari and Noiri [7] in 2003. Several other forms of connectedness can be introduced and studied using it. Tahiliani [8] discussed and studied the characterisations of β-γ-open sets in topological spaces in 2011. This work presents and investigates an additional kind of connectivity that is defined on β-open sets in (X, τ) via operations. Their behviour under is $\beta(\gamma, \delta)$-continuous,as well as their attributes are discussed in this study.

The procedures γ and δ are defined on the set of all β-open sets of topological spaces (X, τ) and (Y, σ) correspondingly during the conversion.For any subset A of X, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ stands for the closure and interior of A, respectively, for any subset A of X.

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2. Preliminaries

Here we lay down the groundwork by defining key terms and showing key findings:

The condition that $A \subseteq \mathrm{Cl}(\mathrm{Int}(Cl(A)))$ merely indicates that subset A of topological space X is β-open [1]. A β-open sets counterpart is a β-closed set, and $\beta O(X)$ [1] is the collection of all β-open sets. $\beta\,\mathrm{Cl}(A)$ [2] is the symbol for intersection of all β-closed sets that include A, while $\beta\,\mathrm{Int}(A))$ [2] is the symbol for union of all β-open sets that contain A.

The condition $V \in V^{\gamma}$ satisfied for each $V \in \beta O(X)$ in an operation $\gamma : \beta O(X) \to P(X)$.The function $V^{id} = V$ for each set $V \in \beta O(X)$ is called the identity operation on $\beta O(X)$.

As γ and δ are always defined on the family of β-open sets in space, We always mean them as operations. From [8], we retrieve the following definitions and findings:

Definition 2.1. *(i): If there exists a β-γ-open set U of X that contains x and $U^{\gamma} \subseteq A$, then for any point $x \in A$, a subset A of X is termed as of β-γ-open set. The β-γ-closed is counterpart of β-γ-open set. The set symbolized by $\beta O(X)\gamma$ includes all β-γ-open sets of (X, τ).*

(ii): A subset A of X is said to be β-γ-closed if and only if all β-γ-closed containing A intersect at point where an operation γ on $\beta O(X)$ is represented by $\beta\gamma\,\mathrm{Cl}(A)$. The $\beta\gamma\,\mathrm{Int}(A)$ notation represents β-γ-interior of A, which is the union of all β-γ-open set included in A. The β-γ-boundary of a set A is represented by $\beta\gamma Bd(A)$ and is defined by $(\beta\gamma\,\mathrm{Cl}(A) - \beta\gamma\,\mathrm{Int}(A))$.

(iii): If, for every element x in X and each β-δ-open set V that contains $f(x)$, there exists a β-γ-open set U such that $x \in U$ and $f(U) \subseteq V$, then we say that $f : (X, \tau) \to (Y, \sigma)$ is $\beta(\gamma, \delta)$-continuous.

(iv): For any β-γ-closed set A of (X, τ), the set $f(A)$ is β-δ-closed in (Y, σ) we say that mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be $\beta(\gamma, \delta)$-closed.

(v): For any β-γ-open set A of (X, τ), the set $f(A)$ is β-δ-open in (Y, σ) we say that mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be $\beta(\gamma, \delta)$-open.

Theorem 2.2. *Suppose X be a subset of a topological space and A is a subset of it. Then*

(i) $x \in \beta\gamma\text{-}\mathrm{Cl}(A)$ *if and only if every $\beta\gamma$-open set U containing x has non empty intersection with A.*

(ii) $\beta\gamma\,\mathrm{Cl}(X - A) = X - \beta\beta\gamma\,\mathrm{Int}(A)$.

3. β-γ-connected spaces

Definition 3.1. *(i): If $(\beta\,\mathrm{Cl}(A) \cap B) \cup (A \cap (\beta\,\mathrm{Cl}(B)) = \emptyset$, then the subsets A and B of a topological space (X, τ) are said to be β-separated.*

(ii): The term "β-γ-separated" is used to describe a pair of subsets A and B of a topological space (X, τ), where

$$(\beta\gamma\,\mathrm{Cl}(A) \cap B) \cup (A \cap (\beta\gamma\,\mathrm{Cl}(B)) = \emptyset.$$

Remark 3.2. *Each two β-γ-separated sets are always disjoint, since $A \cap B \subseteq A \cap \beta\gamma\,\mathrm{Cl}(B) = \emptyset$. The converse may not hold in general.*

Example 3.3. *The set $X = \{a, b, c\}$, and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ are defined as follows: $A^{\gamma} = A$ if $b \in A$, $A^{\gamma} = \mathrm{Cl}(A)$ if $b \notin A$. If X is not β-γ-separated, then $\{a, b\}$ and $\{c\}$ are disjoint subsets of X.*

Given that $\beta\,\mathrm{Cl}(A) \subseteq \gamma\,\mathrm{Cl}(A)$, for all subsets A of X, it follows that every β-γ-separated set is β-separated. The preceding example, however suggests that reverse may not be true. Both $\{a\}$ and $\{b, c\}$ are β-separated in this case,but they are not β-γ-separated.

Theorem 3.4. *The following claims hold if A and B are two non empty subsets of space X*

(1) *If A and B are β-γ-separated and $A1 \subseteq A$ and $B1 \subseteq B$, then $A1$ and $B1$ are also β-γ-separated.*

(2) *If A and B are disjoint and are both β-γ-closed (both β-γ-open), then A and B are β-γ-separated.*

(3) *If A and B are both β-γ-closed (both β-γ-open) then $H = A \cap (X - B)$ and $G = B \cap (X - A)$ are β-γ-separated.*

Proof. 1. Since $A1 \subseteq A$ implies $\beta\gamma\operatorname{Cl}(A1) \subseteq \gamma\operatorname{Cl}(A)$ for every pair of A and $A1$, $\beta\gamma\operatorname{Cl}(A) \cap B = \emptyset$ and $\beta\gamma\operatorname{Cl}(B) \cap A = \emptyset$ implies $\beta\gamma\operatorname{Cl}(A1) \cap B1 = \emptyset$ and $\beta\gamma\operatorname{Cl}(B1) \cap A1 = \emptyset$. Hence $A1$ and $B1$ are β-γ-separated.

2. The equations $A = \beta\gamma\operatorname{Cl}(A)$ and $B = \beta\gamma\operatorname{Cl}(B)$ hold if A and B are both β-γ-closed. Hence because $A \cap B = \emptyset$, it follows that $\beta\gamma\operatorname{Cl}(A) \cap B = \emptyset$ and $\beta\gamma\operatorname{Cl}(B) \cap A = \emptyset$, A and B are β-γ-separated. In other words, the complement of disjoint β-γ-open sets A and B are also β-γ-closed sets. Specifically X-A and X-B are β-γ-separated. If A and B are disjoint and are both, then their complements are disjoint and β-γ-closed. Furthermore, $A \subseteq \beta\gamma\operatorname{Cl}(A) \subseteq \beta\gamma\operatorname{Cl}(X - B) = X - B$ and $B \subseteq \beta\gamma\operatorname{Cl}(B) \subseteq X - A$. Hence by given part (1), A and B are β-γ-separated.

3. Since A and B are β-γ-open, it follows that $X - A$ and $X - B$ are β-γ-closed. Also, $H \subseteq X - B$ means that $X - B$ is equal to $\beta\gamma\operatorname{Cl}(H) \subseteq \beta\gamma\operatorname{Cl}(X - B)$. Then because $\beta\gamma\operatorname{Cl}(H) \cap B = \emptyset$ and it follows that $\beta\gamma\operatorname{Cl}(H) \subseteq G = \emptyset$. Similarly if $H \cap \beta\gamma\operatorname{Cl}(G) = \emptyset$. i.e. H and G are β-γ-separated. $(X - A)$ and $(X - B)$ are β-γ-open if and only if A and B are β-γ-closed. Consequently, H and G are β-γ-separated. ∎

Theorem 3.5. *If there is a set U and set V in $\beta O(X)\gamma$ such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \emptyset$ and $B \subseteq U = \emptyset$, then the subsets A and B of a space X are β-γ-separated and conversely.*

Proof. We have $\beta\gamma\operatorname{Cl}(A) \cap B = \emptyset$ and $\beta\gamma\operatorname{Cl}(B) \cap A = \emptyset$ as A and B are β-γ-separated sets. Therefore the sets $V = X - \beta\gamma\operatorname{Cl}(A)$ and $U = X - \beta\gamma\operatorname{Cl}(B)$ are β-γ-open, such that $A \subseteq U$, $B \subseteq V$ with $A \cap V = \emptyset$ and $B \cap U = \emptyset$. On the other hand, if U and V exists in $\beta O(X)\gamma$ such that $A \subseteq U$, $B \subseteq V$, $A \cap V = \emptyset$ and $B \cap U = \emptyset$, then $X - V$ and $X - U$ are β-γ-closed and $\beta\gamma\operatorname{Cl}(A) \subseteq X - V \subseteq X - B$ and $\beta\gamma\operatorname{Cl}(B) \subseteq X - U \subseteq X = A$ respectively. Hence $\beta\gamma\operatorname{Cl}(A) \cap B = \emptyset$ and $\beta\gamma\operatorname{Cl}(B) \cap A = \emptyset$ were determined. ∎

Theorem 3.6. *In any topological space (X, τ), the following statements are equivalent:*

(1) *$\emptyset$ and X are the only sets which are both β-γ-open and β-γ-closed in X.*

(2) *X is not the union of two disjoint non empty β-γ-open sets.*

(3) *X is not the union of two disjoint non empty β-γ-closed sets.*

(4) *X is not the union of non empty β-γ-separated sets.*

Proof. (1)⇒(2): It is assumed that (2) is not true. Given that A and B are disjoint, non empty and are β-γ-open so let $X = A \cup B$. So $X - A = B$ is a nonempty set which is proper β-γ-open. It follows that (1) is not true, since A is non empty proper β-γ-open and β-γ-closed in X.

(2)⇒(3): Clear.

(3)⇒(4): If (4) is false, then $X = A \cup B$, where A and B are nonempty and β-γ-separated sets. Then $\beta\gamma\operatorname{Cl}(B) \cap A = \emptyset$ implies $\beta\gamma\operatorname{Cl}(B) \subseteq B$ and hence B is β-γ-closed. Similarly A is also β-γ-closed. i.e. (3) is false.

(4)⇒(1). Assuming that (1) is not true, assume that there is a non empty proper subset A of X, that is both β-γ-open and β-γ-closed. If A and B are β-γ-separated and $X = A \cup B$, then (4) is not true since. Then $B = X - A$ is non empty, β-γ-open and β-γ-closed. ∎

Definition 3.7. *The condition that a subset C of a space X is β-γ-disconnected is that $C = A \cup B$ or that C is β-γ-connected if there exists non empty β-γ-separated sets A and B of X such that $C = A \cup B$.*
A pair of sets A and B is referred to as a β-γ-disconnection of C if C is β-γ-disconnected.

In Example 3.3, X is β-γ-disconnected, since $\{c\}$ and $\{a, b\}$ are β-γ-separated sets and hence there union is X.

Example 3.8. *Assume X is a set comprising $\{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let γ be an operation on $\beta O(X)$ such that $A^\gamma = A$ if $b \in A$, $A^\gamma = X$, if $b \notin A$. So X is β-γ-connected, because there is no non empty pair A, B of non empty β-γ-separated subsets of X such that $X = A \cup B$.*

Remark 3.9. (1) *Every indiscrete space is β-γ-connected.*

(2) *Every discrete space with more than one point is β id-disconnected.*

(3) *A space X is β-γ-connected if any of the conditions (1) to (4) in Theorem 3.6 holds.*

(4) *A space X is β-γ-disconnected if $X = A \cup B$, satisfies any one of the following statements:*

(i) *A and B are disjoint, non-empty and β-γ-open sets.*

(ii) *A and B are disjoint, non-empty and β-γ-closed sets.*

(iii) *A and B are disjoint, non-empty and β-γ-separated sets.*

Theorem 3.10. *If there is non empty proper subset A of X which is both β-γ-open and β-γ-closed in X, then we say that space X is β-γ-disconnected.*

Proof. Follows from above remarks. ■

Theorem 3.11. *Every non empty proper subset of X must have a non-empty β-γ-boundary for a space X to be β-γ-connected.*

Proof. Let A be nonempty proper subset of X with $\beta\gamma\,\mathrm{Bd}(A) = \emptyset$. Then $\beta\gamma\,\mathrm{Cl}(A) = \beta\gamma\,\mathrm{Int}(A) \cup \beta\gamma\,\mathrm{Bd}(A)$ implies $\beta\gamma\,\mathrm{Cl}(A) = \beta\gamma\,\mathrm{Int}(A)$. Because A is both β-γ-open and β-γ-closed and $\beta\gamma\,\mathrm{Int}(A) \subseteq A$ is nonempty proper subset of X, by Theorem 3.10, X is β-γ-disconnected, which is a contradictory. Due to this, A has a non-empty β-γ-boundary. On the other hand, let X be β-γ-disconnected. Next, by Theorem 3.10, X contain a valid subset A that is non empty proper subset and is both β-γ-open and β-γ-closed. i.e. $\beta\gamma\,\mathrm{Cl}(A) = A$, $\beta\gamma\,\mathrm{Cl}(X - A) = X - A$ and $\beta\gamma\,\mathrm{Cl}(A) \cap \beta\gamma\,\mathrm{Cl}(X - A) = \beta\gamma\,\mathrm{Bd}(A) = \emptyset$, i.e. A has empty β-γ-boundary, which is again a contradiction. Hence X is β-γ-connected. ■

Lemma 3.12. *Suppose M and N are β-γ-separated subsets of X. If $C \subseteq M \cup N$ and C is β-γ-connected, then $C \subseteq M$ or $C \subseteq N$.*

Proof. Since $C \cap M \subseteq M$ and $C \cap N \subseteq N$ then $C \cap M$ and $C \cap N$ are β-γ-separated sets. Also $C = C \cap (M \cup N) = (C \cap M) \cup (C \cap N)$. Since C is β-γ-connected, so $(C \cap M)$ and $(C \cap N)$ cannot form a β-γ-disconnection of C. Therefore, either $C \cap M = \emptyset$, so $C \subseteq N$ or $C \cap M = \emptyset$ or $C \subseteq M$. ■

Theorem 3.13. *Suppose C and C_i $(i \in I)$ are β-γ-connected but not β-γ-separated subsets of X, then $S = C \cup C_i$ is β-γ-connected for each i.*

Proof. Where M and N are β-γ-separated, then $C \cup C_i$ is equal to $S = M \cup N$ if S is β-γ-disconnected. Either $C \subseteq M$ or $C \subseteq N$ and $C_i \subseteq M$ or $C_i \subseteq N$ are required by Lemma 3.12. Assume $C \subseteq M$ without sacrificing generality. A contradiction would occur if for some i, $C_i \subseteq N$, and C and C_i would be β-γ-separated. Therefore every $C_i \subseteq M$. Therefore $N = \emptyset$. i.e. M and N are not β-γ-connected in S. ■

Following the same line of thinking, we may deduce that for each i, either $C_i \subseteq M$ or $C_i \subseteq N$. But if some $C_i \subseteq N$, then C and C_i would be β-γ-separated. Therefore a contradiction would occur, if for any i, every $C_i \subseteq M$, since M, N are not a β-γ-disconnection of S, we say that $N = \emptyset$.

Corollary 3.14. *Assume that, C_i is β-γ-connected subset of X for every $i \in I$, and if C_i, share a point, then $\bigcup_{i\in I} C_i$ is β-γ-connected.*

Proof. With $I = \emptyset$, the set $\cup C_i = \emptyset$ is obviously β-γ-connected for all i in I. In Theorem 3.13, choose $i_0 \in I$ and C_{i0} be the central set C. If I is not equal to $\emptyset$. It is not true that $C_i \cap C_{i0}$ equal to $\emptyset$ for every $i \in I$. So C_i and C_{i0} are not β-γ-separated. The β-γ-connectedness of $\cup\{C_i : i \in I\}$ is shown by Theorem 3.13. ■

Corollary 3.15. *Suppose that for all $x, y \in X$, there exists a β-γ-connected set $C_{xy} \subseteq X$ with $x, y \in C_{xy}$. Then X is β-γ-connected.*

Proof. Obviously $X = \emptyset$ is β-γ-connected. By hypothesis, there exists a β-γ-connected set C_{ay} that contains both a and y for any $y \in X$ where $X \neq \emptyset$, and let $a \in X$ be a fixed element. The β-γ-connection of $X = \cup\{C_{ay} : y \in X\}$ is established by Corollary 3.14. ■

Corollary 3.16. *Let C be a β-γ-connected subset of X and $A \subseteq X$. If $C \subseteq A \subseteq \beta\gamma\,\mathrm{Cl}(C)$, then A is also β-γ-connected.*

Proof. If $a \in \beta\gamma\,\mathrm{Cl}(C)$ is true for all $a \in A$, then $\{a\} \cap \beta\gamma\,\mathrm{Cl}(C)$ is not equal to $\emptyset$. C and $\{a\}$ are not β-γ-separated. Thus, $A = C \cup \{\{a\} : a \in A\}$ is β-γ-connected by Theorem 3.13. ■

Remark 3.17. *In particular, the β-γ-closure of a β-γ-connected set is β-γ-connected.*

Corollary 3.18. *If for every β-δ-open set V of Y, $f^{-1}(V)$ is β-γ-open in X, then function $f : X \to Y$ is $\beta(\gamma, \delta)$-continuous.*

Proof. Assume that V be β-δ-open in Y. Then $Y - V$ is a set in Y that is β-δ-closed. Following the reasoning in ([8, Theorem 16(ii)]), the set $f^{-1}(Y - V)$ is β-γ-closed set in X. The reason for this is because $f^{-1}(V)$ is β-γ-open set in X, since $f^{-1}(Y - V) = X - f^{-1}(V)$.

On the other side, consider $x \in X$ and V as a β-δ-open subset of Y that contains $f(x)$. Then $x \in f^{-1}(V)$. Given x and $f(f^{-1}(V)) \subseteq V$. It may be inferred that $f^{-1}(V)$ is β-γ-open in X. Hence f is $\beta(\gamma, \delta)$-continuous. ■

Theorem 3.19. *If $f : (X, \tau) \to (Y, \sigma)$ is onto $\beta(\gamma, \delta)$-continuous function and X is β-γ-connected, then Y is β-δ-connected.*

Proof. Y is β-δ-disconnected if and only if A and B give a β-δ-disconnection of Y. A and B are both β-δ-open sets according to Remark 3.9. Both $f^{-1}(A)$ and $f^{-1}(B)$ are both non empty β-γ-open set in X because f is $\beta(\gamma, \delta)$-continuous, according to Corollary 3.18. Now, for function f, $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X$. Remark 3.9 states that $f^{-1}(A)$ and $f^{-1}(B)$ are two β-γ-disconnections of X. Then Y is β-γ-disconnected is contradicted by this. ■

Theorem 3.20. *Let $f : (X, \tau) \to (Y, \sigma)$ be an injective function. Then the following are equivalent:*

(i) *f is $\beta(\gamma, \delta)$-continuous.*

(ii) *$f^{-1}(V) \subseteq \beta\gamma\,\mathrm{Int}(f^{-1}(V))$ for every subset β-γ-open set V of Y.*

(iii) *$f(\beta\gamma\,\mathrm{Cl}(A)) \subseteq \beta\delta\,\mathrm{Cl}(f(A))$ for every subset A of X.*

(iv) *$\beta\gamma\,\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(\beta\delta\,\mathrm{Cl}(B))$ for every subset B of Y.*

(v) *$f^{-1}(\beta\delta\,\mathrm{Int}(B)) \subseteq \beta\gamma\,\mathrm{Int}(f^{-1}(B))$ for every subset B of Y.*

Proof. (i)⇒(ii): Let $x \in f^{-1}(V)$, where V is a β-δ-open subset of Y. Then $f(x) \subseteq V$. Since f is $\beta(\gamma,\delta)$-continuous, there exists β-γ-open set U of X containing x such that $f(U) \subseteq V$ and so $U \subseteq f^{-1}(V)$, this implies that $x \in \beta\gamma\operatorname{Int}(f^{-1}(V))$. Thus $f^{-1}(V) \subseteq \beta\gamma\operatorname{Int}(f^{-1}(V))$ for every β-γ-open subset V of Y.

(ii)⇒(iii): Let A be any subset of X and $f(x) \notin \beta\delta\operatorname{Cl}(f(A))$, then by Theorem 2.2(i), there exists a β-γ-open set V of Y containing $f(x)$ such that $V \cap f(A) = \emptyset$ and hence $f^{-1}(V) \cap A = \emptyset$. Also $f(x) \cap V$ implies $x \in f^{-1}(V)$, which implies $x \in \beta\gamma\operatorname{Int}(f^{-1}(V))$. Hence, there exists a β-γ-open set U of X containing x such that $U \subseteq f^{-1}(V)$. Then $U \cap A = \emptyset$ and so $x \notin \beta\gamma\operatorname{Cl}(A)$ and hence $f(x) \notin (\beta\gamma\operatorname{Cl}(A))$, which is a contradiction. Thus $f(\beta\gamma\operatorname{Cl}(A)) \subseteq \beta\delta\operatorname{Cl}(f(A))$.

(iii)⇒(iv): Let B be any subset of Y. Since $f(f^{-1}(B)) \subseteq B$, so we have $\beta\delta\operatorname{Cl}(f(f^{-1}(B)) \subseteq \beta\delta\operatorname{Cl}(B)$. Also $f^{-1}(B) \subseteq X$. Then by (iii), we have $f(\beta\gamma\operatorname{Cl}(f^{-1}(B)) \subseteq (\beta\delta\operatorname{Cl} f(f^{-1}(B)) \subseteq \beta\delta\operatorname{Cl}(B)$. Thus $\beta\gamma\operatorname{Cl}(f^{-1}(B)) \subseteq f^{-1}(\beta\delta\operatorname{Cl}(B))$.

(iv)⇒(v): Let B be any subset of Y and $x \in f^{-1}(\beta\delta\operatorname{Int}(B))$. Then by Theorem 2.2(ii), $x \notin X - f^{-1}(\beta\gamma\operatorname{Int}(B)) = f^{-1}(\beta\gamma\operatorname{Cl}(Y-B)$. By (iv), $x \in (\beta\gamma\operatorname{Cl}(f^{-1}(Y-B)) = X - (\beta\gamma\operatorname{Int}(f^{-1}(B))$ and hence $x \in \beta\gamma\operatorname{Int} f^{-1}(B)$. Thus $f^{-1}(\beta\delta\operatorname{Int}(B)) \subseteq \beta\gamma\operatorname{Int}(f^{-1}(B))$.

(v)⇒(i): Let $x \in X$ and V be any β-δ-open set of Y containing $f(x)$. Since $V \cap (Y-V) = \emptyset$, we have $f(x) \notin \beta\gamma\operatorname{Cl}(Y-V) = Y - \beta\gamma\operatorname{Int}(V))$ and hence $f(x) \notin \beta\gamma\operatorname{Cl}(Y-B) = Y - \beta\gamma\operatorname{Int}(V)$ and hence $f(x) \notin \beta\gamma\operatorname{Int}(f^{-1}(V))$. ■

Corollary 3.21. *Let $f : X \to Y$ be a $\beta(\gamma,\delta)$-continuous and injective function. If K is β-γ-connected in X, then $f(K)$ is β-δ-connected in Y.*

Proof. Suppose that $f(K)$ is β-δ-disconnected in Y. Then there exists two β-δ-separated sets P and Q of Y such that $f(K) = C \cup Q$. Let $A = K \cap f^{-1}(P)$ and $B = K \cap f^{-1}(Q)$. Since $f(K) \cap P$ is not empty, so is $K \cap f^{-1}(P)$. Hence A and B are non empty. Now $A \cup B = (K \cap f^{-1}(P)) \cup (K \cap f^{-1}(Q)) = K \cap (f^{-1}(P) \cup f^{-1}(Q)) = K \cap (f^{-1}(P \cup Q)) = K \cap (f^{-1}(f(K)) = K$. Since f is $\beta(\gamma,\delta)$-continuous, then by Theorem 3.20, $\beta\gamma\operatorname{Cl}(f^{-1}(Q)) \subseteq f^{-1}(\beta\delta\operatorname{Cl}(Q))$ and this together with $B \subseteq f^{-1}(Q)$, implies $\beta\delta\operatorname{Cl}(B) \subseteq f^{-1}(\beta\gamma\operatorname{Cl}(Q))$. Since $P \cap \beta\gamma\operatorname{Cl}(Q) = \emptyset$, $A \cap \beta\gamma\operatorname{Cl}(B) \subseteq A \cap f^{-1}(\beta\gamma\operatorname{Cl}(Q)) \subseteq f^{-1}(P) \cap f^{-1}(\beta\gamma\operatorname{Cl}(Q)) = \emptyset$. i.e. $A \cap \beta\gamma\operatorname{Cl}(B) = \emptyset$. Similarly $B \cap \beta\gamma\operatorname{Cl}(A) = \emptyset$. Thus A and B are β-γ-separated, therefore K is a β-γ-disconnected, a contradiction. Hence $f(K)$ is β-δ-connected. ■

Theorem 3.22. *A space X is β-γ-disconnected if and only if there exists an $\beta(\gamma, id)$-continuous function from X onto discrete space $\{0,1\}$.*

Proof. Suppose that X is β-γ-disconnected. Then, there exists disjoint β-γ-open sets $G1$ and $G2$ of X such that $X = G1 \cup G2$. Define a function $f : X \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in G1, \\ 1 & \text{if } x \in G2. \end{cases}$$

Now,the only β id-open sets in $\{0,1\}$ are $\emptyset, \{0\}, \{1\}, \{0,1\}$. So, $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{0\}) = G_1$, $f^{-1}(\{1\}) = G_2$ and $f^{-1}(\{0,1\}) = X$, which are β-γ-open sets in X. Thus by Corollary 3.18, f is β id-continuous function from X onto discrete space $\{0,1\}$. Conversely, let the hypothesis holds and if possible suppose that X is β-γ-connected. Therefore by Theorem 3.22, $\{0,1\}$ is β id-connected which is a contradiction by Remark 3.17. So X must be β-γ-disconnected. ■

Theorem 3.23. *A space X is β-γ-connected if and only if every $\beta(\gamma, id)$-continuous function from space X to the discrete space $\{0,1\}$ is constant.*

Proof. Consider X be β-γ-connected and consider any β id-continuous function $f : X \rightarrow \{0,1\}$. Since the space $\{0,1\}$ is discrete, we may say that $\{y\}$ is both β id-open and β id-closed in space $\{0,1\}$. If we let $y \in f(X) \subseteq \{0,1\}$, then $\{y\} \subseteq \{0,1\}$. For any y in $Y f^{-1}(\{y\})$ is both β-γ-open and β-γ-closed in X according to Corollary 3.18 and ([8, Theorem 16(ii)]) since f is β id-continuous function. We may deduce that $f(x) = y$ for every $x \in X$ because $y \in f(X)$, so x is a function of $f^{-1}(\{y\})$. Therefore $f^{-1}(\{y\})$ does not include empty set. If $f^{-1}(\{y\})$ is not equal to X, then $f^{-1}(\{y\})$ is a non empty subset of X which is both β-γ-open and β-γ-closed in X. So there is a contradiction as, X is β-γ-connected. By Theorem 3.10. Therefore if $f^{-1}(\{y\}) = X$, then $f(X) = \{y\}$. This indicates that f is constant since for each $x \in X$, $f(x) = y$. ■

Definition 3.24. *A set C is called maximal β-γ-connected set if it is β-γ-connected and if D is β-γ-connected such that $C \subseteq D \subseteq X$, then $C = D$. A maximal β-γ-connected subset C of a space X is called a β-γ-component of X, if X itself β-γ-connected, then X is only β-γ-component of X.*

Theorem 3.25. *For β-γ-component of X containing x, for each $x \in X$, there is exactly one β-γ-component of X containing x.*

Proof. For any $x \in X$, let $C_x = \cup\{A : x \in A \subseteq X$ and A is β-γ-connected$\}$. Then $\{x\} \in C_x$, since C_x is union of β-γ-connected sets each containing x, is β-γ-connected by Corollary 3.15. If $C_x \subseteq D$ and D is β-γ-connected, then D was one of the sets A in the collection whose union defined C_x. So $D \subseteq C_x$ and therefore $C_x = D$. Therefore C_x is a β-γ-component of X containing x. ■

Corollary 3.26. *Two β-γ-components either are disjoint or coincide.*

Proof. Let C_x and C_y be two β-γ-components and C_x not equal to C_y. If they are not disjoint, let $p \in C_x \cap C_y$. Then by Corollary 3.14, $C_x \cup C_y$ would be a β-γ-connected set strictly larger then C_x. Therefore $C_x \cap C_y = \emptyset$. ■

Theorem 3.27. *Each β-γ-connected subset of X is contained in exactly one β-γ-component of X.*

Proof. Let A be a β-γ-connected subset of X which is not in exactly one β-γ-component of X. Suppose that $C1$ and $C2$ are β-γ-component of X such that, $A \subseteq C1$ and $A \subseteq C2$. Since $C1$ and $C2$ are not disjoint and by Corollary 3.26, $C1 \cup C2$ is another β-γ-connected subset which contain $C1$ and $C2$, a contradiction to the fact that $C1$ and $C2$, are β-γ-components. This proves that A is contained in exactly one β-γ-component of X. ■

Theorem 3.28. *A β-γ-component is a non empty β-γ-connected subset of X that is both β-γ-open and β-γ-closed.*

Proof. Assume that A be a β-γ-connected subset of X which is both β-γ-open and β-γ-closed. A is included in precisely one β-γ-component C of X, according to Theorem 3.27. It is contradictory because if A is proper subset of C, then equation $C = (C \cap A) \cup (C \cap (X - A))$ results in a β-γ-disconnection of C. Thus, $A = C$. ■

Theorem 3.29. *Every β-γ-component of X is β-γ-closed.*

Proof. Assume that C be a β-γ-component of X. according to comment 3.17, $\beta\gamma\,\mathrm{Cl}(C)$ is a β-γ-connected which appropriately includes the β-γ-component C of X. C is therefore β-γ-closed as $C = \beta\gamma\,\mathrm{Cl}(C)$. ■

Definition 3.30. *For every point $x \in X$ and every β-γ-open set U containing x, there exists a β-γ-open β-γ-connected set V such that $x \in V \subseteq U$, we say that X is said to be β-γ-locally connected at x.*

Theorem 3.31. *Let $f : X \rightarrow Y$ be a $\beta(\gamma,\delta)$-continuous, $\beta(\gamma,\delta)$-open and bijective. If X is β-γ-locally connected, then Y is β-δ-locally connected at x.*

Proof. By $y \in Y$, find an element $x \in X$ such that y is equal to $f(x)$. Let U be a β-δ-open set of Y that contains y. According to Corollary 3.18, $f^{-1}(U)$ is β-γ-open in X containing x, because f is $\beta(\gamma,\delta)$-continuous. There is a β-γ-open β-γ-connected set V that contains x such that $x \in V \subseteq f^{-1}(U)$ because X is β-γ-locally connected. This means that $f(x) \in f(V) \subseteq f(f^{-1}(U)) = U$ or $y \in f(V) \subseteq U$. The reason for $f(V)$ is also β-δ-open because f is $\beta(\gamma,\delta)$-open. In addition according to Corollary 3.21, $f(V)$ is β-γ-connected. This establishes that Y is β-δ-locally connected. ■

4. Concluding Remarks and Acknowledgements

Our research in this study focused on β-γ-connected and β-γ-locally connected spaces and we also presented the concept of β-γ-separated sets. There is much scope of further work based on operational approach and variants of open sets. The author would like to express their profound gratitude to the referees who helped me to enhance the paper quality and the findings.

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Approximate and exact solution of Korteweg de Vries problem using Aboodh Adomian polynomial method

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Abstract. This study introduce Aboodh Adomian polynomial Method (AAPM) to solve nonlinear third order KdV problems providing it approximate and exact solution. To get the approximate analytical answers to the issues, the Aboodh transform approach was used. Given that the Aboodh transform cannot handle the nonlinear elements in the equation, the Adomian polynomial was thought to be a crucial tool for linearizing the associated nonlinearities. All of the issues examined demonstrated the strength and effectiveness of the Adomian polynomial and Aboodh transform in solving various nonlinear equations when compared to other well-known methods. To show how this strategy may be applied and is beneficial, three cases were examined.
AMS Subject Classifications: 44A05, 44A20, 35Q53

Keywords: Aboodh transform, Adomian polynomial, Third order KDV, Nonlinear differential equations, Korteweg-De Vries

Contents

1. Introduction

In engineering, physics, and other fields of study, nonlinear models play a crucial role in explaining new phenomena. However, in certain situations, it could be difficult to provide an exact analytical solution for nonlinear problems [3]. To address nonlinear issues, a variety of numerical techniques were employed, and as these techniques improved, so did the analytical techniques. Particular focus has recently been paid to the merging of numerical and analytical methodologies. A technique for solving nonlinear differential equations in series is the homotopy approach, which was developed by He [9, 10]. Easy and straightforward execution are

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the technique's advantages. In engineering, physics, and other disciplines, nonlinear models play a crucial role in explaining new phenomena. However, it might occasionally be challenging to provide a precise analytical solution for nonlinear situations [3]. Numerous numerical techniques were employed to address nonlinear issues, and advancements in these techniques led to advancements in analytical techniques. Combining analytical and numerical approaches has drawn a lot of interest lately. Among these techniques is the homtopy approach, which He developed in order to solve nonlinear differential equations in series [9, 10]. The method's ease of use and effortlessness are advantages.

In the field of nonlinear physics, the KdV problem is a crucial PDE that appears while studying solitons and waves. The KdV equation, which bears the names of the Dutch scientists D.J. Korteweg and G. de Vries who first proposed it in 1895, provides a mathematical explanation for a variety of scientific phenomena, including shallow water waves and electrical pulses in nerve fibers. [6]. The term "soliton," referring to a solution of a non-linear PDE, was initially utilized by Zabusky and Kruskal [13].

A KdV problem in third order is expressed as [4]:

$$\phi_\tau + a\phi\phi_z + b\phi\phi_{zzz} = 0 \tag{1.1}$$

with

$$\phi\,(z,0) = \phi(z,\tau) \tag{1.2}$$

a and b	are arbitrary constants
ϕ_z	Partial derivatives w.r.t z
ϕ_τ	Partial derivatives w.r.t. τ

Various methods, including the combination of LTHPM [8], have been employed to obtain approximate analytical solutions and numerical results for KdV equations and other nonlinear PDE. Achieving exact solutions involved utilizing graphical representation for the KdV equation [4], as well as employing the homotopy perturbation method with Elzaki transform [5, 8] combined with Aboodh transform for approximate solutions of certain third-order KDV equations with initial conditions. Additionally, numerical techniques for partial differential equations [1], methods like the Adomian Polynomial and Elzaki Transform [11], NTHPM [2], and HPM using Mahgoub Transform [6] have been investigated as computational approaches for KdV problem on an unrestricted domain. The combination of Elzaki Transformation with Adomian Polynomial is also considered [11].

In this study, we want to improve the efficiency of the Aboodh transform method by its integration with the Adomian polynomial approach. The combined approach known as "Aboodh Transform and Adomian Polynomial (AAPM) for Solving Third Order Korteweg-De Vries (KDV) Equation" is used in this situation. Usually, this approach takes several stages to get an accurate result, presenting the outcome as an approximation analytic solution in a series structure.

2. Main Results

2.1. Aboodh Transform Method (ATM)

Differential equations are solved with the application of the Aboodh transform and some of its basic characteristics. For exponentially ordered functions, the Aboodh Transform is defined.

Definition 2.1. The Aboodh transform, defined on set A of functions, is the new term for the integral transform.

$$A = \{\phi(\tau)| \exists\, M, k_1, k_2 > 0, |\phi(\tau)| < Me^{-v\tau}\} \tag{2.1}$$

where $k_1, k_2 \subset M$. The symbol for the Aboodh transform is $\mathcal{A}[\phi(\tau)]$. The integral equation of the form is represented by .

$$\mathcal{A}[\phi(\tau)] = q(v) = \frac{1}{v}\int_0^\infty \phi(\tau)e^{-v\tau}d\tau, \quad \tau \geq 0,\ k_1 \leq v \leq k_2 \tag{2.2}$$

Definition 2.2. Aboodh Transform for function $\phi(\tau)$ of exponential order over the set of function is defined as

$$\mathcal{A}\left\{f : |\phi(\tau) < Me_j^k|\tau|,\ \text{if}\ \ \tau \in (-1)^j \times [0,\infty], \quad j = 1, 2, ...(M, k_1, k_2 > 0)\right\} \tag{2.3}$$

where $\phi(\tau)$ is denoted by

$$\mathcal{A}[\phi(\tau)] = \mathcal{H}(v)$$

and defined as

$$\mathcal{A}[\phi(\tau)] = \frac{1}{v}\int_0^\infty \phi(\tau)e^{-v\tau}d\tau = \mathcal{H}(v), \qquad t < 0,\ k_1 \leq v \leq k_2$$

2.2. Aboodh of basic functions

Using definition 2.2, one can show as follows.

1. Let $\phi(z,\tau) = e^{az+b\tau}$, then its Aboodh transform w.r.t τ is given by

$$\begin{aligned}\mathcal{A}\{e^{az+b\tau}\} &= \frac{1}{v}\int_0^\infty e^{az+b\tau}e^{-v\tau}d\tau \\ &= \frac{e^{az}}{v}\int_0^\infty e^{b\tau - v\tau}d\tau \\ &= \frac{e^{az}}{v}.\frac{-1}{b-v} = \frac{e^a z}{v(v-b)}, \quad v > b\end{aligned}$$

2. Let $\phi(z,\tau) = z^m\tau^n$, then its Aboodh transform w.r.t τ is given by

$$\begin{aligned}\mathcal{A}\{z^m\tau^n\} &= \int_0^\infty z^m\tau^n e^{-v\tau}d\tau \\ &= \frac{z^m}{v}\int_0^\infty \tau^n e^{-v\tau}d\tau = \frac{z^m}{v}\int_0^\infty \tau^n e^{-v\tau}d\tau \\ &= \frac{z^m}{v}.\frac{\Gamma(n+1)}{v^(n+1)} \\ &= \frac{n!z^m}{v^{n+2}}, \quad v > 0\end{aligned}$$

3. Let $\phi(z,\tau) = \sin(az + b\tau)$, then its Aboodh transform w.r.t τ is given by

$$\mathcal{A}\{\sin(az+b\tau) = \frac{1}{v}\int_0^\infty \sin(az+b\tau)e^{-v\tau}\}$$

Applying repeated integration by parts leads to

$$\begin{aligned}&= \mathcal{A}\{\sin(az+b\tau)\} = \frac{1}{v}\left(-\frac{e^{-v\tau}(v\sin(b\tau+az) + b\cos(b\tau+az))}{v^2+b^2}\right)\Bigg|_{t=0}^{\infty} \\ &= \frac{v\sin(az) + b\cos(az)}{v(v^2+b^2)}, \qquad v > 0\end{aligned}$$

4. Let $\phi(z,\tau) = \sinh(az + b\tau)$ then its Aboodh transform w.r.t τ is given by

$$\mathcal{A}\{\sinh(az + b\tau)\} = \frac{1}{v}\sinh(az + b\tau)e^{-v\tau}d\tau.$$

By using properties of hyperbolic functions and integration by parts, we have

$$\begin{aligned}\mathcal{A}\{\sinh(az + b\tau)\} &= \frac{1}{v}\int_0^\infty [\cosh(az)\sinh(b\tau) + \sinh(az)\cosh(b\tau)]e^{-v\tau}d\tau \\ &= \frac{1}{v}\left(-\frac{((v+b)e^{2(b\tau+az)} - v + b)e^{(v+b)\tau - az}}{2(v-b)(v+b)}\right)\Bigg|_{t=0}^{\infty} \\ &= \frac{v\sinh(az) + b\cos(az)}{v(v^2 - b^2)}, \quad |v| > |b|\end{aligned}$$

If $\mathcal{A}[\phi(\tau)] = q(v) = \frac{1}{v}\int_0^\infty \phi(\tau)e^{-v\tau}d\tau, \quad \tau \geq 0, \ k_1 \leq v \leq k_2$. Then, the Aboodh and Inverse Aboodh Transform of some Elementary functions are given below.

S/N	$\mathcal{A}^{-1}\{k(v)\} = \phi(\tau)$	$\mathcal{A}\phi(\tau) = k(v)$
1.	1	$\frac{1}{v^2}$
2.	τ^n	$\frac{n!}{v^{n+2}}$
3.	$e^{a\tau}$	$\frac{1}{v^2+a^2}$
4.	$\sin(a\tau)$	$\frac{a}{v(v^2+a^2)}$
5.	$\cos(a\tau)$	$\frac{1}{(v^2+a^2)}$
6.	$\sinh(a\tau)$	$\frac{a}{v(v^2-a^2)}$
7.	$\cosh(a\tau)$	$\frac{1}{v^2-a^2}$

Table 1: Aboodh $\{\mathcal{A}\}$ and inverse Aboodh Transform $\{\mathcal{A}^{-1}\}$ of some functions

also the Aboodh and Inverse Aboodh Transform of some derivatives is given below

S/N	$\mathcal{A}\{u(z,\tau)\} = k(v)$	$\mathcal{A}^{-1}\left[\frac{\partial^n u(z,\tau)}{\partial\tau^n}\right]$
1.	$\mathcal{A}[u(z,\tau)]$	$K(z,v)$
2	$\mathcal{A}\left[\frac{\partial u(z,\tau)}{\partial t}\right]$	$vK(z,v) - \frac{u(z,0)}{v}$
3.	$\mathcal{A}\left[\frac{\partial^2 u(z,\tau)}{d\tau}\right]$	$v^2\left[K(z,v) - u(z,0) - \frac{1}{v}u_t(z,0)\right]$
4.	$\mathcal{A}\left[\frac{\partial^n u(z,\tau)}{\partial\tau^n}\right]$	$v^n K(z,v) - \sum_{k=0}^{n-1}\frac{f^{(k)}(z,0)}{v^{2-n+k}}$

2.3. Properties of Aboodh Transform

The following properties of Aboodh transform are derived from the definition and which will be applied in the following chapter to solve the Schrödinger equation, both linear and nonlinear.

Lemma 2.3 (Linearity Property of Aboodh)**.** Let $\phi(z,\tau)$ and $\varphi(z,\tau)$ be any two functions whose Aboodh transform w.r.t exist. Then for arbitrary constants a and b, we have

$$\mathcal{A}\{a\phi(z,\tau) + b\varphi(z,\tau)\} = a\mathcal{A}\{\phi(z,\tau)\} + b\mathcal{A}\{\varphi(z,\tau)\} \tag{2.4}$$

Proof. By definition of Aboodh transform w.r.t τ, we obtain

$$\begin{aligned}\mathcal{A}\{a\phi(z,\tau)+b\varphi(z,\tau)\} &= \frac{1}{v}\int_0^\infty (a\phi(z,\tau)+b\varphi(z,\tau))e^{-v\tau}d\tau \\ &= a\left(\frac{1}{v}\int_0^\infty (\phi(z,\tau))e^{-v\tau}d\tau\right)+b\left(\frac{1}{v}\int_0^\infty \varphi(z,\tau)e^{-v\tau}d\tau\right) \\ &= a\mathcal{A}\{\phi(z,\tau)\}+b\mathcal{A}\{\varphi(z,\tau)\}\end{aligned}$$

■

2.4. Aboodh Adomian Polynomial Method (AAPM)

This study's main idea is to demonstrate the Adomian polynomial method with the Aboodh Transform by applying it to a broad category of nonlinear partial differential equations.

$$\frac{\partial^q \phi(z,\tau)}{\partial \tau^q}+\mathbb{R}\phi(z,\tau)+\mathbb{N}\phi(z,\tau)=f(z,\tau) \tag{2.5}$$

where $q=1,2,3,\cdots$
with

$$\frac{\partial^{q-1}\phi(z,\tau)}{\partial \tau^{q-1}}(z,0)=g_{q-1}(x) \tag{2.6}$$

Taking Aboodh transform of (2.5)

$\left.\frac{\partial^q\phi(z,\tau)}{\partial\tau^q}\right\vert_{\tau=0}$	q^{th} order partial derivative of $\phi(z,\tau)$
$\mathbb{R}$	linear term
$\mathbb{N}$	nonlinear terms
$\phi(z,\tau)$	represents the source term.

$$\mathcal{A}\left[\frac{\partial^q \phi(z,\tau)}{\partial \tau^q}+R\phi(z,\tau)+N\phi(z,\tau)=f(z,\tau)\right] \tag{2.7}$$

Applying Aboodh linearity property to (2.7)

$$\mathcal{A}\left[\frac{\partial^q \phi(z,\tau)}{\partial \tau^q}\right]+\mathcal{A}[R\phi(z,\tau)]+\mathcal{A}[N\phi(z,\tau)]=\mathcal{A}[f(z,\tau)] \tag{2.8}$$

$$\mathcal{A}\left[\frac{\partial^q \phi(z,\tau)}{\partial \tau^q}\right]=\frac{\mathcal{A}[\phi(z,\tau)]}{v^q}-\sum_{k=0}^{q-1}v^{2-w+k}\frac{\partial^k\phi(z,0)}{\partial\tau^k} \tag{2.9}$$

Substituting equation (2.9) into (2.8)

$$\frac{\mathcal{A}[\phi(z,\tau)]}{v^q}-\sum_{k=0}^{q-1}v^{2-w+k}\frac{\partial^k\phi(z,0)}{\partial\tau^k}+\mathcal{A}[R\phi(z,\tau)]+\mathcal{A}[N\phi(z,\tau)]=\mathcal{A}[f(z,\tau)] \tag{2.10}$$

(2.11)

This result into

$$\frac{\mathcal{A}[\phi(z,\tau)]}{v^q}=\mathcal{A}[f(z,\tau)]+\sum_{k=0}^{q-1}v^{2-w+k}\frac{\partial^k\phi(z,0)}{\partial\tau^k}-\{\mathcal{A}[R\phi(z,\tau)]+\mathcal{A}[N\phi(z,\tau)]\} \tag{2.12}$$

By simplification, equation (2.12) becomes

$$\mathcal{A}\left[\phi\left(z,\tau\right)\right]=v^{q}\left[\mathcal{A}\left[f\left(z,\tau\right)\right]\right]+\sum_{k=0}^{q-1}v^{2+k}\frac{\partial^{k}\phi\left(z,0\right)}{\partial\tau^{k}}-v^{q}\left\{\mathcal{A}\left[R\phi\left(z,\tau\right)\right]+\mathcal{A}\left[N\phi\left(z,\tau\right)\right]\right\} \tag{2.13}$$

Taking Aboodh inverse transform of(2.13), gives

$$\mathcal{A}^{-1}\left[\phi\left(z,\tau\right)\right]=\mathcal{A}^{-1}\left[v^{q}\mathcal{A}\left[f\left(z,\tau\right)\right]+\sum_{k=0}^{q-1}v^{2+k}\frac{\partial^{k}\phi\left(z,0\right)}{\partial\tau^{k}}\right]-\mathcal{A}^{-1}\left[v^{q}\left\{\mathcal{A}\left[R\phi\left(z,\tau\right)\right]+\mathcal{A}\left[N\phi\left(z,\tau\right)\right]\right\}\right] \tag{2.14}$$

$$\phi\left(z,\tau\right)=\mathcal{A}^{-1}\left[v^{q}\mathcal{A}\left[f\left(z,\tau\right)\right]+\sum_{k=0}^{q-1}v^{2+k}\frac{\partial^{k}\phi\left(z,0\right)}{\partial\tau^{k}}\right]-\mathcal{A}^{-1}\left[v^{q}\left\{\mathcal{A}\left[R\phi\left(z,\tau\right)\right]+\mathcal{A}\left[N\phi\left(z,\tau\right)\right]\right\}\right] \tag{2.15}$$

Equation (2.15) is expressed below as

$$\phi\left(z,\tau\right)=F(z,\tau)-\mathcal{A}^{-1}\left[v^{q}\left\{\mathcal{A}\left[R\phi\left(z,\tau\right)\right]+\mathcal{A}\left[N\phi\left(z,\tau\right)\right]\right\}\right] \tag{2.16}$$

$F(z,\tau)$ is obtained from the initial conditions given. The result obtained in (2.16) is

$$\phi\left(z,\tau\right)=\sum_{r=0}^{\infty}\phi_{r}(z,\tau) \tag{2.17}$$

The non-linearity in the equation can be simplified using Adomian polynomial as

$$N\phi\left(z,\tau\right)=\sum_{r=0}^{\infty}A_{r} \tag{2.18}$$

Where A_r represents the Adomian polynomials. It is obtained using the expression in (2.19)

$$A_{r}=\frac{1}{r!}\frac{d^{r}}{d\lambda^{r}}f\left[\sum_{i=0}^{\infty}\lambda^{i}\phi_{i}\right]_{\lambda=0}\quad r=0,\ 1,\ \cdots \tag{2.19}$$

Substituting equation (2.18) and (2.19) into (2.17) leads to

$$\sum_{r=0}^{\infty}\phi_{r}\left(z,\tau\right)=F\left(z,\tau\right)-\mathcal{A}^{-1}\left[v^{q}\left\{\mathcal{A}\left[R\sum_{r=0}^{\infty}\phi\left(z,\tau\right)\right]+\mathcal{A}\left[N\phi\left(z,\tau\right)\right]+\mathcal{A}\left[\sum_{r=0}^{\infty}A_{r}\right]\right\}\right] \tag{2.20}$$

From equation $\phi_0\left(z,\tau\right)=F\left(z,\tau\right)$. Thus, the recursive expression is hereby obtained as

$$\phi_{r+1}=-\mathcal{A}^{-1}\left[v^{q}\left\{\mathcal{A}\left[R\sum_{r=0}^{\infty}\phi\left(z,\tau\right)\right]+\mathcal{A}\left[N\phi\left(z,\tau\right)\right]+\mathcal{A}\left[\sum_{r=0}^{\infty}A_{r}\right]\right\}\right],\quad r\geq 0, \tag{2.21}$$

With truncated series, one can approximate the analytical answer $\phi(z,\tau)$.

$$\phi\left(z,\tau\right)=\lim_{r\to\infty}\sum_{r=0}^{N}\ \phi_{r}\left(z,\tau\right) \tag{2.22}$$

3. Application

In this section, we wll work through the examples that follow to demonstrate the Adomian polynomial method to illustrate how well the third-order nonlinear KDV problems may be solved by using the Aboodh Adomian Polynomial Method..

Example 3.1. *Examine the nonlinear KDV problem [6]*

.

$$\phi_\tau - 6\phi\phi_z + \phi_{zzz}(z,\tau) = 0, \tag{3.1}$$

with

$$\phi(z,0) = 6x \tag{3.2}$$

Applying Aboodh to both sides of the equation (3.1)

$$\mathcal{A}[\phi_\tau] = -\mathcal{A}[\phi_{zzz}(z,\tau) - 6\phi\phi_z] \tag{3.3}$$

Making use of the Aboodh differential properties, equation (3.3) becomes

$$v\mathcal{A}[\phi(z,\tau)] - \frac{1}{v}\phi(z,0) = -\mathcal{A}[\phi_{zzz}(z,\tau) - 6\phi\phi_z] \tag{3.4}$$

Applying the initial condition Equation (3.2) on Equation (3.4), we obtain

$$\mathcal{A}[\phi(z,\tau)] = \frac{6z}{v^2} - \left[\frac{1}{v}\mathcal{A}[\phi_{zzz} - 6\phi\phi_z]\right] \tag{3.5}$$

Equation (3.5), when transformed using the inverse Aboodh Transform, yields

$$\phi(z,\tau) = 6z - \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{zzz} - 6\phi\phi_z]\right] \tag{3.6}$$

$$\phi_0 = 6x \tag{3.7}$$

The following is the recursive relation:

$$\phi_{r+1}(z,\tau) = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6A_r - \phi_{rzzz}]\right] \tag{3.8}$$

Where A_r is the Adomian polynomial.
Let the representation of the nonlinear term be:

$$A_r = \frac{1}{r!}\frac{d^r}{d\lambda^r} f\left[\sum_{i=0}^{\infty}\lambda^i\phi_i\right]_{\lambda=0} \tag{3.9}$$

By using equation (3.9), we obtain

$$\begin{aligned}
A_0 &= \phi_0[\phi_{0z}] \\
A_1 &= \phi_1[\phi_{0z}] + \phi_0\phi_{1z} \\
A_2 &= \phi_2[\phi_{0z}] + \phi_1[\phi_{1z}] + \phi_0[\phi_{2z}]
\end{aligned}$$

From Equation (3.8)
When $r = 0$, $\phi_1 = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6A_0 - \phi_{0zzz}]\right]$

$$\phi_1 = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6\phi_0\phi_{0z} - \phi_{0zzz}]\right]$$
$$\phi_1 = 216z\tau$$

When $r = 1$, $\phi_2 = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6A_1 - \phi_{1zzz}\right]$

$$\phi_2 = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}\left[6\phi_1\phi_{0z} + 6w_0\phi_{1z} - \phi_{1zzz}\right]\right]$$
$$\phi_2 = \frac{1552}{2}z\tau^2$$
$$\phi_2 = 7776z\tau^2$$
$$r = 2,\ \phi_3 = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}\left[6A_2 - \phi_{2zzz}\right]\right]$$
$$\phi_3 = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6\phi_2\phi_{0z} + 6w_1\phi_{1z} + 6\phi_0\phi_{2z} - \phi_{2zzz}]\right]$$
$$\phi_3 = 419904z\tau^3$$

The approximate series solution is:

$$\phi\left(z,\tau\right) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \cdots$$
$$\phi\left(z,\tau\right) = 6z + 216z\tau + 7776z\tau^2 + 419904z\tau^3 + \cdots$$

$$\phi\left(z,\tau\right) = 6z(1 + 36\tau + (36\tau)^2 + (36\tau)^3 + \cdots) \tag{3.10}$$

Equation (3.10) may be expressed in closed form using Taylor's series.:

$$\phi\left(z,\tau\right) = \frac{6z}{1 - 36\tau}\ ,\ |36\tau| < 1 \tag{3.11}$$

The solution obtained in equation (3.10) is in good agreement with the result obtained by Mahgoub Transform method [6].

Example 3.2. *Examine the nonlinear KDV problem [2]*

$$\phi_\tau + \phi\phi_z + \phi_{zzz}\left(z,\tau\right) = 0 \tag{3.12}$$

With

$$\phi\left(z,0\right) = 1 - z \tag{3.13}$$

Applying Aboodh to both sides of the equation (3.12).

$$\mathcal{A}\left[\phi_\tau\right] = -\mathcal{A}[\phi_{zzz}\left(z,\tau\right) + \phi\phi_z] \tag{3.14}$$

Making use of the Aboodh differential properties, equation (3.14) becomes

$$v\mathcal{A}\left[\phi(z,\tau)\right] - \frac{1}{v}\phi(z,0) = -\mathcal{A}[\phi_{zzz}\left(z,\tau\right) + \phi\phi_z] \tag{3.15}$$

Applying the initial condition equation (3.13) on equation (3.15) we obtain,

$$\mathcal{A}\left[\phi(z,\tau)\right] = \frac{z-1}{v^2} - \left[\frac{1}{v}\mathcal{A}[\phi_{zzz}\left(z,\tau\right) + \phi\phi_z]\right] \tag{3.16}$$

Now, we use the Aboodh inverse Transform of (3.16).Thus,

$$\phi(z,\tau) = (z-1) - \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{zzz}\left(z,\tau\right) + \phi\phi_z]\right] \tag{3.17}$$

$$\phi_0 = 1 - z \tag{3.18}$$

The recursive expression is can now be written as

$$\phi_{r+1} = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{rzzz}\left(z,\tau\right) + A_r]\right] \tag{3.19}$$

where the Adomian polynomial denoted by A_r is used to decompose the nonlinear terms.

$$A_r = \frac{1}{r!}\frac{d^r}{d\lambda^r} f\left[\sum_{i=0}^{\infty}\lambda^i\phi_i\right]_{\lambda=0} \tag{3.20}$$

The nonlinear term is represented by

$$f\left(u\right) = \phi\phi_z \tag{3.21}$$

By using equation (3.20) , we obtain

$$A_0 = \phi_0[\phi_{0z}]$$
$$A_1 = \phi_1\left[\phi_{0z}\right] + \phi_0\phi_{1z}$$
$$A_2 = \phi_2\left[\phi_{0z}\right] + \phi_1\left[\phi_{1z}\right] + \phi_0\left[\phi_{2z}\right]$$

From equation (3.19)

When $r = 0$, we obtain

$$\phi_1 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{0zzz}(z,\tau) + A_0]\right]$$
$$\phi_1 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[0 + (1-z)(-1)]\right]$$
$$\phi_1 = (1-z)\tau$$
$$r = 1,\ \ w_2 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{0zzz}(z,\tau) + A_1]\right]$$
$$\phi_2 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{1zzz}(z,\tau) + \phi_1\phi_{0z} + \phi_0\phi_{1z}]\right]$$
$$\phi_2 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[0 + (1-z)\tau(-1) + (1-z)(-\tau)]\right]$$
$$\phi_2 = (1-z)\tau^2$$
$$r = 2,\ \ w_3 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{2zzz}(z,\tau) + A_2]\right]$$
$$w_3 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{2zzz}(z,\tau) + \phi_0\phi_{2z} + \phi_1\phi_{1z} + \phi_2\phi_{0z}]\right]$$
$$w_3 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[0 + (1-z)\left(-\tau^2\right) + (1-z)\left(\tau^2\right) + (1-z)\left(\tau^2\right)(-1)]\right]$$
$$\phi_3 = (1-z)\tau^3$$
$$\phi(z,\tau) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \cdots$$

$$\phi(z,\tau) = (1-z) + (1-z)\tau + (1-z)\tau^2 + (1-z)\tau^3 + \cdots \tag{3.22}$$

$$\phi(z,\tau) = (1-z) + (1 + \tau + \tau^2 + \tau^3 + \cdots) \tag{3.23}$$

Applying Taylor's series, equation (3.23) is expressed in exact form as:

$$\phi(z,\tau) = \frac{1-z}{1-\tau},\ \ |\tau| < 1 \tag{3.24}$$

The solution obtained in equation (3.24) is the same as with the result obtained by Natural Transform and Homotopy Methods [2].

Example 3.3. *Examine the nonlinear KDV problem [11]*

$$\phi_\tau - 6\phi\phi_z + \phi_{zzz} = 0 \tag{3.25}$$

With

$$\phi(z,0) = \frac{2}{(z-3)^2} \tag{3.26}$$

Taking the Aboodh of (3.25)

$$\mathcal{A}[\phi_\tau] = \mathcal{A}[6\phi\phi_z - \phi_{zzz}] \tag{3.27}$$

Using the Aboodh differential properties, we obtain

$$v\mathcal{A}[\phi_\tau] - \frac{1}{v}\phi(z,0) = \mathcal{A}[6\phi\phi_z - \phi_{zzz}] \tag{3.28}$$

putting equation (3.25) into equation (3.28), we get

$$\mathcal{A}\left[\phi(z,\tau)\right]=\frac{2}{v(z-3)^{2}}+\left[\frac{1}{v}\mathcal{A}[6\phi\phi_{z}-w_{zzz}\left(z,\tau\right)]\right] \tag{3.29}$$

Now, we take the Aboodh inverse of (3.29). We have

$$\phi\left(z,\tau\right)=\frac{2}{v\left(z-3\right)^{2}}+\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6\phi\phi_{z}-w_{zzz}\left(z,\tau\right)]\right] \tag{3.30}$$

$$\phi_{0}=\frac{2}{\left(z-3\right)^{2}} \tag{3.31}$$

The recursive relation is given as

$$\phi_{r+1}=\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[A_{r}-\phi_{rzzz}\left(z,\tau\right)]\right] \tag{3.32}$$

Where A_r represents the Adomian polynomials. It is obtained using the expression in

$$A_{r}=\frac{1}{r!}\frac{d^{r}}{d\lambda^{r}}f\left[\sum_{i=0}^{\infty}\lambda^{i}\phi_{i}\right]_{\lambda=0} \tag{3.33}$$

The nonlinear term is represented by

$$f\left(u\right)=\phi\phi_{z} \tag{3.34}$$

By using Equation, we obtain

$$\begin{aligned}
A_{0}&=\phi_{0}\phi_{0z}\\
A_{1}&=\phi_{1}\phi_{0z}+\phi_{0}\phi_{1z}\\
A_{2}&=\phi_{2}\phi_{0z}+\phi_{1}\phi_{1z}+\phi_{0}\phi_{2z}
\end{aligned}$$

From equation (3.33)
When $r=0$, we obtain

$$\begin{aligned}
\phi_{1}&=\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[A_{0}-\phi_{0zzz}\left(z,\tau\right)]\right]\\
\phi_{1}&=\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6w_{0}\phi_{0z}-\phi_{0zzz}]\right]\\
\phi_{1}&=\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[6\left(\frac{2}{(z-3)^{2}}\left(\frac{-4}{(z-3)}\right)^{3}\right)+\frac{48}{(z-3)^{5}}]\right]
\end{aligned}$$

$$\begin{aligned}
&\phi_{1}=0\\
&r=1,\;\; w_{2}=\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[A_{1}-\phi_{1zzz}\left(z,\tau\right)]\right]\\
&\phi_{2}=-\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_{1}\phi_{0z}+\phi_{0}\phi_{1z}-\phi_{1zzz}]\right]\\
&\phi_{2}=-\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[0]\right]
\end{aligned}$$

$$\phi_2 = 0$$
$$r = 2, \;\; w_3 = \mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[A_2 - \phi_{2zzz}(z,\tau)]\right]$$
$$\phi_3 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[\phi_2\phi_{0z} + \phi_1\phi_{1z} + \phi_0\phi_{2z} - \phi_{2zzz}]\right]$$
$$\phi_3 = -\mathcal{A}^{-1}\left[\frac{1}{v}\mathcal{A}[0]\right]$$
$$\phi_3 = 0$$

The approximate series solution is expressed below as

$$\phi(z,\tau) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \cdots$$
$$\phi(z,\tau) = \frac{2}{(z-3)^2} + 0 + 0 + 0 + \cdots$$

The approximate solution $\phi(z,\tau)$ is given by

$$\phi(z,\tau) = \frac{2}{(z-3)^2}. \tag{3.35}$$

The solution obtained in equation (3.35) is in good agreement with result obtained by Adomian Polynomial and Elzaki Transform Method [11]

4. Conclusion

This study presents the solution of third-order nonlinear KdV problem using AAPM. The examples under consideration demonstrated how successful this technique is at solving third-order KDV equations and how well it works as a system to produce outcomes that are realistic and closely aligned with precise solutions after a minimal number of repetitions. The answers found using this approach concur with additional answers found in the cited literature.

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On preserved properties for slant ruled surfaces under homothety in E^3

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Abstract. In mathematics, it is known that if $f : E^3 \to E^3$ represents a homothety and N denotes a surface in E^3, then $f(N) = \bar{N}$ is a surface in E^3. In this study, especially, the surface N is considered a slant ruled surface. Then, it is proved that the image surface $f(N) = \bar{N}$ is a slant ruled surface, too. Moreover, some significant properties are shown to be preserved under homothety in E^3.

AMS Subject Classifications: 53C05, 53B05, 53B15.

Keywords: Slant ruled surface, homothety, connection preserving map.

Contents

1. Introduction and Background

The notation of slant helix, where the normal lines of the curve form a fixed direction with a constant angle, was introduced in [4]. In literature, there are several studies about slant helices such as [10], [11] and [17]. In [10], the slant helices were investigated in E^3. In [11], a thorough analysis was conducted on the spherical images, tangent, and binormal indicatrices of a slant helix. In [17], a system of linear differential equations including an alternative frame was solved. Furthermore, the position vectors of slant helices by means of integration were determined in Minkowski 3-space. Considering the properties of the slant helix, the concept of the slant ruled surface was firstly expressed in [13] as follows: In mathematics, it is commonly known that the orthonormal vectors of a ruled surface are specified by the ruled surface's Frenet frame. In [13], the concept of special ruled surface which is called slant ruled surface was defined by regarding as the Frenet vectors of the ruled surface. Also, these vectors form some fixed directions with a constant angle in the space. Similar definition was adapted to Darboux vector of the ruled surface which is expressed as Darboux slant ruled surface in [14]. Then, in E^3, several substantial characterizations of the slant ruled surface were investigated in [8]. In [12], non-null slant ruled surface was defined and some fundamental theorems of being non-null slant ruled surface were proved. In [5], the concept of the slant ruled surface was defined by exploiting E. Study mapping and the isomorphism between the unit dual sphere and the subset of the tangent bundle of the unit 2-sphere.
The conformal and the properties of connection preserving maps in $n-$ dimensional C^∞ manifold were examined

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in [2]. Moreover, the condition of being connection preserving for conformal maps were proved in terms of homothety. Then, the connection preserving spray maps were investigated in [3]. Besides, the results for connection preserving conformal diffeomeorphisms of spheres were obtained without any restriction in [15]. By using the differential geometric concepts in [6], the normal curvatures of hypersurfaces were calculated under conformal, homothety and isometry maps in [1]. Furthermore, it was demonstrated that if the conformal map is an isometry, then the first and second fundamental forms of hypersurfaces are invariant.
Taking the geodesic Frenet trihedron defined in [16] and the condition of being a surface with the map $f : E^3 \to E^3$ into consideration, several properties for the ruled surfaces were denoted under the homothety in E^3. However, in literature, there is no research about the examination of some properties for the slant ruled surfaces under homothety in E^3. Therefore, we ask that question:"Which properties are preserved for the slant ruled surface under homothety in E^3?" In this study, we answer this question and obtain some results by using homothety in E^3. Thus, the structure of this study is as follows: In Section 2, some fundamental concepts about conformal map, the definition of slant ruled surface and its Frenet apparatus are represented. Moreover, the condition of a connection preserving is mentioned. In Section 3, some important properties are shown for the slant ruled surface under the homothety in E^3.

2. Preliminaries

Assume that M and $\bar{M}$ are two surfaces in E^3 and $f : M \to \bar{M}$ is a C^∞ map. If there exists a C^∞ real valued positive function F on M. Hence, $\forall P \in M$ and $\forall X_p, Y_p \in T_M P$

$$\langle f_*(X_p), f_*(Y_p)\rangle = F(P)\langle X_p, Y_p\rangle \tag{2.1}$$

is satisfied, f is called a conformal map. If F is constant, f is called a homothety of coefficient $F(P)$, where f_* is Jacobian map of f.
Assume that D and $\bar{D}$ are connections on M and $\bar{M}$, respectively. A C^∞ map $f : M \to \bar{M}$ is called connection preserving if

$$f_*(D_X Y) = \bar{D}_{f_*(X)} f_*(Y) \tag{2.2}$$

for all $X, Y \in \chi(M)$, see [2].

Theorem 2.1. *Assume that $f : M \to \bar{M}$ is a conformal map. Then f is a connection preserving iff f is a homothety, [9].*

Assume that I is an open interval in $\mathbb{R}$. The ruled surface N, parametrized as the following equation, is denoted by

$$\vec{r}(u, v) = \vec{\alpha}(u) + v\vec{q}(u). \tag{2.3}$$

Here $\vec{\alpha} = \vec{\alpha}(u)$ is base curve and $\vec{q} = \vec{q}(u)$ is rullings in E^3. Also, the distribution parameter is calculated by

$$P_q = \frac{det(\vec{\alpha}_u, \vec{q}, \vec{q}_u)}{\langle \vec{q}_u, \vec{q}_u\rangle}. \tag{2.4}$$

Here, $\vec{\alpha}_u = \frac{d\vec{\alpha}}{du}$ and $\vec{q}_u = \frac{d\vec{q}}{du}$, [6]. Additionally, the unit normal vector of N is

$$\vec{m} = \frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||} = \frac{(\vec{\alpha}_u + v\vec{q}_u) \times \vec{q}}{\sqrt{\langle \vec{\alpha}_u + v\vec{q}_u, \vec{\alpha}_u + v\vec{q}_u\rangle - \langle \vec{\alpha}_u, \vec{q}\rangle^2}}. \tag{2.5}$$

Along a ruling $u = u_1$, as v infinitely decreases, the surface's unit normal converges in a limiting direction. As the asymptotic normal direction, this direction is determined by Eq. (2.5), which is defined by

$$\vec{a} = lim_{v\to\pm\infty}\vec{m}(u_1, v) = \frac{\vec{q} \times \vec{q}_u}{||\vec{q}_u||}. \tag{2.6}$$

The point where $\vec{m}$ is orthogonal to $\vec{a}$ is defined as the striction point and represented by C. Thus, the striction curve of the surface is defined as the set of striction points of all rulings. The striction curve $\vec{c} = \vec{c}(u)$ is

$$\vec{c}(u) = \vec{\alpha}(u) + v_0\vec{q}(u) = \vec{\alpha}(u) - \frac{\langle \vec{q}_u, \vec{\alpha}_u \rangle}{\langle \vec{q}_u, \vec{q}_u \rangle}\vec{q}(u). \tag{2.7}$$

Here, $v_0 = -\frac{\langle \vec{q}_u, \vec{\alpha}_u \rangle}{\langle \vec{q}_u, \vec{q}_u \rangle}$. Moreover, $\vec{h}$ computed by $\vec{h} = \vec{a} \times \vec{q}$ is defined as normal vector. Hence, the set $\{C; \vec{q}, \vec{h}, \vec{a}\}$ is Frenet frame of N. Here, C denotes the central point and $\vec{q}, \vec{h}$ and $\vec{a}$ also denote unit vectors of ruling, central normal and central tangent, respectively.

For the Frenet formulas of N in terms of the arc-length parameter s are

$$\begin{pmatrix} \frac{d\vec{q}}{ds} \\ \frac{d\vec{h}}{ds} \\ \frac{d\vec{a}}{ds} \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{pmatrix} \begin{pmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{pmatrix}.$$

Here, $k_1 = \frac{ds_1}{ds}$ and $k_2 = \frac{ds_3}{ds}$. s_1 and s_3 denote the arc lengths of the spherical curves bounded by $\vec{q}$ and $\vec{a}$, respectively. For more details, see [6].

Moreover, in [13], the characterizations of being $\vec{q}, \vec{h}$ and $\vec{a}$ slant ruled surfaces are classified as follows:

Assume that N is a ruled surface in E^3 denoted by the parametrization

$$\vec{r}(s,v) = \vec{c}(s) + v\vec{q}(s), \quad ||\vec{q}(s)|| = 1, \tag{2.8}$$

where $\vec{c}(s)$ denotes the striction curve of N and s represents arc length parameter of $\vec{c}(s)$. Let $\{\vec{q}, \vec{h}, \vec{a}, k_1, k_2\}$ be Frenet operators of N. If the rulling (or central normal, central tangent) form a fixed non-zero direction $\vec{u}$ with a constant angle θ, then N is called $\vec{q}-$(or $\vec{h}, \vec{a}$) slant ruled surface, respectively, see [13].

The curve which are drawn by $\vec{q}$ on the unit sphere S^2 is defined as the spherical indicatrix curve. Also, $\vec{q}$ is defined as the spherical indicatrix of N. For the Frenet vectors $\{\vec{q}, \vec{h}, \vec{a}\}$ given above, we can write

$$\begin{aligned} \vec{q}_{s_1} &= \vec{h}, \\ \vec{h}_{s_1} &= -\vec{q} + \frac{k_2}{k_1}\vec{a}, \\ \vec{a}_{s_1} &= -\frac{k_2}{k_1}\vec{h}, \end{aligned} \tag{2.9}$$

where s_1 denotes the arc-parameter of the spherical indicatrix curve $\vec{q}$. Also, $\vec{q}_{s_1} = \frac{d\vec{q}}{ds_1}, \vec{h}_{s_1} = \frac{d\vec{h}}{ds_1}$ and $\vec{a}_{s_1} = \frac{d\vec{a}}{ds_1}$, respectively.

3. Slant ruled surfaces under homothety

Assume that N is a slant ruled surface given in Eq. (2.8) and $f : E^3 \to E^3$ is a homothety of coefficient λ. For the point Z taken on each rulling d of N, we get

$$Z = \vec{c}(s) + v\vec{q}(s). \tag{3.1}$$

according to arc-parameter s. Then, we write

$$f_*(Z) = f_*(\vec{c}(s)) + vf_*(\vec{q}(s)). \tag{3.2}$$

Consequently, $f(d)$ is a striction line passing through the point $f(\vec{c}(s))$ of the image curve $f \circ c$. Hence, $f(N) = \bar{N}$ is a slant ruled surface with the striction curve $f \circ c$ in E^3.

Corollary 3.1. *Slant ruled surfaces transform to slant ruled surfaces under homothety.*

If we exploit the spherical indicatrix vector of $\bar{N}$ as $\vec{\bar{q}}$, we can acquire the parametric representation of $\bar{N}$ as follows:

$$\vec{\bar{r}}(s,v)=\vec{\bar{c}}(s)+v\vec{\bar{q}}(s)$$

or

$$\vec{\bar{r}}(s,v)=f_*(\vec{c}(s))+v\frac{1}{\lambda^{\frac{1}{2}}}f_*(\vec{q}(s)). \tag{3.3}$$

In this study, we will consider $f:E^3\to E^3$ as a homothety with the coefficient λ such that $f(N)=\bar{N}$. Moreover, the rulling (or central normal, central tangent) makes a constant angle θ with a fixed non-zero direction $\vec{u}$. Here, N and $\bar{N}$ are slant ruled surfaces in E^3.

Theorem 3.2. *For all X in $\chi(N)$, we have*

$$[f_*(X)]_s=f_*(X_s). \tag{3.4}$$

For proof of this theorem, see [7].

Let $\vec{\bar{h}}$ and $\vec{\bar{a}}$ be central normal and central tangent vectors of $\bar{N}$, respectively. Thus, we can simply write

$$\vec{\bar{h}}=\frac{\vec{\bar{q}}_s}{||\vec{\bar{q}}_s||},\quad \vec{\bar{a}}=\frac{\vec{\bar{q}}\times\vec{\bar{q}}_s}{||\vec{\bar{q}}_s||}. \tag{3.5}$$

From $\vec{\bar{q}}=\frac{f_*(\vec{q})}{\lambda^{\frac{1}{2}}}$ and Theorem 3.2, we get

$$\vec{\bar{h}}=\frac{f_*(\vec{q}_s)}{||f_*(\vec{q}_s)||} \tag{3.6}$$

and

$$\vec{\bar{a}}=\frac{1}{\lambda^{\frac{1}{2}}}\frac{f_*(\vec{q})\times f_*(\vec{q}_s)}{||f_*(\vec{q}_s)||}. \tag{3.7}$$

By using the following equation

$$f_*(\vec{q})\times f_*(\vec{q}_s)=det f_*.(\vec{q}\times\vec{q}_s), \tag{3.8}$$

we obtain

$$\vec{\bar{a}}=\frac{det f_*}{\lambda}\vec{a}. \tag{3.9}$$

The striction curve $\vec{c}$ is written by

$$\vec{c}(s)=\vec{\alpha}(s)-\frac{\langle\alpha_s,\vec{q}_s\rangle}{\langle\vec{q}_s,\vec{q}_s\rangle}\vec{q}(s). \tag{3.10}$$

Assume that $\vec{\bar{c}}$ is a striction curve of the slant ruled surface $\bar{N}$. Therefore, we get

$$\vec{\bar{c}}(s)=\vec{\bar{\alpha}}(s)+k\vec{\bar{q}}(s) \tag{3.11}$$

or

$$\vec{\bar{c}}(s)=f_*(\vec{\alpha}(s))+k\frac{1}{\lambda^{\frac{1}{2}}}f_*(\vec{q}(s)). \tag{3.12}$$

Due to the definition of the striction point, we acquire

$$\langle\vec{\bar{r}}_s\times\vec{\bar{r}}_v,f_*(\vec{q})\times f_*(\vec{q}_s)\rangle=0. \tag{3.13}$$

By using Eq. (3.3), we obtain

$$k=-\lambda^{\frac{1}{2}}\frac{\langle\vec{\alpha}_s,\vec{q}_s\rangle}{\langle\vec{q}_s,\vec{q}_s\rangle}. \tag{3.14}$$

Then, for the striction curve of $\bar{N}$, we calculate

$$\vec{\bar{c}}(s) = f_*(\vec{\alpha}(s)) - \frac{\langle \vec{\alpha}_s, \vec{q}_s \rangle}{\langle \vec{q}_s, \vec{q}_s \rangle} f_*(\vec{q}(s)). \tag{3.15}$$

Consequently, it is easily seen that $f_*(\vec{c}(s)) = \vec{\bar{c}}(s)$.

Corollary 3.3. *The property of being striction curve is preserved under the homothety.*

Assume that $\beta : I \subseteq \mathbb{R} \to E^3$ is the orthogonal trajectory for N and $\vec{T}$ is the tangent of β. For $s \in I$, we have

$$\langle \vec{T}, \vec{q} \rangle = 0. \tag{3.16}$$

Since f is homothety, we write

$$\langle f_*(\vec{T}), f_*(\vec{q}) \rangle = 0. \tag{3.17}$$

As a result, $f(\beta) = \bar{\beta}$ is a orthogonal trajectory for $\bar{N}$.

Corollary 3.4. *The property of being orthogonal trajectory is preserved under the homothety.*

If the slant ruled surface N is closed, a positive integer t exists such that

$$N(s+t, v) = N(s, v). \tag{3.18}$$

Hence, we write

$$\vec{c}(s+t) + v\vec{q}(s+t) = \vec{c}(s) + v\vec{q}(s). \tag{3.19}$$

Because of the linearity of f_*, we write

$$f_*(\vec{c}(s+t)) + v f_*(\vec{q}(s+t)) = f_*(\vec{c}(s)) + v f_*(\vec{q}(s)). \tag{3.20}$$

If we take the parametrization of $\bar{N}$,

$$\bar{N}(s+t, v) = \bar{N}(s, v). \tag{3.21}$$

Corollary 3.5. *The condition of being closed of the slant ruled surfaces is preserved under the homothety.*

For the distribution parameter P_q is

$$P_{\vec{q}} = \frac{det(\vec{c}_s, \vec{q}, \vec{q}_s)}{||\vec{q}_s||^2}. \tag{3.22}$$

Considering $\vec{q}_s = \frac{d\vec{q}}{ds_1}\frac{ds_1}{ds}$ and Frenet formulas given Eq. (2.9), we obtain

$$P_{\vec{q}} = \frac{1}{\vec{q}_s} \langle \vec{c}_s, \vec{a} \rangle. \tag{3.23}$$

Similarly, for the distribution parameter of $\bar{N}$, we have

$$P_{\vec{\bar{q}}} = \frac{det(\vec{\bar{c}}_s, \vec{\bar{q}}, \vec{\bar{q}}_s)}{||\vec{\bar{q}}_s||^2}. \tag{3.24}$$

Namely, we can write

$$P_{\vec{\bar{q}}} = \frac{det(f_*(\vec{c}(s)), f_*(\vec{q}), f_*(\vec{h}))}{\lambda^{\frac{1}{2}} \vec{q}_s ||f_*(\vec{h})||^2}. \tag{3.25}$$

Since $f_*(\vec{q}) \times f_*(\vec{h}) = \lambda^{\frac{1}{2}} f_*(\vec{a})$, we acquire

$$P_{\vec{q}} = P_{\vec{\bar{q}}}. \tag{3.26}$$

Corollary 3.6. *The property of being developable of the slant ruled surfaces is preserved under the homothety.*

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Uncertainty principles for the continuous wavelet transform associated with a Bessel type operator on the half line

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Abstract. This paper presents uncertainty principles pertaining to generalized wavelet transforms associated with a second-order differential operator on the half line, extending the concept of the Bessel operator. Specifically, we derive a Heisenberg-Pauli-Weyl type uncertainty principle, as well as other uncertainty relations involving sets of finite measure

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Contents

1. Introduction

Al Subaie and Mourou ([4]) have introduced and studied the following second order differential operator $\Delta_{\alpha,n}$, on the half line $(0, +\infty)$,

$$\Delta_{\alpha,n}(u) = u^{''} + \frac{2\alpha+1}{r}u' - \frac{4n(\alpha+n)}{r^2}u,$$

where $\alpha > -\frac{1}{2}$ and $n \in \mathbb{N}$. Its particularity resides in the fact that it generalizes the usual Bessel differential operator, indeed for $n = 0$, we recover the Bessel operator $\ell_\alpha = u^{''} + \frac{2\alpha+1}{r}u'$.

This paper focuses on exploring uncertainty principles concerning the generalized Fourier transform [4] and continuous wavelet transforms [3] associated with $\Delta_{\alpha,n}$. Essentially, a function and its Fourier transform cannot be sharply focused at the same time in harmonic analysis, according to the uncertainty principle. Various mathematical formulations express this principle, involving measurement of sets or norms. For further elaboration, interested readers can refer to [17, 24] and [5, 8, 12, 13, 16, 23]. Recently, similar uncertainty relations have been established for different integral transforms, such as continuous wavelet transforms and

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Gabor transforms, across various contexts. Relevant literature includes [7, 10, 22, 29, 33, 36] and related references.

In this context, we establish, among other, a sharp Heisenberg-Pauli-Weyl type uncertainty principle [35] for the generalized Fourier transform $\mathscr{F}_\Delta$ associated to $\Delta_{\alpha,n}$, which is defined on $\mathbb{R}_+ = [0,+\infty)$ by

$$\mathscr{F}_\Delta(f)(\lambda) = \int_0^{+\infty} f(x)\varphi_\lambda(x)\frac{x^{2\alpha+1}}{2^{\alpha+2n}\Gamma(\alpha+2n+1)}dx; \quad \forall\lambda \in \mathbb{R}_+,$$

where $\varphi_\lambda(x) = x^{2n} j_{\alpha+2n}(\lambda x)$ and j_α is the modified Bessel function (see [27, 34]).

We present Heisenberg-Pauli-Weyl type inequalities applicable to the generalized continuous wavelet transform associated with $\Delta_{\alpha,n}$. These inequalities encompass both the time and frequency variables, as well as their combination. Additionally, we explore other uncertainty relations pertinent to this transform, including Donoho and Stark type principles. Our investigation delves into the concentration of this transform on time-frequency sets, revealing that the generalized wavelet transforms of non-zero functions cannot have arbitrarily large support. Notably, extensive research has been conducted on this generalized Fourier transform, particularly within the realm of uncertainty principles [1, 2, 14, 15].

Numerous studies, including those on time-frequency representations such as Gabor and wavelet transforms, have been thoroughly explored in diverse contexts using various methodologies [6, 10, 18, 20, 22]. For further elucidation, refer to [21].

This document is structured as follows:

The first section revisits some harmonic analysis findings pertinent to the generalized Fourier transform, $\mathscr{F}_\Delta$. The second section focuses on the study of generalized continuous wavelet transforms associated with $\Delta_{\alpha,n}$. In the third section, we present results concerning finite sets of measurements, alongside discussions on Donoho-Stark and Benedicks-type uncertainty principles. Lastly, the fourth section addresses Heisenberg-type uncertainty principles for the generalized continuous wavelet transform.

2. Preliminaries

Within this section, we revisit essential concepts in harmonic analysis pertaining to the Bessel operator ℓ_α, as documented in references [9, 11, 26, 32]. These concepts serve as foundational knowledge for our examination of the Bessel-type operator $\Delta_{\alpha,n}$ (see [4]). For α greater than $-\frac{1}{2}$, the Bessel operator ℓ_α is defined over the interval $(0,+\infty)$ by

$$\ell_\alpha(u) = u'' + \frac{2\alpha+1}{r}u'.$$

Next, considering all values of λ in the complex number set, the following system

$$\ell_\alpha(u) = -\lambda^2 u, \ u(0) = 1, \ u'(0) = 0,$$

admits a unique solution given by the modified Bessel function $x \mapsto j_\alpha(\lambda x)$, where

$$j_\alpha(x) = \frac{2^\alpha\Gamma(\alpha+1)}{x^\alpha}J_\alpha(x) = \Gamma(\alpha+1)\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\Gamma(\alpha+n+1)}(\frac{x}{2})^{2n}, \quad x \in \mathbb{R};$$

and J_α is the Bessel function of the first kind and index α (see [27, 34]).

The Mehler integral representation of the modified Bessel function j_α is expressed as follows:

$$\forall x \in \mathbb{R}; \quad j_\alpha(x) = \begin{cases} \dfrac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\displaystyle\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}}\cos(xt)dt, & \text{if } \alpha > -1/2; \\ \cos(x), & \text{if } \alpha = -1/2. \end{cases}$$

Specifically, for each natural number n and real number x,

$$|j_\alpha^{(n)}(x)| \leq 1. \tag{2.1}$$

On the positive real numbers $\mathbb{R}_+$, define the measure μ_α as follows:

$$d\mu_\alpha(x) = \frac{x^{2\alpha+1}}{c_\alpha}dx; \text{ where } c_\alpha = 2^\alpha\Gamma(\alpha+1), \tag{2.2}$$

and represent the Lebesgue space on $\mathbb{R}_+$ with a focus on the measure μ_α by $L^p_{\mu_\alpha}(\mathbb{R}_+)$, $p \in [1, +\infty]$, and the L^p-norm by $\|.\|_{p,\mu_\alpha}$.

The Bessel translation operators τ_x^α, where $x \geqslant 0$, operate on $L^1_{\mu_\alpha}(\mathbb{R}_+)$, with their definition as follows:

$$\tau_x^\alpha(f)(y) = \begin{cases} \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}\displaystyle\int_0^\pi f(\sqrt{x^2+y^2+2xycos\theta})(sin\theta)^{2\alpha}d\theta, & \text{if } \alpha > -1/2; \\ \dfrac{f(x+y)+f(|x-y|)}{2}, & \text{if } \alpha = -1/2. \end{cases} \tag{2.3}$$

Here, for every $x \in \mathbb{R}_+$, we have

$$\int_0^{+\infty} \tau_x^\alpha(f)(y)d\mu_\alpha(y) = \int_0^{+\infty} f(y)d\mu_\alpha(y).$$

For every $f \in L^p_{\mu_\alpha}(\mathbb{R}_+)$, $p \in [1, +\infty]$ and for every $x \in \mathbb{R}_+$, the function $\tau_x^\alpha(f)$ belongs to the space $L^p_{\mu_\alpha}(\mathbb{R}_+)$ and

$$\| \tau_x^\alpha(f) \|_{p,\mu_\alpha} \leq \| f \|_{p,\mu_\alpha} . \tag{2.4}$$

In $L^1_{\mu_\alpha}(\mathbb{R}_+)$, the convolution operation between two functions f and g is defined by

$$f *_\alpha g(x) = \int_0^{+\infty} f(y)\tau_x^\alpha(g)(y)d\mu_\alpha(y), \quad \forall x \in \mathbb{R}_+.$$

In $L^1_{\mu_\alpha}(\mathbb{R}_+)$, the convolution product "$*_\alpha$" is both commutative and associative.

For the convolution product " $*_\alpha$ ", the Young's inequality states that if p, q, and $r \in [1, +\infty]$ are such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, then for all f in $L^p_{\mu_\alpha}(\mathbb{R}_+)$ and g in $L^q_{\mu_\alpha}(\mathbb{R}_+)$, the function $f *_\alpha g$ belongs to $L^r_{\mu_\alpha}(\mathbb{R}_+)$ and

$$\|f *_\alpha g\|_{r,\mu_\alpha} \leq \| f \|_{p,\mu_\alpha} \| g \|_{q,\mu_\alpha} . \tag{2.5}$$

Furthermore, for any function f and g in $L^2_{\mu_\alpha}(\mathbb{R}_+)$ and we have for each $x \in \mathbb{R}_+$,

$$\tau_x^\alpha(f *_\alpha g) = \tau_x^\alpha(f) *_\alpha g = f *_\alpha \tau_x^\alpha(g).$$

On $L^1_{\mu_\alpha}(\mathbb{R}_+)$, the Hankel transform $\mathscr{H}_\alpha$ is defined, via

$$\mathscr{H}_\alpha(f)(\lambda) = \int_0^{+\infty} f(r)j_\alpha(r\lambda)d\mu_\alpha(r), \quad \forall \lambda \in \mathbb{R}_+. \tag{2.6}$$

The following properties hold

• (Inversion formula for $\mathscr{H}_\alpha$) In $L^1_{\mu_\alpha}(\mathbb{R}_+)$, for any function f, we have for almost all $x \in \mathbb{R}_+$

$$f(x) = \int_0^{+\infty} \mathscr{H}_\alpha(f)(\lambda)j_\alpha(x\lambda)d\mu_\alpha(\lambda).$$

• (Plancherel's theorem for $\mathscr{H}_\alpha$) An isometric isomorphism from $L^2_{\mu_\alpha}(\mathbb{R}_+)$ onto itself may be obtained by extending the Hankel transform $\mathscr{H}_\alpha$. Specifically, the Parseval's formula for each f and g in $L^2_{\mu_\alpha}(\mathbb{R}_+)$ is as follows.

$$\int_0^{+\infty} f(x)\overline{g(x)}d\mu_\alpha(x) = \int_0^{+\infty} \mathscr{H}_\alpha(f)(\lambda)\overline{\mathscr{H}_\alpha(g)(\lambda)}d\mu_\alpha(\lambda).$$

• For every $f \in L^p_{\mu_\alpha}(\mathbb{R}_+)$, $p = 1$ or 2, and $x \in \mathbb{R}_+$, we have

$$\mathscr{H}_\alpha(\tau^\alpha_x(f))(\lambda) = j_\alpha(x\lambda)\mathscr{H}_\alpha(f)(\lambda), \ \ \forall\lambda \in \mathbb{R}_+. \tag{2.7}$$

• For every $f \in L^1_{\mu_\alpha}(\mathbb{R}_+)$ and $g \in L^p_{\mu_\alpha}(\mathbb{R}_+)$, $p = 1, 2$ we have

$$\mathscr{H}_\alpha(f *_\alpha g) = \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g).$$

•Suppose $f, g \in L^2_{\mu_\alpha}(\mathbb{R}_+)$. In $L^2_{\mu_\alpha}(\mathbb{R}_+)$, the function $f *_\alpha g$ is included if and only if, $\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g)$ belongs to $L^2_{\mu_\alpha}(\mathbb{R}_+)$ and in this case, we have

$$\mathscr{H}_\alpha(f *_\alpha g) = \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g).$$

Let us consider the second-order singular differential operator on the half line (see [4])

$$\Delta_{\alpha,n}(u) = u^{''} + \frac{2\alpha+1}{r}u' - \frac{4n(\alpha+n)}{r^2}u$$

where $n \in \mathbb{N}$. We obtain the Bessel operator ℓ_α for $n = 0$.

For all $\lambda \in \mathbb{C}$, the function

$$\varphi_\lambda(x) = x^{2n}j_{\alpha+2n}(\lambda x). \tag{2.8}$$

is solution of $\Delta_{\alpha,n}(u) = -\lambda^2 u$.

The following characteristics apply to the function φ_λ

- For all $\lambda, x \in \mathbb{R}$,

$$\varphi_\lambda(x) = \frac{x^{2n}2\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+\frac{1}{2})}\int_0^1 (1-t^2)^{\alpha+2n-\frac{1}{2}}\cos(\lambda xt)dt.$$

In particular,

$$|\varphi_\lambda(x)| \leqslant x^{2n}. \tag{2.9}$$

- For a measurable function on $\mathbb{R}$, we define the map M by $Mf(x) = x^{2n}f(x)$. Then, for $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, the function φ_λ satisfy the following product formula

$$\varphi_\lambda(x)\varphi_\lambda(y) = \frac{(xy)^{2n}\Gamma(\alpha+2n+1)}{\Gamma(\alpha+2n+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^\pi (M^{-1}\varphi_\lambda)(\sqrt{\lambda^2x^2+\lambda^2y^2+2\lambda^2xycos(\theta)})sin(\theta)^{\alpha+2n}d\theta.$$

In the sequel, we need the following notations.

- $L^p_{(\mu)}(\mathbb{R}_+)$, $p \in [1, +\infty]$, is the space of measurable functions f on $\mathbb{R}_+$ such that $\|M^{-1}f\|_{p,\mu_{\alpha+2n}} < \infty$. The space $L^p_{(\mu)}(\mathbb{R}_+)$ is equipped with the norm $\|.\|_{p,(\mu)}$ given by

$$\|f\|_{p,(\mu)} = \|M^{-1}f\|_{p,\mu_{\alpha+2n}}.$$

From the last product formula, we define the generalized translation operator, $\tau_x^{\Delta},\ x\in\mathbb{R}_+$ by

$$\tau_x^{\Delta}(f)(y)=\frac{(xy)^{2n}\Gamma(\alpha+2n+1)}{\Gamma(\alpha+2n+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^{\pi}(M^{-1}f)(\sqrt{x^2+y^2+2xycos(\theta)})sin(\theta)^{\alpha+2n}d\theta, \tag{2.10}$$

Whenever the integral of the right-hand side is well defined.

- We have the following relation between the generalized and Hankel translation operators

$$\tau_x^{\Delta}(f)(y)=(xy)^{2n}\tau_x^{\alpha+2n}(M^{-1}f)(y),$$

 where $\tau_x^{\alpha+2n}$ is given by the relation (2.3).

- For every $f\in L^p_{(\mu)}(\mathbb{R}_+)$, $p\in[1,+\infty]$ and for every $x\in\mathbb{R}_+$, the function $\tau_x^{\Delta}(f)$ belongs to the space $L^p_{(\mu)}(\mathbb{R}_+)$ and

$$\|\tau_x^{\Delta}(f)\|_{p,(\mu)}\leqslant x^{2n}\|f\|_{p,(\mu)}. \tag{2.11}$$

Given two functions $f,g\in L^1_{(\mu)}(\mathbb{R}_+)$, the generalized convolution product, "$\#$", is defined as

$$f\#g(x)=\int_0^{+\infty}f(y)\tau_x^{\Delta}(g)(y)\frac{y^{2\alpha+1}}{c_{\alpha+2n}}dy,\quad x\geqslant 0, \tag{2.12}$$

where the constant $c_{\alpha+2n}$ is given by the relation (2.2).

We have the following connection between "$\#$" and " $*_{\alpha+2n}$ ",

$$f\#g(x)=M\left(M^{-1}(f)*_{\alpha+2n}M^{-1}(g)\right)(x). \tag{2.13}$$

In $L^1_{(\mu)}(\mathbb{R}_+)$, the convolution product "$\#$" is both commutative and associative.

Young's inequality for the convolution product "$\#$" states that, for all f in $L^p_{(\mu)}(\mathbb{R}_+)$ and g in $L^q_{(\mu)}(\mathbb{R}_+)$, the function $f\#g$ belongs to $L^r_{(\mu)}(\mathbb{R}_+)$ and for all p,q and $r\in[1,+\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$ and

$$\|f\#g\|_{r,(\mu)}\leq\|\ f\ \|_{p,(\mu)}\|\ g\ \|_{q,(\mu)}\ . \tag{2.14}$$

On $L^1_{(\mu)}(\mathbb{R}_+)$, the generalized Fourier transform $\mathscr{F}_{\Delta}$ related to $\Delta_{\alpha,n}$ is defined by

$$\mathscr{F}_{\Delta}(f)(\lambda)=\int_0^{+\infty}f(x)\varphi_{\lambda}(x)\frac{x^{2\alpha+1}}{c_{\alpha+2n}}dx;\quad\forall\lambda\in\mathbb{R}_+, \tag{2.15}$$

where φ_{λ} is given by the relation (2.8).

We have the following properties

- For $f\in L^1_{(\mu)}(\mathbb{R}_+)$,

$$\mathscr{F}_{\Delta}(f)(\lambda)=\mathscr{H}_{\alpha+2n}(M^{-1}f)(\lambda),\quad\lambda\in\mathbb{R}_+.$$

- For each $f\in L^1_{(\mu)}(\mathbb{R}_+)$, $\mathscr{F}_{\Delta}(f)$ is a function that is a part of $\mathscr{C}_{*,0}(\mathbb{R})$ the space of continuous even functions f on $\mathbb{R}$ such that $\lim\limits_{|x|\to\infty}|f(x)|=0$ in addition

$$\|\mathscr{F}_{\Delta}(f)\|_{\infty,\mu_{\alpha+2n}}\leqslant\|f\|_{1,(\mu)}$$

- Since $\mathscr{F}_\Delta(f) \in L^1_{\mu_{\alpha+2n}}(\mathbb{R}_+)$ for every $x \in \mathbb{R}_+$, let $f \in L^1_{(\mu)}(\mathbb{R}_+)$

$$f(x) = \int_0^{+\infty} \mathscr{F}_\Delta(f)(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda).$$

  Applied to $L^1_{(\mu)}(\mathbb{R}_+)$, this demonstrates that $\mathscr{F}_\Delta$ is injective.

- (Plancherel's theorem for $\mathscr{F}_\Delta$) An isometric isomorphism from $L^2_{(\mu)}(\mathbb{R}_+)$ onto $L^2_{\mu_{\alpha+2n}}(\mathbb{R}_+)$ may be obtained by extending the generalized Fourier transform $\mathscr{F}_\Delta$. Moreover, the following Parseval's formula holds for every f and g in $L^2_{(\mu)}(\mathbb{R}_+)$.

$$\langle f|g\rangle_{(\mu)} = \langle \mathscr{F}_\Delta(f)|\mathscr{F}_\Delta(g)\rangle_{\mu_{\alpha+2n}}. \tag{2.16}$$

  where the inner product defined on $L^2_{(\mu)}(\mathbb{R}_+)$ is $\langle\, .|.\, \rangle_{(\mu)}$, via

$$\langle f|g\rangle_{(\mu)} = \langle M^{-1}(f)|M^{-1}(g)\rangle_{\mu_{\alpha+2n}} \tag{2.17}$$

  and the inner product of the Hilbert space $L^2_{\mu_{\alpha+2n}}(\mathbb{R}_+)$ is shown by the notation $\langle\, .|.\, \rangle_{\mu_{\alpha+2n}}$.

- When $x \in \mathbb{R}_+$, $p = 1$ or 2, and $f \in L^p_{(\mu)}(\mathbb{R}_+)$, we obtain

$$\mathscr{F}_\Delta(\tau_x^\Delta(f))(\lambda) = \varphi_\lambda(x)\mathscr{F}_\Delta(f)(\lambda), \;\; \forall\lambda \in \mathbb{R}_+.$$

- For every $f \in L^1_{(\mu)}(\mathbb{R}_+)$ and $g \in L^p_{(\mu)}(\mathbb{R}_+)$, $p = 1, 2$ we have

$$\mathscr{F}_\Delta(f\#g) = \mathscr{F}_\Delta(f)\mathscr{F}_\Delta(g).$$

- Suppose that $f, g \in L^2_{(\mu)}(\mathbb{R}_+)$. If and only if $\mathscr{F}_\Delta(f)\mathscr{F}_\Delta(g)$ belongs to $L^2_{\mu_{\alpha+2n}}(\mathbb{R}_+)$, then the function $f\#g$ belongs to $L^2_{(\mu)}(\mathbb{R}_+)$. In this instance, we have

$$\mathscr{F}_\Delta(f\#g) = \mathscr{F}_\Delta(f)\mathscr{F}_\Delta(g). \tag{2.18}$$

3. Generalized Continuous Wavelet Transforms Associated to $\mathscr{F}_\Delta$.

The theory of generalized continuous wavelet transforms, as studied by R.F. Al Subaie and M.A. Mourou [4], is briefly summarized in this section.

Let $a \in \mathbb{R}^*_+ = (0, +\infty)$. The dilation operator $D_{\alpha,a}$ of a measurable function ψ, is defined by

$$D_{\alpha,a}(\psi)(s) = a^{\alpha+1}\psi(as), \quad \forall s \geqslant 0.$$

We have,

- For every $\psi \in L^2_{(\mu)}(\mathbb{R}_+)$,

$$\|D_{\alpha,a}(\psi)\|_{2,(\mu)} = \|\psi\|_{2,(\mu)}. \tag{3.1}$$

- We obtain for any ψ and $\phi \in L^2_{(\mu)}(\mathbb{R}_+)$

$$\langle D_{\alpha,a}(\psi)|\phi\rangle_{(\mu)} = \langle \psi|D_{\alpha,\frac{1}{a}}(\phi)\rangle_{(\mu)},$$

- For every $\psi \in L^2_{(\mu)}(\mathbb{R}_+)$, we have

$$\mathscr{F}_\Delta(D_{\alpha,a}(\psi)) = D_{\alpha+2n,\frac{1}{a}}\mathscr{F}_\Delta(\psi). \tag{3.2}$$

We indicate by

- $\vartheta_{\alpha,n}$ the measure defined on $\mathbb{R}_+^* \times \mathbb{R}_+$, by

$$d\vartheta_{\alpha,n}(a,x) = d\mu_{\alpha+2n}(a)d\mu_{\alpha+2n}(x),$$

- The Lebesgue space on $\mathbb{R}_+^* \times \mathbb{R}_+$ with regard to the measure $\vartheta_{\alpha,n}$ with the L^p-norm represented by $\| \, . \, \|_{p,\vartheta_{\alpha,n}}$ is $L^p_{\vartheta_{\alpha,n}}(\mathbb{R}_+^* \times \mathbb{R}_+)$, $p \in [1,+\infty]$.
- $\langle \, .| \, . \, \rangle_{\vartheta_{\alpha,n}}$ the inner product of the Hilbert space $L^2_{\vartheta_{\alpha,n}}(\mathbb{R}_+^* \times \mathbb{R}_+)$.
- For a measurable function f on $\mathbb{R}_+^* \times \mathbb{R}_+$, the mapping M_2 is defined by

$$M_2(f)(a,x) = x^{2n} f(a,x).$$

- $L^p_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+)$, $p \in [1,+\infty]$ the space of measurable functions f on $\mathbb{R}_+^* \times \mathbb{R}_+$ such that $\| \, M_2^{-1}(f) \, \|_{p,\vartheta_{\alpha,n}} < +\infty$. The space $L^p_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+)$ is equipped with the norm $\|.\|_{p,(\vartheta)}$ given by

$$\|f\|_{p,(\vartheta)} = \| \, M_2^{-1}(f) \, \|_{p,\vartheta_{\alpha,n}}.$$

- $\langle \, . \mid . \, \rangle_{(\vartheta)}$ the inner product of the Hilbert space $L^2_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+)$ defined by

$$\langle \, f | \, g \, \rangle_{(\vartheta)} = \langle \, M_2^{-1}(f) | \, M_2^{-1}(g) \, \rangle_{\vartheta_{\alpha,n}}.$$

A generalized admissible wavelet is defined as $\psi \in L^2_{(\mu)}(\mathbb{R}_+)\backslash\{0\}$ if

$$0 < C_\psi^\Delta = \frac{1}{c_{\alpha+2n}} \int_0^\infty |\mathscr{F}_\Delta(\psi)(a)|^2 \frac{da}{a} < \infty. \tag{3.3}$$

The generalized continuous wavelet transform $\mathcal{W}_\psi^\Delta$, for such ψ, is defined on $L^2_{(\mu)}(\mathbb{R}_+)$ by

$$\mathcal{W}_\psi^\Delta(f)(a,x) = \int_0^\infty f(s)\overline{\psi_{a,x}^\Delta(s)} \frac{s^{2\alpha+1}}{c_{\alpha+2n}} ds, \quad (a,x) \in \mathbb{R}_+^* \times \mathbb{R}_+ \tag{3.4}$$

where

$$\psi_{a,x}^\Delta(s) = \tau_x^\Delta D_{\alpha,a}(\psi)(s). \tag{3.5}$$

Another way to express the transform $\mathcal{W}_\psi^\Delta$ is

$$\begin{aligned} \mathcal{W}_\psi^\Delta(f)(a,x) &= f\#D_{\alpha,a}(\overline{\psi})(x) \\ &= \langle \, f \, | \psi_{a,x}^\Delta \, \rangle_{(\mu)}. \end{aligned} \tag{3.6}$$

Then, in virtue of relations (3.6), (2.14) and (3.1), we deduce that the function $\mathcal{W}_\psi^\Delta(f)$ belongs to the space $L^\infty_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+)$ and

$$\left\|\mathcal{W}_\psi^\Delta(f)\right\|_{\infty,(\vartheta)} \leqslant \|f\|_{2,(\mu)} \|\psi\|_{2,(\mu)}. \tag{3.7}$$

Theorem 3.1. *Let $\psi \in L^2_{(\mu)}(\mathbb{R}_+)$ be a generalized admissible wavelet.*

(i) (Plancherel's formula for $\mathcal{W}^{\Delta}_{\psi}$) For every function $f \in L^2_{(\mu)}(\mathbb{R}_+)$, the function $\mathcal{W}^{\Delta}_{\psi}(f)$ belongs to the space $L^2_{(\vartheta)}(\mathbb{R}^*_+ \times \mathbb{R}_+)$ and we have

$$\|\mathcal{W}^{\Delta}_{\psi}(f)\|_{2,(\vartheta)} = \sqrt{C^{\Delta}_{\psi}}\|f\|_{2,(\mu)}. \tag{3.8}$$

(ii) (Parseval's formula for $\mathcal{W}^{\Delta}_{\psi}$) For all functions $f, g \in L^2_{(\mu)}(\mathbb{R}_+)$ we have

$$\langle\, f \mid g \,\rangle_{(\mu)} = \frac{1}{C^{\Delta}_{\psi}}\langle\, \mathcal{W}^{\Delta}_{\psi}(f) \mid \mathcal{W}^{\Delta}_{\psi}(g) \,\rangle_{(\vartheta)}, \tag{3.9}$$

Proof. (i) Let $f \in L^2_{(\mu)}(\mathbb{R}_+)$, we have from relations (2.16), (2.18) and (3.6),

$$\begin{aligned}
\|\mathcal{W}^{\Delta}_{\psi}(f)\|^2_{2,(\vartheta)} &= \int_0^\infty \int_0^\infty |M_2^{-1}(\mathcal{W}^{\Delta}_{\psi}(f))(a,x)|^2 d\vartheta_{\alpha,n}(a,x) \\
&= \int_0^\infty \left[\int_0^\infty |f \# D_{\alpha,a}(\overline{\psi})(x)|^2 x^{-4n} d\mu_{\alpha+2n}(x)\right] d\mu_{\alpha+2n}(a) \\
&= \int_0^\infty \left[\int_0^\infty |\mathscr{F}_{\Delta}(f \# D_{\alpha,a}(\overline{\psi}))(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right] d\mu_{\alpha+2n}(a) \\
&= \int_0^\infty \left[\int_0^\infty |\mathscr{F}_{\Delta}(f)(\lambda)|^2 |\mathscr{F}_{\Delta}(D_{\alpha,a}(\overline{\psi}))(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right] d\mu_{\alpha+2n}(a).
\end{aligned}$$

Now using relations (3.2) and (3.3) we get

$$\begin{aligned}
\|\mathcal{W}^{\Delta}_{\psi}(f)\|^2_{2,(\nu)} &= \int_0^\infty |\mathscr{F}_{\Delta}(f)(\lambda)|^2 \left[\int_0^\infty |D_{\alpha+2n,\frac{1}{a}}\mathscr{F}_{\Delta}(\overline{\psi})(\lambda)|^2 d\mu_{\alpha+2n}(a)\right] d\mu_{\alpha+2n}(\lambda) \\
&= \int_0^\infty |\mathscr{F}_{\Delta}(f)(\lambda)|^2 \left[\frac{1}{c_{\alpha+2n}}\int_0^\infty |\mathscr{F}_{\Delta}(\overline{\psi})(a)|^2 \frac{da}{a}\right] d\mu_{\alpha+2n}(\lambda) \\
&= C^{\Delta}_{\psi}\int_0^\infty |\mathscr{F}_{\Delta}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = C^{\Delta}_{\psi}\|f\|^2_{2,(\mu)}.
\end{aligned}$$

(ii) The outcome is derived from the polarization identity and (i).

■

Theorem 3.2. *Let ψ be a generalized admissible wavelet. For every $f \in L^2_{(\mu)}(\mathbb{R}_+)$, the function $\mathcal{W}^{\Delta}_{\psi}(f) \in L^p_{(\vartheta)}(\mathbb{R}^*_+ \times \mathbb{R}_+)$, $p \in [2,\infty]$ and we have*

$$\|\mathcal{W}^{\Delta}_{\psi}(f)\|_{p,(\vartheta)} \leqslant (C^{\Delta}_{\psi})^{\frac{1}{p}}\|\psi\|^{1-\frac{2}{p}}_{2,(\mu)}\|f\|_{2,(\mu)}. \tag{3.10}$$

Proof. According to the relation (3.8), the Plancherel's theorem for the generalized continuous wavelet transform for $p = 2$ produces

$$\|\mathcal{W}^{\Delta}_{\psi}(f)\|_{2,(\vartheta)} = \sqrt{C^{\Delta}_{\psi}}\|f\|_{2,(\mu)}.$$

For $p = \infty$, we have by the relation (3.7)

$$\left\|\mathcal{W}^{\Delta}_{\psi}(f)\right\|_{\infty,(\vartheta)} \leqslant \|f\|_{2,(\mu)}\|\psi\|_{2,(\mu)}.$$

The outcome of the Riez-Thorin Theorem is obtained. ■

Proposition 3.3. *For $\psi \in L^2_{(\mu)}(\mathbb{R}_+)$ be a generalized admissible wavelet. Then, $\mathcal{W}^{\Delta}_{\psi}(L^2_{(\mu)}(\mathbb{R}_+))$ is a reproducing kernel Hilbert space in $L^2_{(\vartheta)}(\mathbb{R}^*_+ \times \mathbb{R}_+)$, with kernel*

$$\mathscr{K}^{\Delta}_{\psi}((a,x),(a',x')) = \frac{1}{C^{\Delta}_{\psi}} \langle\, \psi^{\Delta}_{a,x} \mid \psi^{\Delta}_{a',x'} \,\rangle_{(\mu)}. \tag{3.11}$$

The kernel $\mathscr{K}^{\Delta}_{\psi}$ satisfies the following

$$\forall (a,x),(a',x') \in \mathbb{R}^*_+ \times \mathbb{R}_+, \quad \left|\mathscr{K}^{\Delta}_{\psi}((a,x),(a',x'))\right| \leqslant \frac{(xx')^{2n}}{C^{\Delta}_{\psi}} \|\psi\|^2_{2,(\mu)}.$$

Proof. From the relation (3.6), we have for all $(a,x),(a',x') \in \mathbb{R}^*_+ \times \mathbb{R}_+$,

$$\mathscr{K}^{\Delta}_{\psi}((a,x),(a',x')) = \frac{1}{C^{\Delta}_{\psi}} \mathcal{W}^{\Delta}_{\psi}(\psi^{\Delta}_{a,x})(a',x').$$

Thus, from Theorem 3.1, we deduce that for all $(a,x) \in \mathbb{R}^*_+ \times \mathbb{R}_+$ the function $\mathscr{K}^{\Delta}_{\psi}((a,x),(.,.))$ belongs to $L^2_{(\vartheta)}(\mathbb{R}^*_+ \times \mathbb{R}_+)$.

Let $F \in \mathcal{W}^{\Delta}_{\psi}(L^2_{(\mu)}(\mathbb{R}_+))$; $F = \mathcal{W}^{\Delta}_{\psi}(f)$, $f \in L^2_{(\mu)}(\mathbb{R}_+)$, by relations (3.6) and (3.9), we have for all $(a,x) \in \mathbb{R}^*_+ \times \mathbb{R}_+$

$$\begin{aligned} F(a,x) = \mathcal{W}^{\Delta}_{\psi}(f)(a,x) &= \langle\, f \mid \psi^{\Delta}_{a,x} \,\rangle_{(\mu)} \\ &= \frac{1}{C^{\Delta}_{\psi}} \langle\, \mathcal{W}^{\Delta}_{\psi}(f) \mid \mathcal{W}^{\Delta}_{\psi}(\psi^{\Delta}_{a,x}) \,\rangle_{(\vartheta)} \\ &= \langle\, \mathcal{W}^{\Delta}_{\psi}(f) \mid \mathscr{K}^{\Delta}_{\psi}((a,x),(.,.)) \,\rangle_{(\vartheta)} \end{aligned}$$

This demonstrates that given the Hilbert space $\mathcal{W}^{\Delta}_{\psi}(L^2_{(\mu)}(\mathbb{R}_+))$, $\mathscr{K}^{\Delta}_{\psi}$ is a reproducing Kernel.

Then, we obtain from relations (3.5), (2.11), and (3.1),

$$\begin{aligned} \left|\mathscr{K}^{\Delta}_{\psi}((a,x),(a',x'))\right| &= \frac{1}{C^{\Delta}_{\psi}} |\langle\, \psi^{\Delta}_{a,x} \mid \psi^{\Delta}_{a',x'} \,\rangle_{(\mu)}| \\ &\leqslant \frac{1}{C^{\Delta}_{\psi}} \|\psi^{\Delta}_{a,x}\|_{2,(\mu)} \|\psi^{\Delta}_{a',x'}\|_{2,(\mu)} \\ &\leqslant \frac{(xx')^{2n}}{C^{\Delta}_{\psi}} \|\psi\|^2_{2,(\mu)}. \end{aligned}$$

This achieves the proof. ■

4. Approximate Concentration

In this part, we introduce a weak uncertainty principle [16], which is adapted for the generalized continuous wavelet transforms. It is a Donoho and Stark type uncertainty principle. Such results were first reported by Gröchenig in [22], first for the Gabor transform. We also examine how concentrated these generalized continuous wavelet transformations are on subsets of $\mathbb{R}^*_+ \times \mathbb{R}_+$ with finite measures. Finally, we present a Benedicks-type uncertainty principle, subject to some assumptions on the wavelet function. Comparable outcomes are reported in [7, 36].

Proposition 4.1. *Consider a generalized wavelet ψ with the property that $\|\psi\|_{2,(\mu)} = 1$. For any function f belonging to the space $L^2_{(\mu)}(\mathbb{R}_+)$ satisfying the condition $\|f\|_{2,(\mu)} = 1$, and for any subset Ω of $\mathbb{R}^*_+ \times \mathbb{R}_+$ and $\xi \geq 0$, the following holds:*

$$1 - \xi \leq \iint_{\Omega} |M_2^{-1} \mathcal{W}^{\Delta}_{\psi}(f)(a,x)|^2 d\vartheta_{\alpha,n}(a,x),$$

we obtain,

$$1-\xi \leq \vartheta_{\alpha,n}(\Omega).$$

Proof. Based on equation (3.7), we obtain the following relation:

$$1-\xi \leq \iint_{\Omega} |M_2^{-1}\mathcal{W}_{\psi}^{\Delta}(f)(a,x)|^2 d\vartheta_{\alpha,n}(a,x) \leq \|\mathcal{W}_{\psi}^{\Delta}(f)\|_{\infty,(\vartheta)}\vartheta_{\alpha,n}(\Omega) \leq \vartheta_{\alpha,n}(\Omega).$$

■

Theorem 4.2. *Suppose ψ represents a generalized wavelet such that its norm, denoted by $\|\psi\|_{2,(\mu)} = 1$, and let Ω be a subset of $\mathbb{R}_+^* \times \mathbb{R}_+$ satisfying*

$$C_{\psi}^{\Delta} > \vartheta_{\alpha,n}(\Omega),$$

Therefore, given a function f in $L^2_{(\mu)}(\mathbb{R}_+)$, we get

$$\|\chi_{\Omega^c}\mathcal{W}_{\psi}^{\Delta}(f)\|_{2,(\vartheta)} \geq \sqrt{1-\frac{\vartheta_{\alpha,n}(\Omega)}{C_{\psi}^{\Delta}}}\sqrt{C_{\psi}^{\Delta}}\|f\|_{2,(\mu)}.$$

Proof. According to equation (3.7), it follows that for any function f belonging to the space $L^2_{(\mu)}(\mathbb{R}_+)$

$$\begin{aligned}\|\mathcal{W}_{\psi}^{\Delta}(f)\|^2_{2,(\vartheta)} &= \|\chi_{\Omega}\mathcal{W}_{\psi}^{\Delta}(f)\|^2_{2,(\vartheta)} + \|\chi_{\Omega^c}\mathcal{W}_{\psi}^{\Delta}(f)\|^2_{2,(\vartheta)} \\ &\leq \vartheta_{\alpha,n}(\Omega)\|\mathcal{W}_{\psi}^{\Delta}(f)\|^2_{\infty,(\vartheta)} + \|\chi_{\Omega^c}\mathcal{W}_{\psi}^{\Delta}(f)\|^2_{2,(\vartheta)} \\ &\leq \vartheta_{\alpha,n}(\Omega)\|f\|^2_{2,(\mu)}\|\psi\|^2_{2,(\mu)} + \|\chi_{\Omega^c}\mathcal{W}_{\psi}^{\Delta}(f)\|^2_{2,(\vartheta)}.\end{aligned}$$

We obtain the necessary result by using Plancherel's formula to $\mathcal{W}_{\psi}^{\Delta}$ as stated in relation (3.8) and the inequality $\vartheta_{\alpha,n}(\Omega) < C_{\psi}^{\Delta}$. ■

Remark 4.3. *It implies that the generalized wavelet transform $\mathcal{W}_{\psi}^{\Delta}(f)$ cannot be substantially focused on a set whose volume is smaller than the minimum C_{ψ}^{Δ} for any non-zero function f. In particular, we have*

$$\vartheta_{\alpha,n}(supp\mathcal{W}_{\psi}^{\Delta}(f)) < C_{\psi}^{\Delta} \;\Rightarrow\; f=0.$$

We take into account the following orthogonal projections:

1. P_{ψ}: This projection operates from $L^2_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+)$ to $\mathcal{W}_{\psi}^{\Delta}(L^2_{(\mu)}(\mathbb{R}_+))$. Its range is denoted by $\text{Im}P_{\psi}$.
2. P_{Ω}: Defined as the orthogonal projection onto $L^2_{(\vartheta)}(\mathbb{R}^*+\times\mathbb{R}+)$, given by

$$P_{\Omega}F = \chi_{\Omega}F, \qquad F \in L^2_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+),$$

where $F \in L^2_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+)$, and Ω is a subset of $\mathbb{R}_+^* \times \mathbb{R}_+$. The range of P_{Ω} is denoted by $\text{Im}P_{\Omega}$.

We define

$$\|P_{\Omega}P_{\psi}\| = \sup\{\|P_{\Omega}P_{\psi}(F)\|_{2,(\vartheta)}, F \in L^2_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+); \|F\|_{2,(\vartheta)} = 1\}.$$

Proposition 4.4. *Consider ψ be a generalized wavelet with a unit norm. A Hilbert Schmidt operator $P_{\Omega}P_{\psi}$ is defined for each subset $\Omega \subset \mathbb{R}_+^* \times \mathbb{R}_+$ of a finite measure $\vartheta_{\alpha,n}(\Omega)$ and we have*

$$\|P_{\Omega}P_{\psi}\|^2 \leq \frac{\vartheta_{\alpha,n}(\Omega)}{C_{\psi}^{\Delta}}. \tag{4.1}$$

Proof. For each function $F \in L^2_{(\vartheta)}(\mathbb{R}_+^* \times \mathbb{R}_+)$, we obtain since P_ψ is a projection onto a reproducing kernel Hilbert space

$$P_\psi(F)(a,x) = \langle\, F \mid \mathscr{K}_\psi^\Delta((a,x),(.,.))\, \rangle_{(\vartheta)}$$

as defined by (3.11) for $\mathscr{K}_\psi^\Delta$. Thus,

$$P_\Omega P_\psi(F)(a,x) = \langle\, F \mid \chi_\Omega(a,x)\mathscr{K}_\psi^\Delta((a,x),(.,.))\, \rangle_{(\vartheta)}$$

Now, using the definition of the kernel provided by the relation (3.11), Fubini's theorem, relations (3.5), Plancherel's formula for the generalized wavelet transform (3.8), (2.11), and (3.1), we obtain

$$\begin{aligned}
\|P_\Omega P_\psi\|_{HS}^2 &= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (xx')^{-4n}|\chi_\Omega(a,x)|^2|\mathscr{K}_\psi^\Delta((a,x),(a',x'))|^2 d\vartheta_{\alpha,n}(a',x')|d\vartheta_{\alpha,n}(a,x)\\
&= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty x^{-4n}\chi_\Omega(a,x)|\frac{1}{C_\psi^\Delta}M_2^{-1}\mathcal{W}_\psi^\Delta(\psi_{a,x})(a',x')|^2 d\vartheta_{\alpha,n}(a',x')d\vartheta_{\alpha,n}(a,x)\\
&= \frac{1}{C_\psi^\Delta}\int\int_\Omega x^{-4n}\frac{1}{C_\psi^\Delta}\|\mathcal{W}_\psi^\Delta(\psi_{a,x})\|_{2,(\vartheta)}^2 d\vartheta_{\alpha,n}(a,x)\\
&= \frac{1}{C_\psi^\Delta}\int\int_\Omega x^{-4n}\|\psi_{a,x}\|_{2,(\mu)}^2 d\vartheta_{\alpha,n}(a,x) = \frac{1}{C_\psi^\Delta}\int\int_\Omega x^{-4n}\|\tau_x^\Delta D_{\alpha,a}(\psi)\|_{2,(\mu)}^2 d\vartheta_{\alpha,n}(a,x)\\
&\leq \frac{\|\psi\|_{2,(\mu)}^2}{C_\psi^\Delta}\vartheta_{\alpha,n}(\Omega) = \frac{\vartheta_{\alpha,n}(\Omega)}{C_\psi^\Delta}.
\end{aligned}$$

The integral operator $P_\Omega P_\psi$ has a Hilbert Schmidt kernel as a result. The fact that $\|P_\Omega P_\psi\| \leq \|P_\Omega P_\psi\|_{HS}$ implies the outcome. ■

According to Havin and Jöricke [25, 1.A, p.88], we have the following

Proposition 4.5. *Let Ω be a subset of $\mathbb{R}_+^* \times \mathbb{R}_+$ and let ψ be a generalized wavelet. The following is our equivalency*

1. *In $L^2_{(\mu)}(\mathbb{R}_+)$, there is a constant c= c(Ω, ψ) > 0 such that for any function f*

$$\sqrt{C_\psi^\Delta}\|f\|_{2,(\mu)} \leq c\|\chi_{\Omega^c}\mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}. \tag{4.2}$$

2. $\|P_\Omega P_\psi\| < 1$.

Remark 4.6. *1. If the relation (4.2) is met, then (P_Ω, P_ψ) is considered a strong a-pair.*

2. *If $\|P_\Omega P_\psi\| < 1$, then*

$$\sqrt{C_\psi^\Delta}\|f\|_{2,(\mu)} \leq \frac{1}{\sqrt{1-\|P_\Omega P_\psi\|^2}}\|\chi_{\Omega^c}\mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}. \tag{4.3}$$

3. *Relative to (4.1) and (4.3), Theorem 4.2 can be obtained.*

Theorem 4.7. ***(Benedicks-type uncertainty principle for $\mathcal{W}_\psi^\Delta$)*** *For each generalized wavelet ψ, allow $\mu_{\alpha+2n}(\{\mathscr{F}_\Delta(\psi) \neq 0\}) < \infty$. Let $\int_0^\infty \chi_\Omega(a,x)d\mu_{\alpha+2n}(x) < \infty$ be any subset Ω of $\mathbb{R}_+^* \times \mathbb{R}_+$ such that for virtually every $a > 0$, we have*

$$\mathcal{W}_\psi^\Delta(L^2_{(\mu)}(\mathbb{R}_+)) \cap ImP_\Omega = \{0\}. \tag{4.4}$$

Proof. If F is a non-trivial function in $\mathcal{W}_\psi^\Delta(L^2_{(\mu)}(\mathbb{R}_+)) \cap ImP_\Omega$, then $F = \mathcal{W}_\psi(f)$ and $SuppF \subset \Omega$ exist for some function f in $L^2_{(\mu)}(\mathbb{R}_+)$. Suppose $a > 0$. Then, $\int_0^\infty \chi_\Omega(a,x) d\mu_{\alpha+2n}(x) < \infty$. Examine the function

$$F_a(x) = \mathcal{W}_\psi^\Delta(f)(a,x), \qquad x \geq 0.$$

After that,

$$suppF_a \subset \{x \geq 0; (a,x) \in \Omega\},$$

additionally

$$\mu_{\alpha+2n}(suppF_a) < \infty.$$

Currently, we have by utilizing the relation (2.18),

$$\mathscr{F}_\Delta(F_a)(\lambda) = \mathscr{F}_\Delta(f)(\lambda)\mathscr{F}_\Delta(D_a^\alpha \psi)(\lambda), \qquad a.e.$$

Consequently

$$\{\mathscr{F}_\Delta(F_a) \neq 0\} \subset \{\mathscr{F}_\Delta(\psi) \neq 0\}.$$

Furthermore, we derive the following from the hypothesis: $\mu_{\alpha+2n}(\{\mathscr{F}_\Delta(F_a) \neq 0\}) < \infty$. By applying the Fourier-Bessel transform Benedicks-type theorem [19], we may infer that, for any $a > 0$, F(a,.) = 0, implying that F=0. ■

The outcome that follows is a direct result of [[24], 2. A) p. 90].

Proposition 4.8. *If ψ is a generalized wavelet with $\mu_{\alpha+2n}(\{\mathscr{F}_\Delta(\psi) \neq 0\}) < \infty$, and Ω is a subset of $\mathbb{R}_+^* \times \mathbb{R}_+$ with $\vartheta_{\alpha,n}(\Omega) < \infty$, then $c(\Omega,\psi) > 0$, such that the inequality (4.2) holds.*

Here, we rephrase the proof given in [5].

Proof. Because P_Ω, P_ψ are projections, the equation $\|P_\Omega P_\psi(F)\|_{2,(\vartheta)} = \|F\|_{2,(\vartheta)}$, implies P_Ω(F)= P_ψ(F) =F. Now, the fact that

$$\vartheta_{\alpha,n}(\Omega) = \int_0^\infty \int_0^\infty \chi_\Omega(a,x) d\vartheta_{\alpha,n}(a,x) < \infty,$$

implies that for almost every $a > 0$,

$$\int_0^\infty \chi_\Omega(a,x) d\mu_\alpha(a,x) < \infty.$$

Then, from relation (4.4), we get $F = 0$ and therefore, for F $\neq 0$ we have $\|P_\Omega P_\psi(F)\|_{2,(\vartheta)} < \|F\|_{2,(\vartheta)}$. Using the fact that $P_\Omega P_\psi$ is a Hilbert-Schmidt operator, we deduce that its largest eigenvalue λ satisfies $|\lambda| < 1$ and $\|P_\Omega P_\psi\| = |\lambda| < 1$.

The result follows from Proposition 4.5. ■

5. Heisenberg-Pauli-Weyl Type Inequalities for $\mathcal{W}_\psi^\Delta$.

The primary findings of this study, the Heisenberg-Pauli-Weyl type inequality for $\mathscr{F}_\Delta$ and the generalised wavelet transform $\mathcal{W}_\psi^\Delta$, are presented in this section. We consult Rassias [30] for his study on the classical Fourier transform. Rösler and Voit demonstrated the Heisenberg-Pauli-Weyl uncertainty principle for the Hankel transform in [31]. It asserts that for any function $f \in L^2_{\mu_\alpha}(\mathbb{R}_+)$,

$$\|rf\|_{2,\mu_\alpha} \|\lambda \mathscr{H}_\alpha(f)\|_{2,\mu_\alpha} \geqslant (\alpha+1)\|f\|^2_{2,\mu_\alpha},$$

with equality for any $d \in \mathbb{C}$ and $b > 0$, if and only if $f(r) = de^{-br^2/2}$.

The previous inequality was extended by Ma in his paper [28] to a more general setting, namely the Chébli-Triméche hypergroups. Specifically, he established that, for $s, t > 0$, there exists a constant $c = c(\alpha, s, t) > 0$ such that, for every function $f \in L^2_{\mu_\alpha}(\mathbb{R}_+)$, we have

$$\|r^s f\|_{2,\mu_\alpha}^{\frac{t}{s+t}} \|\lambda^t \mathscr{H}_\alpha(f)\|_{2,\mu_\alpha}^{\frac{s}{s+t}} \geqslant c\|f\|_{2,\mu_\alpha}.$$

Subsequently, Soltani provided the constant c in the cases $s \geqslant 1$ and $t \geqslant 1$ explicitly in his article [33], which is $c = (\alpha+1)^{\frac{st}{s+t}}$. If and only if $s = t = 1$ and $f(r) = de^{-br^2/2}$ for some $d \in \mathbb{C}$ and $b > 0$, then we have equality.

Combining these outcomes, we obtain

Theorem 5.1. *Let $t, s > 0$. For any $f \in L^2_{\mu_\alpha}(\mathbb{R}_+)$, there is a constant $c = c(\alpha, s, t) > 0$ such that*

$$\|r^s f\|_{2,\mu_\alpha}^{\frac{t}{s+t}} \|\lambda^t \mathscr{H}_\alpha(f)\|_{2,\mu_\alpha}^{\frac{s}{s+t}} \geqslant c\|f\|_{2,\mu_\alpha}, \tag{5.1}$$

Moreover, for $s, t \geq 1$ the constant $c = (\alpha+1)^{\frac{st}{s+t}}$ with equality if and only if $s = t = 1$ and $f(r) = de^{-br^2/2}$ for some $d \in \mathbb{C}$ and $b > 0$.

In the following theorem we give the Heisenberg-Pauli-Weyl type inequality for $\mathscr{F}_\Delta$.

Theorem 5.2. *Assume $s, t > 0$. There is a constant $c = c(\alpha, n, s, t) > 0$, for any function $f \in L^2_{(\mu)}(\mathbb{R}_+)$ such that*

$$\|r^s f\|_{2,(\mu)}^{\frac{t}{s+t}} \|\lambda^t \mathscr{F}_\Delta(f)\|_{2,\mu_{\alpha+2n}}^{\frac{s}{s+t}} \geq c\|f\|_{2,(\mu)}. \tag{5.2}$$

Moreover, for $s, t \geq 1$ the constant c is given by $(\alpha+2n+1)^{\frac{st}{s+t}}$ with equality if and only if $s = t = 1$ and $f(r) = dr^{2n} e^{\frac{-br^2}{2}}$ for some $d \in \mathbb{C}$ and $b > 0$.

Proof. Assume $f \in L^2_{(\mu)}(\mathbb{R}_+)$. Using the relation (5.1) to apply the Heisenberg-Pauli-Weyl inequality for Hankel transform with index $\alpha + 2n$, we obtain

$$\begin{aligned}
\|r^s f\|_{2,(\mu)}^{\frac{t}{s+t}} \|\lambda^t \mathscr{F}_\Delta(f)\|_{2,\mu_{\alpha+2n}}^{\frac{s}{s+t}} &= \|M^{-1}(r^s f)\|_{2,\mu_{\alpha+2n}}^{\frac{t}{s+t}} \|\lambda^t \mathscr{H}_{\alpha+2n}(M^{-1}f)\|_{2,\mu_{\alpha+2n}}^{\frac{s}{s+t}} \\
&= \|r^s M^{-1}(f)\|_{2,\mu_{\alpha+2n}}^{\frac{t}{s+t}} \|\lambda^t \mathscr{H}_{\alpha+2n}(M^{-1}f)\|_{2,\mu_{\alpha+2n}}^{\frac{s}{s+t}} \\
&\geq c\|M^{-1}f\|_{2,\mu_{\alpha+2n}} \\
&\geq c\|f\|_{2,(\mu)}.
\end{aligned}$$

If and only if $s = t = 1$ and $f(r) = dr^{2n} e^{\frac{-br^2}{2}}$, then $c = (\alpha+2n+1)^{\frac{st}{s+t}}$ with equality for $s, t \geq 1$. ■

In the next theorems, we establish inequalities that we will use to prove Heisenberg-Pauli-Weyl type inequality for $\mathcal{W}^\Delta_\psi$.

Theorem 5.3. *Let ψ be a generalized admissible wavelet in $L^2_{(\mu)}(\mathbb{R}_+)$ and $s, t > 0$. Then, for any function $f \in L^2_{(\mu)}(\mathbb{R}_+)$, there is a constant $c = c(\alpha, n, s, t) > 0$, such that*

$$\|x^s \mathcal{W}^\Delta_\psi(f)\|_{2,(\nu)}^{\frac{t}{s+t}} \|\lambda^t \mathscr{F}_\Delta(f)\|_{2,\mu_{\alpha+2n}}^{\frac{s}{s+t}} \geq c(\sqrt{C_\psi})^{\frac{t}{s+t}} \|f\|_{2,(\mu)},$$

Furthermore, $c = (\alpha+2n+1)^{\frac{st}{s+t}}$ if $s, t \geq$1.

Proof. Considering that both of the integrals on the left-hand side are finite is a non-trivial situation. The function $x \mapsto \mathcal{W}_\psi^\Delta(f)(a,x)$ may be obtained by using the Heisenberg-Pauli-Weyl type inequality for $\mathscr{F}_\Delta$. For every $a \in \mathbb{R}_+^*$,

$$\left(\int_0^\infty x^{2s}|\mathcal{W}_\psi^\Delta(f)(a,x)|^2 \frac{x^{2\alpha+1}}{c_{\alpha+2n}}dx\right)^{\frac{t}{s+t}} \left(\int_0^\infty \lambda^{2t}|\mathscr{F}_\Delta(\mathcal{W}_\psi^\Delta(f)(a,.))(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{s}{s+t}}$$
$$\geq c^2 \int_0^\infty |\mathcal{W}_\psi^\Delta(f)(a,x)|^2 \frac{x^{2\alpha+1}}{c_{\alpha+2n}}dx.$$

Therefore, by integrating over $d\mu_{\alpha+2n}(a)$ and using Plancherel's theorem and Hölder's inequality for $\mathcal{W}_\psi^\Delta$, we obtain

$$\|x^s\mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^{\frac{2t}{s+t}} \left(\int_0^\infty \int_0^\infty \lambda^{2t}|\mathscr{F}_\Delta(\mathcal{W}_\psi^\Delta(f)(a,.))(\lambda)|^2 d\mu_{\alpha+2n}(a)d\mu_{\alpha+2n}(\lambda)\right)^{\frac{s}{s+t}}$$
$$\geq c^2\|\mathcal{W}_\psi^\Delta(f)\|_{(\vartheta)}^2 = c^2 C_\psi^\Delta \|f\|_{2,(\mu)}^2.$$

But, relations (3.6) and (2.18) yield

$$\int_0^\infty \int_0^\infty \lambda^{2t}|\mathscr{F}_\Delta(\mathcal{W}_\psi^\Delta(f)(a,.))(\lambda)|^2 d\mu_{\alpha+2n}(a)d\mu_{\alpha+2n}(\lambda)$$
$$= \int_0^\infty \int_0^\infty \lambda^{2t}\left|\mathscr{F}_\Delta\big(f\#D_{\alpha,a}(\overline{\psi})\big)(\lambda)\right|^2 d\mu_{\alpha+2n}(a)d\mu_{\alpha+2n}(\lambda)$$
$$= \int_0^\infty \lambda^{2t}|\mathscr{F}_\Delta(f)(\lambda)|^2\left(\int_0^\infty |\mathscr{F}_\Delta(D_{\alpha,a}(\overline{\psi}))|^2(\lambda)d\mu_{\alpha+2n}(a)\right)d\mu_{\alpha+2n}(\lambda).$$

Then, from relations (3.2) and (3.3) it follows

$$\int_0^\infty \int_0^\infty \lambda^{2t}|\mathscr{F}_\Delta(\mathcal{W}_\psi^\Delta(f)(a,.))(\lambda)|^2 d\mu_{\alpha+2n}(a)d\mu_{\alpha+2n}(\lambda) = C_\psi^\Delta\|\lambda^t\mathscr{F}_\Delta(f)\|_{\alpha,\mu_{\alpha+2n}}^2.$$

Then,

$$\|x^s\mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^{\frac{t}{s+t}}\|\lambda^t\mathscr{F}_\Delta(f)\|_{2,\mu_{\alpha+2n}}^{\frac{s}{s+t}} = \|x^s\mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^{\frac{s}{s+t}}(\sqrt{C_\psi^\Delta})^{\frac{s}{s+t}}\|\lambda\mathscr{F}_\Delta(f)\|_{\alpha,\mu_{\alpha+2n}}^{\frac{s}{s+t}}$$
$$\geq c\sqrt{C_\psi^\Delta}\|f\|_{2,(\mu)}.$$

it yields the outcome. ■

Theorem 5.4. *Let ψ be a generalized admissible wavelet in $L^2_{(\mu)}(\mathbb{R}_+)$ and $s,t>0$. Then, there exists a constant $c=c(\alpha,n,s,t)>0$, such that*

$$\|r^s f\|_{2,(\mu)}^{\frac{t}{s+t}}\|a^t\mathcal{W}_\psi^\Delta(f)\|_{2,(\vartheta)}^{\frac{s}{s+t}} \geq c\left(\sqrt{\frac{1}{c_{\alpha+2n}}\mathcal{M}\big(|\mathscr{F}_\Delta(\psi)|^2\big)(2t)}\right)^{\frac{s}{s+t}}\|f\|_{2,(\mu)},$$

for every function $f \in L^2_{(\mu)}(\mathbb{R}_+)$, where $\mathcal{M}: f \mapsto \mathcal{M}(f)(z) = \int_0^\infty f(x)\frac{dx}{x^{z+1}}$ denotes the classical Mellin transform and $c_{\alpha+2n}$ is the constant given in (2.2).

Moreover, if $s,t \geqslant 1$ then $c=(\alpha+2n+1)^{\frac{st}{s+t}}$ and we have equality if and only if $s=t=1$ and $f(r) = dr^{2n}e^{-br^2/2}$, $d\in\mathbb{C}$, $b>0$.

Proof. Let us assume the non-trivial case that both integrals on the left-hand side are finite.

Using Fubini's theorem, Plancherel's theorem for $\mathscr{F}_\Delta$ given by (2.16) and the relation (2.18), we get

$$\begin{aligned}\|a^t\mathcal{W}_\psi^\Delta(f)\|^2_{2,(\nu)} &= \int_0^\infty a^{2t}\left(\int_0^\infty |\mathcal{W}_\psi^\Delta f(a,x)|^2\frac{x^{2\alpha+1}}{c_{\alpha+2n}}\right)d\mu_{\alpha+2n}(a)\\ &= \int_0^\infty a^{2t}\left(\int_0^\infty |\mathscr{F}_\Delta(f\# D_{\alpha,a}(\overline{\psi}))(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)d\mu_{\alpha+2n}(a)\\ &= \int_0^\infty |\mathscr{F}_\Delta(f)(\lambda)|^2\left(\int_0^\infty a^{2t}|\mathscr{F}_\Delta(D_{\alpha,a}(\overline{\psi}))(\lambda)|^2 a^{2\alpha+4n+1}da\right)d\mu_{\alpha+2n}(\lambda)\\ &= \int_0^\infty |\mathscr{F}_\Delta(f)(\lambda)|^2\left(\int_0^\infty a^{2t}|D_{\alpha+2n,\frac{1}{a}}\mathscr{F}_\Delta(\overline{\psi})(\lambda)|^2 d\mu_{\alpha+2n}(a)\right)d\mu_{\alpha+2n}(\lambda)\\ &= \int_0^\infty |\mathscr{F}_\Delta(f)(\lambda)|^2\left(\frac{1}{c_{\alpha+2n}}\int_0^\infty a^{2t}|\mathscr{F}_\Delta(\overline{\psi})(\frac{\lambda}{a})|^2\frac{da}{a}\right)d\mu_{\alpha+2n}(\lambda),\end{aligned}$$

by a change of variables $b=\frac{\lambda}{a}$, it follows

$$\begin{aligned}\|a^t\mathcal{W}_\psi^\Delta(f)\|^2_{2,(\vartheta)} &= \int_0^\infty \lambda^{2t}|\mathscr{F}_\Delta(f)(\lambda)|^2\left(\frac{1}{c_{\alpha+2n}}\int_0^\infty |\mathscr{F}_\Delta(\overline{\psi})(b)|^2\frac{db}{b^{2t+1}}\right)d\mu_{\alpha+2n}(\lambda)\\ &= \big(\frac{1}{c_{\alpha+2n}}\mathcal{M}(|\mathscr{F}_\Delta(\psi)|^2)(2t)\big)\|\lambda^t\mathscr{F}_\Delta(f)\|^2_{2,\mu_{\alpha+2n}}\end{aligned} \tag{5.3}$$

Now, applying Heisenberg-Pauli-Weyl inequality for $\mathscr{F}_\Delta$ given in the relation (5.2), we get

$$\begin{aligned}\|r^sf\|^{\frac{t}{t+s}}_{2,(\mu)}\|a^t\mathcal{W}_\psi^\Delta(f)\|^{\frac{s}{t+s}}_{2,(\vartheta)} &= \left(\frac{1}{c_{\alpha+2n}}\sqrt{\mathcal{M}(|\mathscr{F}_\Delta(\psi)|^2)(2t)}\right)^{\frac{s}{t+s}}\|r^sf\|^{\frac{t}{t+s}}_{2,(\mu)}\|\lambda^t\mathscr{F}_\Delta(f)\|^{\frac{s}{t+s}}_{2,\mu_{\alpha+2n}}\\ &\geq c\left(\sqrt{\frac{1}{c_{\alpha+2n}}\mathcal{M}(|\mathscr{F}_\Delta(\psi)|^2)(2t)}\right)^{\frac{s}{t+s}}\|f\|_{2,(\mu)}.\end{aligned}$$

■

The next theorem proves the Heisenberg-Pauli-Weyl uncertainty principle for $\mathcal{W}_\psi^\Delta$ which involves the two variables of the time-frequency plan.

Theorem 5.5. *Let $s,t>0$ and ψ be a generalized admissible wavelet in $L^2_{(\mu)}(\mathbb{R}_+)$. Then, there exists a constant $c=c(\alpha,n,s,t)>0$, such that*

$$\|x^s\mathcal{W}_\psi^\Delta(f)\|^{\frac{t}{s+t}}_{2,(\vartheta)}\|a^t\mathcal{W}_\psi^\Delta(f)\|^{\frac{s}{s+t}}_{2,(\vartheta)} \geq c\left(\sqrt{\frac{1}{c_{\alpha+2n}}\mathcal{M}\big(|\mathscr{F}_\Delta(\psi)|^2\big)(2t)}\right)^{\frac{s}{s+t}}\big(\sqrt{C_\psi^\Delta}\big)^{\frac{t}{s+t}}\|f\|_{2,(\mu)},$$

for every function $f\in L^2_{(\mu)}(\mathbb{R}_+)$. Moreover, if $s,t\geq 1$ then $c=(\alpha+2n+1)^{st/(s+t)}$.

Proof. From the equality (5.3),

$$\|x^s\mathcal{W}_\psi^\Delta(f)\|^{\frac{t}{s+t}}_{2,(\vartheta)}\|a^t\mathcal{W}_\psi^\Delta(f)\|^{\frac{s}{s+t}}_{2,(\vartheta)} = \|x^s\mathcal{W}_\psi^\Delta(f)\|^{\frac{t}{s+t}}_{2,(\vartheta)}\left(\sqrt{\frac{1}{c_{\alpha+2n}}\mathcal{M}(|\mathscr{F}_\Delta(\psi)|^2)(2t)}\right)^{\frac{s}{s+t}})\|\lambda^t\mathscr{F}_\Delta(f)\|^{\frac{s}{s+t}}_{2,\mu_{\alpha+2n}},$$

thus, using Theorem 5.3, we get

$$\|x^s\mathcal{W}_\psi^\Delta(f)\|^{\frac{t}{s+t}}_{2,(\vartheta)}\|a^t\mathcal{W}_\psi^\Delta(f)\|^{\frac{s}{s+t}}_{2,(\nu)} \geq c\left(\sqrt{\frac{1}{c_{\alpha+2n}}\mathcal{M}\big(|\mathscr{F}_\Delta(\psi)|^2\big)(2t)}\right)^{\frac{s}{s+t}}\big(\sqrt{C_\psi^\Delta}\big)^{\frac{t}{s+t}}\|f\|_{2,(\mu)}.$$

■

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Formal derivation and existence of global weak solutions of an energetically consistent viscous sedimentation model

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Abstract. The purpose of this paper is to derive a viscous sedimentation model from the Navier-Stokes system for incompressible flows with a free moving boundary. The derivation is based on the different properties of the fluids; thus, we perform a multiscale analysis in space and time, and a different asymptotic analysis to derive a system coupling two different models: the sediment transport equation for the lower layer and the shallow water model for the upper one. We finally prove the existence of global weak solutions in time for model containing some additional terms.
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Contents

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1. Introduction

Sediment transport models are used to model watercourse beds. They are bilayer models of two immiscible layers that have a model of the shallow water in the first layer and Reynolds lubrication equation at the second layer. In the literature, many works has been done on sediment transport, proposing models to stimulate sediment transport by water. We can quote [6, 7, 22, 24].
Recently in [7], Fernandez and al. are derived a non-viscous sediment model. In their work, they are limited to a first-order approximation for obtaining the model of shallow water which does not allow to obtain a viscous model. To carry out our work we relied on the papers [6, 7, 19, 21].

From theoretical point of view, many studies have been done, particularly for the existence of global weak solutions of shallow-water equations model. As an example, we refer to [5], where such results were given for an isothemal model of compressible fluids with capillarity.

In [21], only the stability of weak solutions has been proved, since the construction of approximate weak solutions which preserve the 'mathematical BD entropy' seems to be an open problem. In the present work this problem does not exist, as we do not need the multiplier $|u|^k u$ to get the 'BD entropy'.

In the analysis we propose in this work, our contribution is twofold. On the one hand, we propose a constructive approach inspired by [7, 16] to arrive at a viscous sedimentation model. Our purpose is to study the evolution of this system which consists of two layers of Newtonian viscous fluids with different properties. On the other hand, our study is concerned with the existence of global weak solutions of a model similar to the one we obtained. This is done in a bounded domain of $\mathbb{R}^2$ with periodic boundary conditions.

In our model we add some additional regularizing terms, namely
$-\kappa\nabla\cdot(1+\dfrac{h_2}{rh_1})\nabla(h_1+\dfrac{1}{r}h_2)$, the cold pressure $\delta h_1\nabla h_1^{-\alpha}$ and the interface tension $\bar{\kappa}h_1\nabla\Delta^{2s+1}h_1$ with $\alpha,\kappa,\bar{\kappa}$ positive constants and $\alpha\neq 0$. Those terms are useful to bound h_1 away form zero (see [3, 11, 24]).

Our paper is organized as follows. In the section 2, we did the formal derivation of the model. First of all we write the equations in non-dimensional variables. Next, we perform the hydrostatic approximation and use am asymptotic analysis to deduce the shallow water system for the upper layer. Also by an asymptotic analysis, we deduce the transport equation for the lower layer. In addition in the section 3, we present our final model. To finish, in Section 3 we study the existence of global weak solutions for a model similar to that obtained in Section 2. We start by giving the definition of global weak solutions, next we establish a classical energy equality and the 'mathematical BD entropy', which entail some regularities on the unknowns. We also give an existence theorem of global weak solutions.

2. Formal derivation

2.1. Physical domain and governing equations

This section is devoted to the formal derivation of the model. Thus, we consider a superposition of two immiscible layers of different materials. The upper layer contains water and the lower layer is formed of sediment. Each layer is governed by the incompressible three dimensional Navier Stokes equations. We consider a cartesian coordinate system where x represents the horizontal 2D direction and z the vertical one. Taking into account the different physical properties for each layer, we derive shallow water model for the upper layer and the Reynolds lubrification equation for the lower layer. Let us define the physical domain for the fluid and sediment by $\Omega_1(t)$ and $\Omega_2(t)$ respectively; t being the time variable. Here, we suppose that the sediment domain is composed by a one layer. We assume that the bottom is defined by the function $b(x)$ and we denote by $\eta(t,x)$ the interface. The free surface is given by $\xi(t,x)$. The global domain $\Omega(t)$ is defined as

$$\Omega(t) = \Omega_1(t) \cup \Omega_2(t) \cup \Gamma_b(t) \cup \Gamma_{1,2}(t) \cup \Gamma_s(t),$$

$\Omega_1(t) = \{(x,z) \in \mathbb{R}^3 : x \in \omega, \eta(x,t) < z < \xi(x,t)\}$,

$\Omega_2(t) = \{(x,z) \in \mathbb{R}^3 : x \in \omega, b(x) < z < \eta(x,t)\}$,

$\Gamma_{1,2}(t) = \{(x,z) \in \mathbb{R}^3 : x \in \omega, z = \eta(x,t)\}$,

$\Gamma_s(t) = \{(x,z) \in \mathbb{R}^3 : x \in \omega, z = \xi(x,t)\}$,

and
$\Gamma_b = \{(x,z) \in \mathbb{R}^3 : x \in \omega, z = b(x)\}$.

The domain $\Omega(t) \subset \mathbb{R}^3$ is periodic. For each layer $(i = 1,2)$, we start from the 3D Navier-Stokes equations for incompressible fluid and sediment components see [6, 7, 15]

$$\text{div}(U_i) = 0, \tag{2.1a}$$
$$\rho_i \partial_t(U_i) + (\rho_i U_i \nabla) U_i - \text{div}(\sigma_i) = -\rho_i g \vec{e}_z, \tag{2.1b}$$

where we denote by $U_i = {}^t(\mathbf{u}_i, w_i)$ the velocity field with $\mathbf{u}_i = (u_i, v_i)$, σ_i the stress tensor associated to each layer, ρ_i the density and g the gravitational vector with $\vec{e}_z =^t (0,0,1)$.
If we rewrite the equation for each component of the velocity, the previous system is equivalent to the following one:

$$\text{div}_x \mathbf{u}_i + \partial_z w_i = 0, \tag{2.2a}$$
$$\rho_i \partial_t \mathbf{u}_i + \rho_i \mathbf{u}_i \nabla \mathbf{u}_i + \rho_i w_i \partial_z(\mathbf{u}_i) + \nabla p_i = 2\nu_i \text{div}(D(\mathbf{u}_i)) + \nu_i \partial_z^2 \mathbf{u}_i + \nu_i \nabla_x(\partial_z w_i), \tag{2.2b}$$
$$\rho_i \partial_t w_i + \rho_i \mathbf{u}_i \nabla w_i + \rho_i w_i \partial_z w_i = \nu_i \Delta w_i + 2\nu_i \partial_z^2 w_i + \nu_i \partial_z(\text{div}\mathbf{u}_i) - \partial_z p_i - \rho_i g. \tag{2.2c}$$

for $i = 1,2$,
where ρ_i is the density, p_i the pressure and g the gravity constant.Moreover μ_i and $\nu_i = \mu_i/\rho_i$, denote the dynamic and kinematic viscosity coefficients respectively. We also introduce the ratio of the densites r, respectively the stress tensor given by

$$r = \frac{\rho_1}{\rho_2}, \qquad \sigma_i(\mathbf{u}_i) = 2\nu_i D(\mathbf{u}_i) - p_i Id, \quad \text{where} \quad D(\mathbf{u}_i) = \frac{\nabla \mathbf{u}_i +^t \nabla \mathbf{u}_i}{2},$$

and Id is the identity matrix.
From now on, subscript 1 will correspond to the layer located on the top and subscript 2 to those located belox. We denote by $h_1(t,x) = \xi(t,x) - b(x)$ the tichness of the layer 1 and by $h_2(t,x) = \eta(t,x) - b(x)$ the tichness of the sediment layer. See Figure 1.
The system (2.2a)-(2.2c) is completed by the following boundaries conditions:
• At the free surface $z = \xi(x,t) = b(x) + h_2(x,t) + h_1(x,t)$:

- The surface tension condition. Let N_s the unitary outward normal vector to the free surface and k the mean curvature of the surface with $k = -\text{div}(N_s)$. The surface tension is given by the equality

$$\sigma_1 N_s = -\delta k N_s, \tag{2.3}$$

where $N_s = \dfrac{1}{\sqrt{1 + |\nabla_x \xi|^2}} \begin{pmatrix} -\nabla_x \xi \\ 1 \end{pmatrix}$

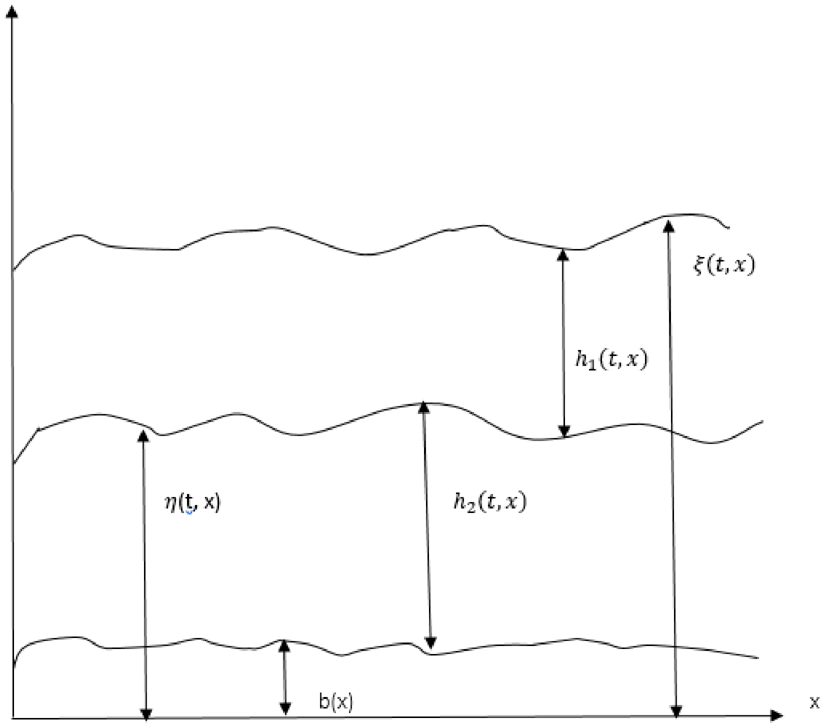

Figure 1: Sediment and water heights

and δ being a constant.

- The kinematic condition:

$$\partial_t \xi = U_1.N_s. \tag{2.4}$$

- At the fluid/sediment interface, $\eta(t,x) = b(x) + h_2(x,t)$:
  - The kinematic conditions corresponding to both velocities:

$$\partial_t \eta = U_1.N_\eta = U_2.N_\eta \tag{2.5}$$

where $N_\eta = \dfrac{1}{\sqrt{1+|\nabla_x \eta|^2}} \begin{pmatrix} -\nabla_x \eta \\ 1 \end{pmatrix}$.

  - The continuity of the normal component of the tensors:

$$(\sigma_1 N_\eta)_n - (\sigma_2 N_\eta)_n = (\delta_\eta k_\eta N_\eta), \tag{2.6}$$

where δ_η is the interfacial tension coefficient, $k_\eta = -\text{div} N_\eta$ is the mean curvature of the interface.

  - The friction law (Navier-slip boundary condition) at the fluid-sediment interface asserting that:

$$(\sigma_i N_\eta)_\tau = \text{fric}(U_1 - U_2)_\tau. \tag{2.7}$$

We note that the friction coefficient is denoted by c and the subscript τ is the tangential component of the vector.
In the sequel we denote by $\text{fric}(U_1 - U_2) = C\rho_1(U_1 - U_2)$ the friction term between the two layers.

• At the bottom, $z = b(x)$:

- The no penetration condition:

$$U_2.N_b = 0, \tag{2.8}$$

where the unitary normal vector to the bottom is
$N_b = \frac{1}{\sqrt{1+|\nabla_x b|^2}}\begin{pmatrix}-\nabla_x b\\ 1\end{pmatrix}$.

Remark 2.1. 1. In [7], a coulomb condition is considered between the static and the moving sediment particules. Here, we consider this condition at the interface $z = \eta(t, x)$.

2. To obtain the model, firstly we shall write these equations under a dimensionless. Secondly we shall develop the vertical integration in each layer to obtain the shallow water system. In addition, we shall perform the asymptotic analysis studding both, first and second order approximative for the the shallow water system. Finally, we will find for the sediment layer, the transport equation.

2.2. Dimensionless equations

In order to compare the terms that occur in the equations, we introduce dimensionless variables. For this, we note by H, and L the characteristic height and length respectively. In the considered flows, we assume that the characteristic height is very small compared to the characteristic length and we note by $\varepsilon = \frac{H}{L}$ the aspect ratio between the characteristic height and length. The characteristic velocities are U for the layer 1 and U_2 for the sediment layer. Consequently, the characteristic times are respectively $T = \frac{L}{U_1}$ and $T_2 = \frac{L}{V}$ for each layer. In particular we assume that

$$U_2 = \varepsilon^2 U, \quad \text{so consequently,} \quad T_2 = \frac{L}{U_2} = \frac{1}{\varepsilon^2}T.$$

This hypothesis also affects the definitions of the Froude and Reynolds numbers. For the sake of clarity we indicate separately these variables. We consider the ”asterisk” notation for the dimensionless variables.
General dimensionless variables:

$$x = L\bar{x}, \qquad z = H\bar{z}, \qquad \text{fric} = \rho_1 U^2 \bar{\text{fric}}$$

Non-dimensionalization for layer 1:

$$\mathbf{u}_1 = U\bar{\mathbf{u}}_1, \qquad w_1 = \varepsilon U\bar{w}_1, \qquad t_1 = \frac{L}{U}\bar{t}_1, \qquad p_1 = \rho_1 U^2 \bar{p}_1$$

$$F_{r_1} = \frac{U}{\sqrt{gH}}, \qquad Re_1 = \frac{UL}{\nu_1}, \qquad h_1 = H\bar{h}_1$$

Non-dimensionalization for layer 2:

$$\mathbf{u}_2 = \varepsilon^2 U\bar{\mathbf{u}}_2, \qquad w_2 = \varepsilon^3 U\bar{w}_2, \qquad t = \frac{1}{\varepsilon^2}T\bar{t}_2, \qquad p_2 = \frac{\rho_2\nu_2 U}{\varepsilon H}\bar{p}_2$$

$$F_{r_2} = \frac{\varepsilon^2 U}{\sqrt{gH}}, \qquad Re_2 = \frac{\varepsilon^2 UL}{\nu_2}, \qquad h_2 = H_2\bar{h}_2 \qquad \text{with} \quad H_2 = \varepsilon H.$$

We also define the ratio of the densities,

$$r = \frac{\rho_1}{\rho_2} \qquad \text{with} \qquad r < 1.$$

Remark 2.2. We set $C = U\bar{C}$.
Assuming that H is the characteristic height for the bottom, $b = H\bar{b}$.

Thus, the equations and the boundary conditions written in dimensionless form read as follows (we omit the "asterisk" to simplify the notation):

• Layer 1:

$$\mathrm{div}_x \mathbf{u}_1 + \partial_z w_1 = 0, \tag{2.9a}$$

$$\partial_{t_1}\mathbf{u}_1 + \mathbf{u}_1\nabla_x\mathbf{u}_1 + w_1\partial_z\mathbf{u}_1 + \nabla_x p_1 = \frac{1}{Re_1}(2\mathrm{div}_x(D_x(\mathbf{u}_1)) + \frac{1}{\varepsilon^2}\partial_z^2\mathbf{u}_1 + \nabla_x(\partial_z w_1)), \tag{2.9b}$$

$$\varepsilon^2(\partial_{t_1}w_1 + \mathbf{u}_1\nabla_x w_1 + w_1\partial_z w_1) = \frac{1}{Re_1}(\varepsilon^2\Delta_x w_1 + 2\partial_z^2 w_1 + \partial_z(\mathrm{div}_x\mathbf{u}_1)) - \partial_z p_1 - \frac{1}{Fr_1^2}. \tag{2.9c}$$

• Layer 2:

$$\mathrm{div}_x \mathbf{u}_2 + \partial_z w_2 = 0, \tag{2.10a}$$

$$\varepsilon^8 Re_2(\partial_{t_2}\mathbf{u}_2 + \mathbf{u}_2\nabla_x\mathbf{u}_2 + w_2\partial_z\mathbf{u}_2) + \nabla_x p_2 = 2\varepsilon^4\mathrm{div}_x(D_x(\mathbf{u}_2)) + \partial_{z^2}^2\mathbf{u}_2 + \varepsilon^4\nabla(\partial_z w_2) \tag{2.10b}$$

$$\varepsilon^8 Re_2(\partial_{t_2}w_2 + \mathbf{u}_2\nabla_x w_2 + w_2\partial_z w_2) = \varepsilon^4(\varepsilon^4\Delta w_2 + \partial_z(\mathrm{div}_x\mathbf{u}_2) + 2\partial_z^2 w_2)$$
$$- \varepsilon^4\frac{Re_2}{Fr_2^2} - \partial_z p_2. \tag{2.10c}$$

• Conditions at the free surface

$$\partial_{t_1}\xi + \mathbf{u}_1.\nabla_x\xi = w_1, \tag{2.11a}$$

$$\left(\frac{-2}{Re_1}D_x(\mathbf{u}_1) + \rho_1 p_1 - \rho_1\frac{\varepsilon}{Re_1}C^{-1}\Delta\xi\right)\nabla_x\xi + \frac{1}{Re_1}\nabla_x w_1 + \frac{1}{\varepsilon^2}\frac{1}{Re_1}\partial_z\mathbf{u}_1 = 0, \tag{2.11b}$$

$$-\frac{1}{Re_1}(\varepsilon^2\nabla_x w_1 + \partial_z\mathbf{u}_1)\nabla_x\xi + \frac{2}{Re_1}\partial_z w_1 + \rho_1\varepsilon\frac{1}{Re_1}C^{-1}\Delta\xi - \rho_1 p_1 = 0. \tag{2.11c}$$

• Conditions at the interface

$$\partial_{t_1}\eta + \mathbf{u}_1.\nabla_x\eta = w_1, \tag{2.12a}$$

$$\partial_{t_1}\eta + \varepsilon^2\mathbf{u}_2.\nabla_x\eta = \varepsilon^3 w_2, \tag{2.12b}$$

$$\partial_{t_2}\eta + \mathbf{u}_2.\nabla_x\eta = w_2, \tag{2.12c}$$

$$\frac{1}{Re_1}\left(\nabla w_1 + \frac{1}{\varepsilon^2}\partial_z\mathbf{u}_1\right) = -r\frac{1}{\varepsilon}\mathrm{fric}(\mathbf{u}_1 - \varepsilon^2\mathbf{u}_2)\sqrt{1+\varepsilon^2|\nabla\eta|^2}, \tag{2.12d}$$

$$\frac{1}{Re_1}\left(\varepsilon^3\nabla w_2 + \varepsilon\partial_z\mathbf{u}_2\right) = -\frac{1}{\varepsilon}\mathrm{fric}(\mathbf{u}_1 - \varepsilon^2\mathbf{u}_2)\sqrt{1+\varepsilon^2|\nabla\eta|^2}, \tag{2.12e}$$

$$\frac{1}{Re_1}\left(-2D(\mathbf{u}_1)\cdot\nabla\eta + (\nabla w_1 + \frac{1}{\varepsilon^2}\partial_z\mathbf{u}_1)(1-\varepsilon^2|\nabla\eta|^2) + 2\partial_z w_1\nabla\eta\right)$$
$$= r\frac{1}{\varepsilon}\mathrm{fric}\big((\mathbf{u}_1 - \varepsilon^2\mathbf{u}_2) + \varepsilon^2(w_1 - \varepsilon^2 w_2)\nabla\eta\big)\sqrt{1+\varepsilon^2|\nabla\eta|^2}, \tag{2.12f}$$

$$\frac{1}{Re_2}\left(-2\varepsilon^3 D(\mathbf{u}_2)\cdot\nabla\xi_\varepsilon + (\varepsilon^3\nabla w_2 + \partial_z\mathbf{u}_2)(1-\varepsilon^2|\nabla\eta|^2) + 2\varepsilon^2\partial_z w_2\nabla\eta\right)$$
$$= \frac{1}{\varepsilon}\mathrm{fric}\big((\mathbf{u}_1 - \varepsilon^2\mathbf{u}_2) + \varepsilon^2(w_1 - \varepsilon^2 w_2)\nabla\eta\big)\sqrt{1+\varepsilon^2|\nabla\eta|^2}, \tag{2.12g}$$

$$\rho_1\varepsilon^2[\frac{2}{Re_1}D(\mathbf{u}_1) - p_1]|\nabla\eta|^2 - 2\rho_1(\partial_z\mathbf{u}_1 + \varepsilon^2\nabla w_1)\nabla\eta + \rho_1(\frac{2}{Re_1}\partial_z w_1 - p_1)$$
$$= \varepsilon^2\frac{\rho_2}{Re_2}(\varepsilon^4 D(\mathbf{u}_2) - p_2)|\nabla\eta|^2 - 2\rho_2\frac{1}{Re_2}\varepsilon^3(\partial_z\mathbf{u}_2 + \varepsilon^3\nabla w_2)\nabla\eta$$
$$+ \frac{1}{Re_2}\rho_2(2\varepsilon^3\partial_z w_2 - p_2) - \varepsilon\rho_1\frac{C_\eta^{-1}}{Re_1}\mathrm{div}(\eta)(1+\varepsilon^2|\nabla\eta|^2). \tag{2.12h}$$

• Condition at the bottom

$$-\mathbf{u}_2\nabla_x b + w_2 = 0. \tag{2.13}$$

2.3. Layer Ω_1: Shallow water

To get the Saint-Venant-Exner system, we first take the hydrostatic approximation and then develop the asymptotic analysis of equations.

2.3.1. Hydrostatic approximation

Since the length of the flow is supposed to be very large compared to the depth of the water, we assume that ε to be small. Let us take the formal expression of system (2.2a)-(2.8) at $O(\varepsilon^2)$ (see [1, 9, 10, 12] for the usual derivations of hydrostatic approximations), and keep the terms of order zero and one. We obtain successively,

• Layer 1:

$$\mathrm{div}_x\mathbf{u}_1 + \partial_z w_1 = 0, \tag{2.14a}$$

$$\partial_t\mathbf{u}_1 + \mathbf{u}_1\nabla\mathbf{u}_1 + \partial_z(w_1\mathbf{u}_1) + \nabla p_1 = \frac{1}{Re_1}(2\mathrm{div}(D(\mathbf{u}_1)) + \frac{1}{\varepsilon^2}\partial_z^2\mathbf{u}_1 + \nabla(\partial_z w_1), \tag{2.14b}$$

$$\partial_z p_1 = -\frac{1}{Fr_1^2} + \frac{1}{Re_1}(2\partial_z^2 w_1 + \partial_z(\mathrm{div}\mathbf{u}_1)). \tag{2.14c}$$

• Layer 2:

$$\mathrm{div}_x\mathbf{u}_2 + \partial_z w_2 = 0, \tag{2.15a}$$

$$\nabla_x p_2 = \partial_z^2\mathbf{u}_2, \tag{2.15b}$$

$$\partial_z p_2 = O(\varepsilon). \tag{2.15c}$$

• Conditions at the free surface

$$\partial_{t_1}\xi + \mathbf{u}_1.\nabla_x\xi = w_1, \tag{2.16a}$$

$$\left(\frac{-2}{Re_1}D_x(\mathbf{u}_1) + \rho_1 p_1 - \rho_1\frac{\varepsilon}{Re_1}C^{-1}\nabla\xi\right)\nabla_x\xi + \frac{1}{Re_1}\nabla_x w_1 + \frac{1}{\varepsilon^2}\frac{1}{Re_1}\partial_z\mathbf{u}_1 = 0, \tag{2.16b}$$

$$-\frac{1}{Re_1}\partial_z\mathbf{u}_1\nabla_x\xi + \frac{2}{Re_1}\partial_z w_1 + \frac{\rho_1\varepsilon C^{-1}\Delta\xi}{Re_1} - \rho_1 p_1 = 0. \tag{2.16c}$$

• Conditions at the interface

$$\partial_{t_1}\eta + \mathbf{u}_1.\nabla_x\eta = w_1, \tag{2.17a}$$

$$\partial_{t_1}\eta = O(\varepsilon), \tag{2.17b}$$

$$\partial_{t_2}\eta + \mathbf{u}_2.\nabla_x\eta = w_2, \tag{2.17c}$$

$$\frac{1}{Re_1}\left(\nabla w_1 + \frac{1}{\varepsilon^2}\partial_z\mathbf{u}_1\right) = -r\frac{1}{\varepsilon}\text{fric}(\mathbf{u}_1 - \varepsilon^2\mathbf{u}_2)\sqrt{1+\varepsilon^2|\nabla\eta|^2}, \tag{2.17d}$$

$$\frac{1}{Re_1}\left(\varepsilon^4\nabla w_2 + \varepsilon\partial_z\mathbf{u}_2\right) = -\frac{1}{\varepsilon}\text{fric}(\mathbf{u}_1 - \varepsilon^2\mathbf{u}_2)\sqrt{1+\varepsilon^2|\nabla\eta|^2}, \tag{2.17e}$$

$$\begin{aligned}\frac{1}{Re_1}\Big(&-2D(\mathbf{u}_1)\cdot\nabla\eta + (\nabla w_1 + \frac{1}{\varepsilon^2}\partial_z\mathbf{u}_1)(1-\varepsilon^2|\nabla\eta|^2) + 2\partial_z w_1\nabla\eta\Big)\\ &= r\frac{1}{\varepsilon}\text{fric}\big((\mathbf{u}_1-\varepsilon^2\mathbf{u}_2) + \varepsilon^2(w_1-\varepsilon^3 w_2)\nabla\eta\big)\sqrt{1+\varepsilon^2|\nabla\eta|^2},\end{aligned} \tag{2.17f}$$

$$\begin{aligned}\frac{1}{Re_2}\Big(&-2\varepsilon^3 D(\mathbf{u}_2)\cdot\nabla\eta + (\varepsilon^3\nabla w_2 + \partial_z\mathbf{u}_2)(1-\varepsilon^2|\nabla\eta|^2) + 2\varepsilon^2\partial_z w_2\nabla\eta\Big)\\ &= \frac{1}{\varepsilon}\text{fric}\big((\mathbf{u}_1-\varepsilon^2\mathbf{u}_2) + \varepsilon^2(w_1-\varepsilon^2 w_2)\nabla\eta\big)\sqrt{1+\varepsilon^2|\nabla\eta|^2},\end{aligned} \tag{2.17g}$$

$$\begin{aligned}\varepsilon^2[&\frac{2}{Re_1}D(\mathbf{u}_1) - p_1]|\nabla\eta|^2 - 2r(\partial_z\mathbf{u}_1 + \varepsilon^2\nabla w_1)\nabla\eta + (\frac{2}{Re_1}\partial_{z_1}w_1 - p_1)\\ &= \varepsilon^2\frac{r}{Re_2}(\varepsilon^4 D(\mathbf{u}_2) - p_2)|\nabla\eta|^2 - 2r\frac{1}{Re_2}\varepsilon^3(\partial_{z_2}\mathbf{u}_2 + \varepsilon^3\nabla w_2)\nabla\eta\\ &\quad + \frac{r}{Re_2}(2\varepsilon^3\partial_z w_2 - p_2) - \varepsilon\frac{C_\eta^{-1}}{Re_1}\text{div}(\eta)(1+\varepsilon^2|\nabla\eta|^2).\end{aligned} \tag{2.17h}$$

• Conditions at the bottom

$$-\mathbf{u}_2\nabla_x b + w_2 = 0. \tag{2.18}$$

2.3.2. Asymptotic analysis and shallow water system

To obtain the shallow water equation, we assume that the height is small with respect to the length of the domain, that is $\varepsilon \ll 1$.
We first integrate each equations of (2.14a)-(2.14c) from the layer 1 from η to ξ. For equation (2.14a), by using (2.11a) (2.17a) and (2.17b), we get

$$\partial_{t_1}h_1 + \text{div}\int_\eta^\xi \mathbf{u}_1 dz = 0. \tag{2.19}$$

The condition (2.17a) allows us by integrating the equation (2.14b) to get

$$\begin{aligned}&\partial_{t_1}\int_\eta^\xi \mathbf{u}_1 dz + \text{div}\int_\eta^\xi \mathbf{u}_1\otimes\mathbf{u}_1 dz + \nabla_x\int_\eta^\xi p_1 - \frac{2}{Re_1}\text{div}\int_\eta^\xi D(\mathbf{u}_1)dz\\ &= \frac{1}{\varepsilon^2 Re_1}\partial_z\mathbf{u}_{1|z=\xi} - \frac{1}{\varepsilon^2 Re_1}\partial_z\mathbf{u}_{1|z=\eta} + \frac{1}{Re_1}\nabla_x w_{1|z=\xi} - \frac{1}{Re_1}\nabla_x w_{1|z=\eta}\\ &+ (w_1\mathbf{u}_1)_{|z=\xi} - (w_1\mathbf{u}_1)_{|z=\eta} - \mathbf{u}_1\partial_{t_1}\xi_{|z=\xi} + \mathbf{u}_1\partial_{t_1}\eta_{|z=\eta} - (\mathbf{u}_1.\nabla\xi)\mathbf{u}_{1|z=\xi}\\ &+ (\mathbf{u}_1.\nabla\eta)\mathbf{u}_{1|z=\eta} + p_1\nabla_x\xi_{|_{z=\xi}} - p_1\nabla_x\eta_{|_{|z=\eta}} - \frac{2}{Re_1}D(\mathbf{u}_1)\nabla_x\xi_{|z=\xi} + \frac{2}{Re_1}D(\mathbf{u}_1)\nabla_x\eta_{|z=\eta}\end{aligned} \tag{2.20a}$$

The expression of the pressure in (2.14c) is given by

$$\partial_z p_1 = -\frac{1}{Fr_1^2} + \frac{1}{Re_1}(2\partial_z^2 w_1 + \partial_z(\text{div}\mathbf{u}_1)).$$

By integrating this equation from z to ξ for $z \in [\eta, \xi]$, to obtain,

$$p_1 = p_{1|z=\xi} - \frac{1}{Fr^2}(z-\xi) + \frac{1}{Re_1}[2\partial_z w_1 + \text{div}(\mathbf{u}_1)] - \frac{1}{Re_1}[2\partial_z w_1 + \text{div}(\mathbf{u}_1)]_{|z=\xi}.$$

We use the divergence free condition, we get the following expression for P_1:

$$p_1 = p_{1|z=\xi} - \frac{1}{Fr^2}(z-\xi) - \frac{1}{Re_1}[\text{div}(\mathbf{u}_1) - \text{div}(\mathbf{u}_1)_{|z=\xi}]. \tag{2.21}$$

Due to conditions (2.16a), (2.17a), we can write

$$(\partial_{t_1}\xi + \mathbf{u}_1.\nabla_x \xi - w_1)\mathbf{u}_1|_{z=\xi} = 0 \quad \text{and} \quad (\partial_{t_1}\eta + \mathbf{u}_1.\nabla_x \eta - w_1)\mathbf{u}_1|_{z=\eta} = 0.$$

Thanks to conditions (2.16b), (2.16c), we have

$$\begin{aligned}\frac{1}{Re_1}\Big[-2D_x(\mathbf{u}_1)\nabla\xi + (\nabla_x w_1 + \frac{1}{\varepsilon^2}\partial_z \mathbf{u}_1)\Big] &= -\frac{1}{Re_1}\Big[\rho_1 p_1 - \rho_1 C^{-1}\Delta\xi\Big]\nabla\xi,\\ &= -\frac{1}{Re_1}\big[\partial_z u_1 - 2\partial_z w_1\big].\nabla\xi.\end{aligned} \tag{2.22a}$$

By using (2.17f), we have

$$\frac{1}{Re_1}\Big[-2D_x(\mathbf{u}_1)\nabla\eta + (\nabla_x w_1 + \frac{1}{\varepsilon^2}\partial_z \mathbf{u}_1)\Big] = \frac{1}{Re_1}\big[\partial_z u_1\nabla\eta - 2\partial_z w_1\big].\nabla\eta - r\frac{1}{\varepsilon}\mathbf{u}_1\text{fric}. \tag{2.23}$$

So, for the first layer, we get the equation

$$\begin{aligned}&\partial_{t_1}\int_\eta^\xi \mathbf{u}_1 dz + \text{div}\int_\eta^\xi \mathbf{u}_1\otimes\mathbf{u}_1 dz + \nabla_x\int_\eta^\xi p_1 - \frac{2}{Re_1}\text{div}\int_\eta^\xi D(\mathbf{u}_1)dz\\ &-p_1\nabla_x\eta_{|z=\eta} + p_1\nabla_x\xi_{|z=\xi} - \frac{1}{Re_1}(\partial_z\mathbf{u}_1\nabla\eta - 2\partial_z w_1)|_{z=\eta}.\nabla\eta\\ &= -\frac{1}{Re_1}\big(\partial_z u_1\nabla\xi - 2\partial_z w_1\big)|_{z=\xi}.\nabla\xi - r\mathbf{u}_1\frac{1}{\varepsilon}\text{fric}\end{aligned} \tag{2.24}$$

2.3.3. Asymptotic analysis

We assume the problem to be in an asymptotic regime by supposing the following hypotheses on the data

$$\frac{1}{Re_i} = \varepsilon\mu_{01}, \quad \text{fric} = \varepsilon\text{fric}_0, \quad \nu_2 = \varepsilon^{-1}\bar{\nu}_2. \tag{2.25}$$

Thanks to the definition of the dimensionless variables for the layer 2, we have $Re_2 = \dfrac{\varepsilon^2 UL}{\nu_2}$,

$$Re_2 = \frac{\varepsilon^3}{\nu_{02}}, \quad \text{where} \quad \nu_{02} = \frac{\bar{\nu}_{02}}{UL} = O(1).$$

Since we look for a second-order approximation, we develop the unknowns at order 1 and define

$$\begin{aligned}&\mathbf{u}_1 = \mathbf{u}_1^0 + \varepsilon\mathbf{u}_1^1 + O(\varepsilon^2), \quad w_1 = w_1^0 + \varepsilon w_1^1 + O(\varepsilon^2), \quad p_1 = p_1^0 + \varepsilon p_1^1 + O(\varepsilon^2),\\ &\eta = \eta^0 + \varepsilon\eta^1 + O(\varepsilon^2), \quad \xi = \xi^0 + \varepsilon\xi^1 + O(\varepsilon^2), \quad \mathbf{u}_2 = \mathbf{u}_2^0 + \varepsilon\mathbf{u}_2^1 + O(\varepsilon^2),\\ &w_2 = w_2^0 + \varepsilon w_2^1 + O(\varepsilon^2), \quad p_2 = p_2^0 + \varepsilon p_2^1 + O(\varepsilon^2).\end{aligned} \tag{2.26}$$

For the development of h_2, we take into account that $\eta = h_2 + b$, so we can write

$$h_2 = h_2^0 + \varepsilon h_2^1 + O(\varepsilon^2), \tag{2.27}$$

where $h_2^0 = \eta^0 - b$ and $h_2^1 = \eta^1 - b$. In the some way, we can write

$$h_1 = h_1^0 + \varepsilon h_1^1 + O(\varepsilon^2), \tag{2.28}$$

with $h_1^0 = \xi^0 - \eta^0$ and $h_1^1 = \xi^1 - \eta^1$ (remember that $\xi = \eta + h_1$).

(a) First approximation

If we consider the terms of principal order (ε^0), we deduce from (2.9b), (2.11b) and (2.12f) the following expressions:

$$\partial_z^2 \mathbf{u}_1 = O(\varepsilon), \qquad \partial_z \mathbf{u}_{1|_{z=\xi}} = O(\varepsilon), \qquad \partial_z \mathbf{u}_{1|_{z=\eta}} = O(\varepsilon). \tag{2.29}$$

Then $\mathbf{u}_1$ does not depend on z at first order, so we can write $\mathbf{u}_1^0(t,x,z) = \mathbf{u}_1^0(t,x)$. This implies that we can rewrite the expressions above up to order one, to obtain the final equation for layer 1 at the first order. To begin with, by using the conditions (2.16a), (2.17a) and (2.17b), we write (2.19) as

$$\partial_{t_1} h_1^0 + \text{div}(h_1^0 \mathbf{u}_1^0) = 0. \tag{2.30}$$

To get the momentum equation, we simplify (2.21) by using the free surface condition (2.16a)-(2.16c) to have

$$p_1^0(z) = -\frac{1}{Fr_1^2}(z - \xi^0) - 2\varepsilon\nu_{01}\text{div}_x \mathbf{u}_1^0 + O(\varepsilon^2). \tag{2.31}$$

Therefore, computing the integral appearing in (2.24) yields

$$\nabla \int_\eta^{\xi^0} p_1^0 dz = h_1^0 \nabla(p_1^0(\xi^0)) + p_1^0(\xi^0)\nabla h_1 + \frac{1}{2}\frac{1}{Fr_1^2}\nabla(h_1^0)^2. \tag{2.32}$$

If we inject this expression into (2.24) and consider only the principal order terms, we obtain

$$\begin{aligned}\partial_{t_1}(h_1\mathbf{u}_1^0) &+ \text{div}(h_1\mathbf{u}_1^0 \otimes \mathbf{u}_1^0) \\ &= -h_1\nabla(p_1^0(\xi^0)) - p_1^0(\xi^0)\nabla h_1 \\ &\qquad -\frac{1}{2}\frac{1}{Fr_1^2}\nabla(h_1)^2 - p_1^0\nabla\eta_{|_{z=\eta}} + p_1^0\nabla\xi|_{z=\xi^0} + \text{fric}_0.\end{aligned} \tag{2.33}$$

Therefore, the final equation reads

$$\begin{aligned}\partial_{t_1}(h_1^0\mathbf{u}_1^0) + \text{div}(h_1^0\mathbf{u}_1^0 \otimes \mathbf{u}_1^0) = \\ -h_1^0\nabla(p_1^0(\xi^0)) - \frac{1}{2}\frac{1}{Fr_1^2}\nabla(h_1^0)^2 - \frac{1}{Fr_1^2}h_1^0\nabla\eta + \text{fric}_0,\end{aligned} \tag{2.34}$$

where the friction term fric_0 (see [7]) is given by

$$\text{fric}_0 = \frac{1}{r}\frac{1}{Fr_1^2}h_2^0\Big((1-r)\text{sgn}(\mathbf{u}_2)\tan\delta_0 + \big(r\nabla_x h_1^0 + \nabla_x\eta^0\big)\Big). \tag{2.35}$$

Remark 2.3. Notice that the equation (2.34) does not contain the viscous effect. To recover it, we will derive the second-order approximation. To do so, we must take into account the terms of order ε ignored before and perform a parabolic correction of the velocity.

(b) **Approximation de Saint-Venant au second ordre**

Let us define the average of the velocity $\mathbf{u}_1$ as $\overline{\mathbf{u}}_1 = \frac{1}{h_1}\int_\eta^\xi \mathbf{u}_1 dz$.
We go back to (2.24) to write

$$\begin{aligned}&\partial_{t_1}(h_1\overline{\mathbf{u}}_1) + \mathrm{div}(h_1\overline{\mathbf{u}}_1 \otimes \overline{\mathbf{u}}_1)\\ &= \frac{2}{Re_1}\mathrm{div} h_1 D(\overline{\mathbf{u}}_1) - \int_\eta^\xi \nabla_x p_1 - \frac{1}{Re_1}(\partial_z \mathbf{u}_1 \nabla\eta - 2\partial_z w_1)|_{z=\eta}.\nabla\eta\\ &- r\mathbf{u}_1\frac{1}{\varepsilon}\mathrm{fric} - \frac{1}{Re_1}(\partial_z\mathbf{u}_1\nabla\xi - 2\partial_z w_1)|_{z=\xi}.\nabla\xi. \end{aligned} \tag{2.36a}$$

We have $\overline{\mathbf{u}_1^2} = \overline{\mathbf{u}}_1^2 + O(\varepsilon^2)$, and $\overline{\mathbf{u}_1 \otimes \mathbf{u}_1} = \overline{\mathbf{u}}_1 \otimes \overline{\mathbf{u}}_1 + O(\varepsilon^2)$. See [22] for details.
Now we consider the approximation up to order 2 for unknowns

$$\tilde{\mathbf{u}}_1 = \mathbf{u}_1^0 + \varepsilon\mathbf{u}_1^1, \qquad \tilde{p}_1 = p_1^0 + \varepsilon p_1^1, \qquad \tilde{\xi}_1 = \xi_1^0 + \varepsilon\xi_1^1, \qquad \tilde{h}_1 = h_1^0 + \varepsilon h_1^1, \tag{2.37}$$

We consider equations defined in (2.14a)-(2.14c) and write them up to second order. For (2.14a), we get

$$\partial_{t_1}\tilde{h}_1 + \mathrm{div}(\tilde{h}_1\tilde{\mathbf{u}}_1) = O(\varepsilon^2). \tag{2.38}$$

Now, we use the asymptotic hypothesis (2.26) and previous calculations to simplify (2.36a). Using the pressure expression (2.32), gives

$$\nabla\int_\eta^\xi p_1 dz - p_{1|_{z=\xi}}\nabla\xi + p_{1|_{z=\eta}}\nabla\eta = \frac{1}{2}\frac{1}{Fr_1^2}\nabla(h_1^2) + \frac{1}{Fr_1^2}h_1\nabla\eta + h_1\nabla p_{1|_{z=\xi}}. \tag{2.39}$$

Thanks to condition (2.16c), we can write:

$$h_1\nabla p_{1|_{z=\xi}} = -2\varepsilon\mu_{01}\nabla(h_1\mathrm{div}(\mathbf{u}_1^0)) + O(\varepsilon^2). \tag{2.40}$$

Finally, we insert (2.39) and (2.40) into (2.36a) and simplify the terms on the bottom and on the interface ξ. Thus, we get the second-order approximation of the momentum equation for layer 1 as follows:

$$\begin{aligned}&\partial_{t_1}(h_1\overline{\mathbf{u}}_1) + \mathrm{div}(h_1\overline{\mathbf{u}}_1 \otimes \overline{\mathbf{u}}_1)\\ &\qquad = 2\varepsilon\mu_{01}\mathrm{div}[h_1 D(\overline{\mathbf{u}}_1)] - \frac{1}{2}\frac{1}{Fr_1^2}\nabla(h_1^2)\\ &\qquad\qquad - \frac{1}{Fr_1^2}h_1\nabla\eta + 2\varepsilon\mu_{01}\nabla(h_1\mathrm{div}(\overline{\mathbf{u}}_1)). \end{aligned} \tag{2.41}$$

2.4. Layer Ω_2: Reynolds

As for the first layer, we look for a second-order approximation, so we develop each unknown at the first order. We set $\tilde{h}_2 = h_2^0 + \varepsilon h_2^1$, $\tilde{\mathbf{u}_2} = \mathbf{u}_2^0 + \varepsilon\mathbf{u}_2^1$, $\tilde{p}_2 = p_2^0 + \varepsilon p_2^1$. The asymptotic regime for layer 2 affects the viscosity and capillary constants. When the surface tension effects are strong, it is essential to have them at the leading order, thus we assume

$$\nu_2 = O(\varepsilon), \qquad \delta = O(\varepsilon^{-2}). \tag{2.42}$$

Consequently, $Re_2 = \frac{\varepsilon U H}{\nu_2} = O(1)$ and $C^{-1} = \frac{\delta}{\varepsilon^2 U\rho_2\nu_2} = O(\varepsilon^{-5})$ and for simplicity we write $C^{-1} = \varepsilon^{-5}C_0^{-1}$.
Now, we study the velocity equation in (2.15a)-(2.15c), which can be written as follows:

$$\partial_z^2\mathbf{u}_2 - \nabla p_2 = O(\varepsilon^4), \tag{2.43}$$

$$\partial_z p_2 = -\varepsilon^4 \frac{Re_2}{Fr_2^2} + O(\varepsilon^4). \tag{2.44}$$

From the definitions of Re_2 and Fr_2, we have $\varepsilon^2 \frac{Re_2}{Fr_2^2} = \frac{gLH}{U\nu_2} = O(\varepsilon)$, so for the simplicity we introduce

$$\beta_0 = \varepsilon \frac{Re_2}{Fr_2^2} = \varepsilon \frac{1}{\nu_{02} Fr_1^2}. \tag{2.45}$$

.

The equation for the pressure reads

$$\partial_z p_2 = -\varepsilon\beta_0 = o(\varepsilon^4). \tag{2.46}$$

The next step is to find the transport equation for the sediment. To do so, we start to look for $\tilde{\mathbf{u}_2}$ in (2.43), after we compute $\tilde{p_2}$ and $\tilde{\mathbf{u}_2}|_{z=\eta}$ that appear into the expression of $\tilde{\mathbf{u}_2}$.

Integrating the divergence-free equation, we obtain

$$\nabla \cdot \int_b^\eta \tilde{\mathbf{u}}_2 \mathbf{d}z - \tilde{\mathbf{u}}_{2|_{z=\eta}} \nabla\eta + \tilde{\mathbf{u}}_{2|_{z=b}} \nabla b + \tilde{w}_{2|_{z=\eta}} - \tilde{w}_{2|_{z=b}} = 0.$$

If we take into account the conditions (2.17c), (2.18), the mass equation for the second layer is

$$\partial_{t_2} \tilde{h}_2 + \nabla . \int_b^\eta \tilde{\mathbf{u}}_2 dz = 0. \tag{2.47}$$

We integrate (2.46) from z to η to obtain

$$\tilde{p}_2(z) = \tilde{p}_2(\eta) - \varepsilon\beta_0(z-\eta)$$

We use the condition at the interface (2.12h) and the condition (2.45) to write

$$\tilde{p}_2|_\eta = \varepsilon \frac{r}{\nu_{02} Fr_1^2} h_1^0.$$

Thus, $\tilde{p}_2(z) = \varepsilon \frac{r}{\nu_{02} Fr_1^2} h_1^0 - \varepsilon\beta_0(z-\eta)$ and $\nabla_x \tilde{p}_2 = \varepsilon \frac{r}{\nu_{02} Fr_1^2} \nabla h_1^0 + \varepsilon\beta_0 \nabla\eta$ does not depend on z.

Integrating now (2.43) from z to η, we get

$$\partial_z \tilde{\mathbf{u}}_2 = \partial_z \tilde{u}_{2|_{z=\eta}} + \nabla \tilde{p}_2 (z-\eta) = \partial_z \tilde{u}_{2|_{z=\eta}} + O(\varepsilon).$$

We use a generalized law based on the work [15], that reads

$$\text{fric} = C(\mathbf{u}_1 - \mathbf{u}_2)|_{z=\eta} \tag{2.48}$$

We must also take into account the adimensionalization for this friction term. Thus we assume the following dimension and asymptotic to the coefficient C:

$$C = U\bar{C}; \qquad \bar{C} = \varepsilon C^0.$$

Then, we have

$$\text{fric}_0 = C^0(\mathbf{u}_1^0 - \varepsilon^2 \mathbf{u}_2^0|_{z=\eta}) \tag{2.49}$$

From this expression, we get the value of $\mathbf{u}_2^0$

$$\begin{aligned} \mathbf{u}_2^0 = \mathbf{u}_2|_{z=\eta} &= \frac{1}{\varepsilon^2}\mathbf{u}_1^0 - \frac{1}{\varepsilon^2 C^0}\text{fric}_0 \\ &= \frac{1}{\varepsilon^2}\mathbf{u}_1^0 - \frac{1}{r\varepsilon^2 C^0}\frac{h_2^0}{Fr_1^2}\Big((1-r)\text{sgn}(\mathbf{u}_2)\tan\delta_0 + \big(r\nabla_x h_1^0 + \nabla_x \eta^0\big)\Big). \end{aligned}$$

Considering the equation (2.47) we have

$$\partial_{t_2} h_2^0 + \text{div}_x\bigg(\frac{1}{\varepsilon^2} h_2^0 \mathbf{u}_1^0 - \frac{1}{r\varepsilon^2 C^0}\frac{(h_2^0)^2}{Fr_1^2}\Big((1-r)\text{sgn}(\mathbf{u}_2)\tan\delta_0 + \big(r\nabla_x h_1^0 + \nabla_x \eta^0\big)\Big)\bigg) = 0 \tag{2.50}$$

2.5. Final model

In this section, we expose the final model obtained in the previous section as a formal second-order approximation of the initial problem (2.2a)-(2.8). For that, we write this system in dimensional variables.
The final model is given in non-dimensional variables by (2.35), (2.38), (2.41) and (2.50). The model is composed of three equations, mass and momentum for the shallow water flow and lubrification Reynolds equation for the sediment layer. We recover the system in dimensional variables

$$\begin{cases} \partial_t h_1 + \text{div}(h_1 \mathbf{u}_1) = 0, \\ \partial_t (h_1 \mathbf{u}_1) + \text{div}(h_1 \mathbf{u}_1 \otimes \mathbf{u}_1) + \frac{1}{2} g \nabla (h_1^2) + g h_1 \nabla (b + h_2) - 2\nu_1 \text{div}[h_1 D(\mathbf{u}_1)] \\ -2\nu_1 \nabla (h_1 \text{div}(\mathbf{u}_1)) + \dfrac{g h_2}{r} \mathcal{P} = 0, \\ \partial_t h_2 + \text{div}_x (h_2 v_b \sqrt{(\frac{1}{r} - 1) g d_s}) = 0, \end{cases} \tag{2.51}$$

with $\mathcal{P} = \nabla_x (r h_1 + h_2 + b) + (1-r)\text{sgn}(\mathbf{u}_2^0)\tan\delta_0$
and
$v_b = \dfrac{1}{\sqrt{(\frac{1}{r} - 1) g d_s}} \mathbf{u}_1 - \dfrac{v}{1-r} \mathcal{P}.$
We note that we were inspired by [6] for the expression of v_b. Note that in this paper, we do not decompose the sediment layer into two entities. We suppose it one. We refer the readers to [6] for the meaning of d_s, v and v_b.

3. Existence of weak solutions

In this section we assume that bottom vanish in the model (i.e $b(x,y) = 0$)and that the velocities of the sediment and the water are identical. We also needed a regularizing term of the form $-\kappa \nabla \cdot (1 + \dfrac{h_2}{r h_1}) \nabla (h_1 + \dfrac{1}{r} h_2)$ on the transport equation. The model studied is as follow:

$$\partial_t h_1 + \text{div}(h_1 u_1) = 0, \tag{3.1}$$

$$\partial_t (h_1 u_1) + \text{div}(h_1 u_1 \otimes u_1) + g h_1 \nabla h_1 + g h_1 \nabla h_2 - 2\nu_1 \text{div}(h_1 D(u_1)) + g h_2 \nabla (h_1 + \frac{1}{r} h_2)$$
$$-\beta h_1 \nabla \Delta h_1 + \delta h_1 \nabla h_1^{-\alpha} + \bar{\kappa} h_1 \nabla \Delta^{2s+1} h_1 = 0, \tag{3.2}$$

$$\partial_t h_2 + \text{div}(h_2 u_1) - \kappa \nabla \cdot \left[(1 + \frac{h_2}{r h_1}) \nabla (h_1 + \frac{1}{r} h_2) \right] = 0, \tag{3.3}$$

where $\alpha, \kappa, \bar{\kappa}$ are a positive constants $\alpha \neq 0$.The term $\delta h_1 \nabla h_1^{-\alpha}$ represente the cold presure, while $\bar{\kappa} h_1 \nabla \Delta^{2s+1} h_1$ is the interface tension.
The initial data are

$$h_1(0,x) = h_{1_0}, \quad h_2(0,x) = h_{2_0}, \quad (h_1 u_1)(0,x) = \mathbf{m}_0(x) \quad \text{in } \Omega, \tag{3.4}$$

and we assume that h_{1_0}, h_{2_0} and $\mathbf{m}_0$ are such that

$$h_{1_0} \in L^2(\Omega), \quad h_{2_0} \in L^2(\Omega), \quad 0 < h_{1_0}, \quad 0 \le h_{2_0}, \quad \nabla(\sqrt{h_{1_0}}) \in L^2(\Omega),$$
$$\nabla \Delta^s h_{1_0} \in L^2(\Omega), \quad h_{1_0}^{\frac{1-\alpha}{2}} \in L^2(\Omega), \quad \nabla \mathbf{m}_0 \in L^2(\Omega), \quad \mathbf{m}_0 = 0 \quad \text{if} \quad h_{1_0} = 0, \tag{3.5}$$
$$\frac{\mathbf{m}_0}{h_{1_0}} \in L^2(\Omega)$$

3.1. Mains results

Definition 3.1. We say that (h_1, h_2, u_1) is weak solutions of $(3.1) - (3.3)$, with the initial condition (3.4) satisfying (3.5), if
• the initial condition (3.4) hold in $D'((0,T)\times\Omega)$,
• the energy inequalities defined in the **Proposition** 3.2 and **Proposition** 3.4 are satisfied, and the regularities properties obtained in **Corollary** 3.3 and **Corollary** 3.5 hold,
• for all smooth test functions $\varphi = \varphi(t,x)$ with $\varphi(T,\cdot) = 0$, we have:

$$-h_{1_0}\varphi(0,\cdot) - \int_0^T\int_\Omega h_1\partial_t\varphi - \mathbf{m}_0(x)\varphi(0,\cdot) - \int_0^T\int_\Omega h_1u_1 \text{ div } (\varphi) = 0, \tag{3.6}$$

$$h_{2_0}\varphi(0,\cdot) - \int_0^T\int_\Omega h_2u_1\nabla\varphi + \kappa\int_0^T\int_\Omega \left(1+\frac{h_1}{rh_2}\right)\nabla(h_1+h_2/r)\nabla\varphi = 0, \tag{3.7}$$

$$\begin{aligned}-h_{1_0}u_{1_0}\varphi(0,\cdot) &- \int_0^T\int_\Omega (h_1u_1)\partial_t\varphi - \int_0^T\int_\Omega \sqrt{h_1}u_1\otimes\sqrt{h_1}u_1 : D(\varphi) + 2\nu_1\int_0^T\int_\Omega h_1[D(u_1):D(\varphi)]\\ &-g\int_0^T\int_\Omega h_1^2\text{div}(\varphi) - g\int_0^T\int_\Omega h_1h_2 \text{ div}(\varphi) - \frac{g}{2r}\int_0^T\int_\Omega h_2^2\text{div}(\varphi) + \delta\int_0^T\int_\Omega h_1\nabla h_1^{-\alpha}\varphi\\ &-\beta\int_0^T\int_\Omega \Big[h_1\Delta h_1\Big]\text{div}(\varphi) - \beta\int_0^T\int_\Omega\Big[\Delta h_1\nabla h_1\Big]\varphi + \bar\kappa\int_0^T\int_\Omega\Big[h_1\nabla\Delta^{2s+1}h_1\Big]\varphi = 0.\end{aligned} \tag{3.8}$$

3.2. Estimates.

Proposition 3.2. *Let* (h_1, h_2, u_1) *be a smooth solution of* $(3.1) - (3.3)$. *then the following energy inequality holds*

$$\begin{aligned}\frac12\frac{d}{dt}\int_\Omega\Big[h_1|u_1|^2 + g|h_1+h_2|^2 + g(\frac{1-r}{r})|h_2|^2 + \frac12\beta|\nabla h_1|^2 + \frac{\delta}{\alpha-1}|h_1^{\frac{1-\alpha}{2}}|^2 + \frac{\bar\kappa}{2}|\nabla\Delta^s h_1|^2\Big]\\ +\frac{\nu_1}{2}\int_\Omega h_1|\nabla u_1 +^t\nabla u_1|^2 + g\kappa\int_\Omega(1+\frac{h_2}{rh_1})|\nabla(h_1+r^{-1}h_2)|^2 = 0\end{aligned} \tag{3.9}$$

Proof. First, we multiply the momentum equation (3.2) by u_1 and we integrate on Ω. We use the mass conservation equation for simplification. Then, we obtain

$$\begin{aligned}\bullet\int_\Omega(\partial_th_1u_1)u_1 + \int_\Omega\text{div}(h_1u_1\otimes u_1)u_1 &= -\int_\Omega\text{div}(h_1u_1)u_1^2 + \int_\Omega h_1u_1\partial_tu_1 + \int_\Omega(h_1u_1\cdot\nabla)u_1\cdot u_1\\ &= \frac12\frac{d}{dt}\int_\Omega h_1|u_1|^2,\end{aligned}$$

$$\begin{aligned}\bullet\quad g\int_\Omega(h_1u_1\nabla(h_1+h_2) + g\int_\Omega h_2u_1\nabla(h_1+\frac1r h_2) &= g\int_\Omega(h_1+h_2)\partial_th_1 - g\int_\Omega(h_1+\frac1r h_2)\text{div}(h_2u_1)\\ &= \frac12 g\frac{d}{dt}\int_\Omega h_1^2 + g\int_\Omega h_2\partial_th_1 - g\int_\Omega(h_1+\frac1r h_2)\text{div}(h_2u_1)\end{aligned}$$

$$\bullet -\int_\Omega 2\nu_1\text{div}(h_1D(u_1)u_1 = 2\nu_1\int_\Omega h_1D(u_1):\nabla u_1 = \frac{\nu_1}{2}\int_\Omega h_1|\nabla u_1 +^t\nabla u_1|^2$$

$$\bullet -\delta\int_\Omega(h_1\nabla h_1^{-\alpha})u_1 = \frac{\delta}{\alpha-1}\frac{d}{dt}\int_\Omega|h_1^{\frac{1-\alpha}{2}}|^2$$

- $-\int_\Omega h_1 u_1 \nabla \Delta^{2s+1} h_1 = \int_\Omega \partial_t \Delta^{2s+1} h_1 = \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \Delta^s h_1|^2$
- $\beta \int_\Omega h_1 u_1 \nabla \Delta h_1 = \beta \int_\Omega \partial_t h_1 \Delta h_1 = -\frac{1}{2}\beta \frac{d}{dt}\int_\Omega |\nabla h_1|^2$

We get the following equality:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h_1|u_1|^2 + \frac{1}{2}g\frac{d}{dt}\int_\Omega h_1^2 + \frac{\nu_1}{2}\int_\Omega h_1|\nabla u_1 +^t \nabla u_1|^2 + \frac{1}{2}g\frac{d}{dt}\int_\Omega h_1^2$$

$$+g\int_\Omega h_2\partial_t h_1 + \frac{\delta}{\alpha-1}\frac{d}{dt}\int_\Omega |h_1^{\frac{1-\alpha}{2}}|^2$$

$$+\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \Delta^s h_1|^2 - \frac{1}{2}\beta\frac{d}{dt}\int_\Omega |\nabla h_1|^2 - g\int_\Omega (h_1 + \frac{1}{r}h_2)\text{div}(h_2 u_1) = 0 \tag{3.10}$$

Now, we multiply the transport equation by $g(h_1 + \frac{1}{r}h_2)$ to have:

$$\frac{1}{2r}\frac{d}{dt}\int_\Omega g h_2^2 + \int_\Omega g h_1 \partial_t h_2 - \int_\Omega h_2 u_1 \nabla(h_1 + r^{-1}h_2)$$

$$+g\kappa\int (1 + \frac{h_2}{rh_1})|\nabla(h_1 + r^{-1}h_2)|^2 = 0. \tag{3.11}$$

To end, we add the equations (3.10) and (3.11) and with a simple calculation, we have the proclamed equality. ■

Corollary 3.3. *For (h_1, h_2, u_1) solution of the system $(3.1) - (3.3)$ the following bound holds:*

$$\sqrt{h_1}u_1$$

$$\textit{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \quad \sqrt{h_1}|\nabla u_1 +^t \nabla u_1| \quad \textit{is bounded in} \quad L^2(0,T;L^2(\Omega)),$$

$$h_1 \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \quad h_2 \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(\Omega)),$$

$$\sqrt{1 + h_2/rh_1}|\nabla(h_1 + r^{-1}h_2| \quad \textit{is bounded in} \quad L^2(0,T;L^2(\Omega)),$$

$$\nabla h_1 \quad \textit{is bounded in} \quad L^\infty(0,T;(L^2(\Omega))^2), \quad h_1^{\frac{1-\alpha}{2}} \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(\Omega)),$$

$$\nabla \Delta^s h_1 \quad \textit{is bounded in} \quad L^\infty(0,T;(L^2(\Omega))^3).$$

Proposition 3.4. For (h_1, h_2, u_1) solution of model $(3.1) - (3.3)$, we show the following relation :

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left[h_1|u_1 + 2\nu_1\nabla \text{log}h_1|^2 + g|h_1+h_2|^2 + g(r^{-1}-1)|h_2|^2 + \beta|\nabla h_1|^2 + \frac{2\delta}{\alpha-1}|h_1^{\frac{1-\alpha}{2}}|^2\right]$$

$$+2\nu_1\int_\Omega h_1(A(u_1):A(u_1)) + \nu_1\beta\int_\Omega |\Delta h_1|^2 + \frac{\bar{\kappa}}{2}\int_\Omega |\nabla\Delta^s h_1|^2 + 2\nu_1\bar{\kappa}\int_\Omega |\Delta^{s+1}h_1|^2$$

$$+\frac{8\nu_1\delta\alpha}{(\alpha-1)^2}\int_\Omega |\nabla h_1^{\frac{1-\alpha}{2}}|^2 + 2\nu_1 g\int_\Omega (1 + h_2/rh_1)|\nabla h_1|^2$$

$$\leq r\nu_1 g\int_\Omega (1 + h_2/rh_1)|\nabla(h_1 + r^{-1}h_2|^2. \tag{3.12}$$

Proof. **Proposition** 3.4
The proof of the **Proposition** 3.4 follows the techniques used in [2, 4, 5, 17, 21].

We consider the mass equation:

$$\partial_t h_1 + \text{div}(h_1 u) = 0.$$

We derive this equation with respect to x, y and we make the sum. We have:

$$\partial_t \nabla h_1 + \text{div}(h_1^t \nabla u_1) + \text{div}(u_1 \otimes \nabla h_1) = 0.$$

By Remplacing ∇h_1 by $h_1 \nabla \log h_1$ and multiply by the viscoity $2\nu_1$,we get:

$$2\nu_1 \partial_t(h_1 \nabla \log h_1) + 2\nu_1 \text{div}(h_1^t \nabla u_1) + 2\nu_1 \text{div}(h_1 u_1 \otimes \nabla \log h_1) = 0.$$

Next, we add this equation to the momentum equation to have:

$$\partial_t[h_1(u_1 + 2\nu_1 \nabla \log h_1)] + \text{div}[h_1 u_1 \otimes (u + 2\nu_1 \nabla \log h_1)] - 2\nu_1 \text{div}(h_1(\mathbf{D}(u_1) - \nabla^t u_1)$$
$$+ g h_1 \nabla(h_1 + h_2) + g h_2 \nabla(h_1 + r^{-1} h_2) = 0.$$

We multiply the above equation by $(u + 2\nu_1 \nabla \log h_1)$ and we integrate the result obtained on Ω. We will now transform each term in the previous equation.
We have :

$$\int_\Omega [\partial_t[h_1(u_1 + 2\nu_1 \nabla \log h_1)] + \text{div}[h_1 u_1 \otimes (u_1 + 2\nu_1 \nabla \log h_1)]](u_1 + 2\nu_1 \nabla \log h_1)$$
$$= \frac{1}{2}\frac{d}{dt}\int_\Omega h_1 |u_1 + 2\nu_1 \nabla \log h_1|^2.$$

Using the definition of the tensor of contraint, we get:

$$-2\nu_1 \int_\Omega \text{div}(h_1(\mathbf{D}(u_1) - \nabla^t u_1)(u + 2\nu_1 \nabla \log h_1) = 2\nu_1 \int_\Omega h_1(\mathbf{A}(u_1) : \mathbf{A}(u_1)),$$

where $A(u_1) = \dfrac{\nabla u_1 -^t \nabla u_1}{2}$.
For the terms pressure, surface tension and friction, we only look at those that do not appear in the **Proposition** 3.2. We modify their expressions essentially using integrations by parts. We have:

- $\frac{1}{2} g \int_\Omega h_1 \nabla(h_1 + h_2)(2\nu_1 \nabla \log h_1) = \nu_1 g \int_\Omega |\nabla h_1|^2 + \nu_1 g \int_\Omega \nabla h_1 \nabla h_2,$
- $g \int_\Omega h_2(\nabla(h_1 + r^{-1} h_2)(2\nu_1 \nabla \log h_1) = 2\nu_1 g \int_\Omega \frac{h_2}{h_1}|\nabla h_1|^2 + \frac{2\nu_1}{r}\int_\Omega \frac{h_2}{h_1}\nabla h_1 \nabla h_2.$

The sum of these two terms gives:

$$\frac{1}{2} g \int_\Omega h_1 \nabla(h_1 + h_2)(2\nu_1 \nabla \log h_1) + g \int_\Omega h_2(\nabla(h_1 + r^{-1} h_2)(2\nu_1 \nabla \log h_1)$$
$$= 2\nu 1 g \int_\Omega (1 + \frac{h_2}{h_1})|\nabla h_1|^2$$
$$+ 2\nu_1 g \int_\Omega (1 + \frac{h_2}{r h_1})\nabla h_1 \nabla h_2.$$

Now, we change the tension term as follows:

- $-\beta \int_\Omega h_1 \nabla \Delta h_1 (2\nu_1 \nabla \log h_1) = 2\nu_1 \beta \int_\Omega |\Delta h_1|^2.$

For the cold presure term:

- $, -\delta \int_\Omega h_1 \nabla (h_1^{-\alpha})[2\nu_1 \nabla \log h_1] = \dfrac{8\nu_1 \delta\alpha}{(\alpha-1)^2} \int_\Omega |\nabla h_1^{\frac{1-\alpha}{2}}|^2.$

Also we have

- $-\bar{\kappa} \int_\Omega [h_1 \nabla \Delta^{2s+1} h_1][2\nu_1 \nabla \log h_1] = 2\nu_1 \bar{\kappa} \int_\Omega |\Delta^{s+1} h_1|^2.$

By bringing these results together and integrating between 0 and T, we deduce the stated inequality. Which completes the proof. ■

Corollary 3.5. *For (h_1, h_2, u_1) solution of the system $(3.1) - (3.3)$ the following bound holds:*

$$\nabla\sqrt{h_1} \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \quad \sqrt{h_1}A(u_1) \quad \textit{is bounded in} \quad L^2(0,T;L^2(\Omega)),$$

$$\Delta h_1 \quad \textit{is bounded in} \quad L^2(0,T;L^2(\Omega)), \quad \nabla h_2 \quad \textit{is bounded in} \quad L^2(0,T;(L^2(\Omega))^2),$$

$$\Delta^{s+1} h_1 \quad \textit{is bounded in} \quad L^2(0,T;L^2(\Omega)), \quad \nabla h_1^{\frac{1-\alpha}{2}} \quad \textit{is bounded in} \quad L^2(0,T;(L^2(\Omega))^2).$$

Proposition 3.6. *If h_1 has the regularities established in **Corollaire** 3.3 and **Corollaire** 3.5, then there exist constants c and $\bar{c}$ dependent on δ, $\bar{\kappa}$ such that*

$$c \le h_1(t,x) \le \bar{c} \tag{3.13}$$

Remark 3.7. This result was first proved in [23] and also used in [11],[16].

Remark 3.8. In this paper, we impose a physical condition that is

$$\frac{h_2}{h_1} \le C, \quad \text{where} \quad C \in [0,1], see[24].$$

It implies that the tickness of the sediment layer is small compared to that of the fluid. Using this physical condition, proposition 3.6 and the results in [16], we can prove the existence of solutions of our model.

Remark 3.9. Sobolev's injections give us thanks to the estimentions of the **Corollary** 3.9 and **corollairy** 3.12 that

$$h_1 \text{ and } u_1 \quad \text{are bounded in} \quad L^\infty(0,T;L^p(\Omega)) \text{ for } p \ge 2. \tag{3.14}$$

Theorem 3.10. *There exists a global weak solutions to system $(3.1) - (3.3)$ with initial data $(3.4) - (3.5)$ and satisfying the inequalities denined in the **Proposition** 3.2 and **Proposition**3.4.*

3.3. Proof of Theorem 3.10

This section is devoted to the proof of theorem 3.10, in six steps. We can, thanks to the preceding estimations, the convergence of the various terms which intervene in the equation. We exploit the ideas presented in [16].

3.4. Step 1: Convergence of the sequences $(\sqrt{h_{1_n}})_{n\ge1}$, $(h_{1_n})_{n\ge1}$, u_{1_n} and $(h_{2_n})_{n\ge1}$

From the mass equation, we derive:

$$\frac{d}{dt}\int_\Omega \left|\sqrt{h_{1_n}}\right|^2 = -\int_\Omega h_{1_n} \nabla u_{1_n} - \int_\Omega u_{1_n} \nabla h_{1_n},$$

which allows us to have

$$(\sqrt{h_{1_n}})_{n\ge1} \text{ bounded in } L^\infty(0,T;L^2(\Omega)).$$

Corollary 3.5 gives us that $\| \nabla\sqrt{h_{1_n}} \|_{L^{+\infty}} (0,T;(L^2(\Omega))^2) \leq c$, so we obtain:

$$(\sqrt{h_{1_n}})_{n\geq 1} \text{ is bounded in } L^\infty(0,T;H^1(\Omega)). \tag{3.15}$$

We still use the mass equation to have:

$$\partial_t\sqrt{h_{1_n}} = \frac{1}{2}\sqrt{h_{1_n}}\text{div}u_{1_n} - \text{div}(\sqrt{h_{1_n}}u_{1_n}),$$

which gives that $\partial_t\sqrt{h_{1_n}}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$.
Applying Aubin-Simon lemma, we can extract a subsequence, still denoted $(h_{1_n})_{n\geq 1}$, such that

$$\sqrt{h_{1_n}} \text{ converges strongly to } \sqrt{h_1} \text{ in } C^0(0,T;L^2(\Omega)).$$

Thanks to the **Remark** 3.9 and Sobolev embeddings, we know that, for all finite p, $\sqrt{h_{1_n}}$ is bounded in $L^\infty(0,T;L^p(\Omega))$ with $p\geq 4$, and this to ensure that $(h_{1_n})_n$ is in $L^\infty(0,T;L^2(\Omega))$.
Equality $\nabla h_{1_n} = 2\sqrt{h_{1_n}}\nabla\sqrt{h_{1_n}}$ enables us to bound the sequence $(\nabla h_{1_n})_n$ in $L^\infty(0,T;(L^{\frac{2p}{2+P}}(\Omega))^2)$ and consequently, we have:

$$(h_{1_n})_n \text{ is bounded in } L^\infty(0,T;W^{1,\frac{2p}{P+2}}(\Omega)).$$

Let us now look at some properties of the derivative in time of h_{1_n}. The mass equation reads:

$$\partial_t h_n = -\text{div}(h_n u_n) = -\sqrt{h_{1_n}}u_{1_n}\nabla\sqrt{h_{1_n}} - \sqrt{h_{1_n}}\text{div}\sqrt{h_{1_n}}u_{1_n}.$$

So, we get

$$(h_{1_n}u_{1_n})_n \text{ bounded in } L^\infty(0,T;(L^{\frac{2p}{p+2}}(\Omega))^2) \text{ and } (\partial_t h_{1_n})_n \text{ bounded in}$$
$$L^\infty(0,T;W^{-1,\frac{2p}{P+2}}(\Omega))$$

Thanks to Aubin-Simon lemma again, we find:

$$h_{1_n} \longrightarrow h_1 \quad \text{dans} \quad C^0(0,T;L^{\frac{2p}{2+P}}(\Omega)).$$

Last, we consider the bottom term h_{2_n}: with **Corollary** 3.5 and the bound on $(\sqrt{h_{2_n}})_n$ in $L^\infty(0,T;L^2(\Omega))$, we know that the sequence $(\nabla h_{2_n})_n$ is bounded in $L^2(0;T;(L^2(\Omega))^2)$, which gives:

$$(h_{2_n})_n \text{ is bounded in } L^\infty(0,T;H^1(\Omega)).$$

For the time derivative of h_{2_n}, we restart from Equation (3.3). We have:

$$\partial_t h_{2_n} = -\text{div}(h_{2_n}u_{1_n}) + \kappa\nabla\cdot\left[(1+\frac{h_{2_n}}{rh_{1_n}})\nabla(h_{1_n}+\frac{1}{r}h_{2_n})\right]. \tag{3.16}$$

According to the Sobolev embeddings, the first term is in $W^{-1,\frac{2p}{p+2}}(\Omega)$, since h_{2_n} is bounded in $L^2(\Omega)$ and u_{2_n} is bounded in $L^p(\Omega)$. The last term is in $W^{-1,1}(\Omega)$.
We then deduce that

$$\partial_t h_{2_n} \text{ is bounded in } W^{-1,1}(\Omega).$$

Therfore, thanks to the Aubin Simon Lemma, we get

$$h_{2_n} \longrightarrow h_2 \text{ Strongly in } W^{-1,\frac{2p}{p+2}}(\Omega).$$

Now we are interested in the velocity u_{1_n}. Thanks to the **Corollary** 3.3, **Corollary** 3.5 and the **Remark** 3.9 we have

$$u_{1_n} \text{ is bounded in } L^\infty(0,T;H^1(\Omega)).$$

Also we have $\partial_t u_{1_n} = \dfrac{1}{h_{1_n}}\partial_t(h_{1_n}u_{1_n}) + u_{1_n}\nabla u_{1_n} + u_{1_n}^2\dfrac{\nabla h_{1_n}}{h_{1_n}}$, thanks to the **Proposition**3.6 and the **Remark** 3.9, we have

$$\partial_t u_{1_n} \text{ is bounded in } W^{-1,1}(\Omega).$$

The Aubin Simon **Lemma** ensures that

$$u_{1_n} \longrightarrow u_1 \text{ Strongly in } \mathcal{C}^0(0,T;W^{-1,1}(\Omega)).$$

3.5. Step 2: Convergence of the sequences $\frac{h_{2_n}}{h_{1_n}}$ and $(1+\dfrac{h_{2_n}}{rh_{1_n}})\nabla(h_{1_n}+\frac{1}{r}h_{2_n})$

We have

$$\left|\frac{h_{2_n}}{h_{1_n}} - \frac{h_2}{h_1}\right|^2 = \left|\frac{h_{2_n}h_1 - h_2h_1 + h_2h_1 - h_2h_{1_n}}{h_{1_n}h_1}\right|^2 \leq \mathbf{K}|h_{2_n}-h_2|^2 + |h_{1_n}-h_1|^2$$

thanks to the **Proposition** 3.6. According to the **Step 1**, we have

$$\left|\frac{h_{2_n}}{h_{1_n}} - \frac{h_2}{h_1}\right|^2 \to 0, \quad \text{then} \quad \frac{h_{2_n}}{h_{1_n}} \longrightarrow \frac{h_2}{h_1} \quad \text{strongly in } L^2(0,T;L^2(\Omega))$$

consequently,

$$(1+\frac{h_{2_n}}{rh_{1_n}})\nabla(h_{1_n}+\frac{1}{r}h_{2_n}) \longrightarrow (1+\frac{h_2}{rh_1})\nabla(h_1+\frac{1}{r}h_2) \text{ weakly in } L^1(0,T;(L^1(\Omega)).$$

3.6. Step 3: Weak convergences of $h_{1_n}\nabla\Delta^{2s+1}h_{1_n}$ and $h_{1_n}\nabla\left[h_{1_n}^{-\alpha}\right]$

Concerning the two terms, we have

$$h_{1_n}\nabla\Delta^{2s+1}h_{1_n}\text{bounded in}$$
$$L^2(0,T;W^{-1,1}(\Omega)) \text{ and } h_{1_n}\nabla\left[h_{1_n}^{-\alpha}\right] \text{ bounded in } L^2(0,T;L^{\frac{2p}{p+2}}(\Omega))$$

So, we have

$$h_{1_n}\nabla\Delta^{2s+1}h_{1_n} \text{ converges weakly to } h_1\nabla\Delta^{2s+1}h_1 \text{ in } L^2(0,T;W^{-1,1}(\Omega)),$$

and

$$h_{1_n}\nabla\left[h_{1_n}^{-\alpha}\right]\text{converges weakly to } h_1\nabla\left[h_1^{-\alpha}\right] \text{ in } L^2(0,T;L^{\frac{2p}{p+2}}(\Omega)).$$

3.7. Step 4: Convergence of ∇h_{1_n} and Δh_{1_n}

As Δh_{1_n} and ∇h_{1_n} are bounded respectively in $L^2(0,T;L^2(\Omega))$ and $L^\infty(0,T;(L^2(\Omega))$, so we have:

$$\nabla h_{1_n} \text{ bounded in } L^2(0,T;H^1(\Omega)).$$

Using the mass equation, one has $\partial_t\nabla h_{1_n} = -\nabla\text{div}h_{1_n}u_{1_n}$, as $h_{1_n}u_{1_n}$ is bounded in $L^2(0,T;L^2(\Omega))$, we have

$$\partial_t\nabla h_{1_n} \text{ is bounded in } L^2(0,T;H^{-2}(\Omega)).$$

Then, applying Aubin-Simon Lemma, it follows that

$$\nabla h_{1_n} \longrightarrow \nabla h_1 \text{ strongly in } L^2(0,T;(L^q(\Omega))^2), \quad q\in[1,+\infty[.$$

But as we have shown that ∇h_{1_n} is bounded in $L^\infty(0,T;(L^2(\Omega))$, hence

$$\nabla h_{1_n} \longrightarrow \nabla h_1 \text{ strongly in } L^2(0,T;(L^2(\Omega))^2).$$

Thanks to the **Corrolary** 3.5, we have finally

$$\Delta h_{1_n} \longrightarrow \Delta h_1 \text{ weakly in } L^2(0,T;L^2(\Omega)).$$

3.8. Step 5: Convergence of $(h_{1_n}u_{1_n})_{n\geq 1}$

In the previous part, we proved that the sequence $(h_{1_n}u_{1_n})_n$ is bounded in $L^\infty(0,T;(L^{\frac{2p}{p+2}}(\Omega))^2)$ where p is an integer greater than four.Writing the gradient as follows:

$$\nabla(h_{1_n}u_{1_n}) = 2\sqrt{h_{1_n}}u_{1_n}\nabla\sqrt{h_{1_n}} + \sqrt{h_{1_n}}\sqrt{h_{1_n}}\nabla u_{1_n},$$

since the first term is in $L^\infty(0,T;L^1(\Omega))$ and the second one belongs to $L^2(0,T;L^{\frac{2p}{p+2}}(\Omega))$, we have: $(h_{1_n}u_{1_n})_n$ bounded in $L^2(0,T;W^{1,1}(\Omega))$.
Moreover, the momentum equation (3.2) enables us to write the time derivative of the water discharge:

$$\partial_t(h_{1_n}u_{1_n}) = -\text{div}(h_{1_n}u_{1_n}\otimes u_{1_n}) - gh_{1_n}\nabla(h_{1_n}+h_{2_n}) + 2\nu_1\text{div}(h_{1_n}D(u_{1_n}))$$

$$-gh_{2_n}\nabla(h_{1_n}+\frac{h_{2_n}}{r}) + \beta\nabla\Delta h_{1_n} - \delta h_{1_n}\nabla\left[h_{1_n}^{-\alpha}\right] - \bar{\kappa}h_{1_n}\nabla\Delta^{2s+1}h_{1_n}$$

We then study each term:
• $\text{div}(h_{1_n}u_{1_n}\otimes u_{1_n}) = \text{div}(\sqrt{h_{1_n}}u_{1_n}\otimes\sqrt{h_{1_n}}u_{1_n})$ which is in $L^\infty(0,T;W^{-1,1}(\Omega))$,

• as h_{1_n} is bounded in $L^\infty(0,T;L^p(\Omega))$ and $\nabla(h_{1_n}+h_{2_n})$ is in $L^2(0,T;L^2(\Omega))$, the we have: $h_{1_n}\nabla(h_{1_n}+h_{2_n})$ bounded in $L^2(0,T;L^{\frac{2p}{p+2}}(\Omega))$

• remark that

$$h_{1_n}\nabla u_{1_n} = \nabla(h_{1_n}u_{1_n}) - u_{1_n}\otimes\nabla h_{1_n} = \nabla(\sqrt{h_{1_n}}\sqrt{h_{1_n}}u_{1_n}) - 2\sqrt{h_{1_n}}u_{1_n}\nabla\sqrt{h_{1_n}}, \tag{3.17}$$

we know that the first term is in $L^\infty(0,T;W^{-1,\frac{2p}{p+2}}(\Omega))$ and the second one in $L^\infty(0,T;(L^1\Omega))$. So we have $h_nD(u_n)$ bounded in $L^2(0,T;W^{-1,\frac{2p}{p+2}}(\Omega))$.

• Also, h_{2_n} is bounded in $L^\infty(0,T;L^2(\Omega))$ and $\nabla(h_{1_n}+\frac{h_{2_n}}{r})$ is bounded in $L^2(0,T;L^2(\Omega))$, therefore $h_{2_n}\nabla(h_{1_n}+\frac{h_{2_n}}{r})$ is bounded in $L^2(0,T;L^1(\Omega))$.

• We have Δh_{1_n} is bounded in $L^2(0,T;L^2(\Omega))$, so that $h_{1_n}\nabla\Delta h_{1_n}$ is bounded in $L^2(0,T;W^{-1,1}(\Omega))$.

• One knowns that $\nabla\Delta^s h_{1_n}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and $\Delta^{s+1}h_{1_n}$ is bounded in $L^2(0,T;L^2(\Omega))$. Thus $h_{1_n}\nabla\Delta^{2s+1}h_{1_n}$ is bounded in
$L^2(0,T;L^1(\Omega)) \subset L^2(0,T;W^{-1,1}(\Omega))$.

• Thanks to the **Proposition** 3.6, h_{1_n} is bounded in $L^\infty(0,T;L^\infty))$, hence $h_{1_n}\nabla\left[h_{1_n}^{-\alpha}\right]$ is bounded in $L^2(0,T;W^{-1,1}(\Omega))$.

Finally, note that this terms are included in $L^2(0,T;W^{-1,1}(\Omega))$, which means that $\partial_t(h_{1_n}u_{1_n})$ is also in this space. Then, applying Aubin-Simon lemma, we obtain:

$$(h_{1_n}u_{1_n})_n \text{strongly converges to } h_1u_1 \text{ in } C^0(0,T;W^{-1,1}(\Omega)).$$

3.9. Step 6: Convergence of $(\sqrt{h_{1_n}}u_{1_n})_{n\geq 1}$.

As we have $\mathbf{m}_n = h_{1_n}u_{1_n}$, so,we have $\sqrt{h_{1_n}u_{1_n}} = \dfrac{\mathbf{m}_n}{\sqrt{h_{1_n}}}$

We will show the convergence of this term. We know that $\dfrac{\mathbf{m}_n}{\sqrt{h_{1_n}}}$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Consequently Fatou lemma reads:

$$\int_\Omega \liminf \frac{\mathbf{m}_n}{h_{1_n}} \leq \liminf \int_\Omega \frac{\mathbf{m}_n^2}{h_{1_n}} < +\infty$$

Then, we can define the limit velocity taking $u_{1_n}(t,x) = \frac{\mathbf{m}_n(t,x)}{h(t,x)}$ ($h_{1_n}(t,x) \neq 0$). So we have a link between the limits $\mathbf{m}_n(t,x) = h_{1_n}(t,x)u_{1_n}(t,x)$ and:

$$\int_\Omega \frac{\mathbf{m}_n^2}{h_{1_n}} = \int_\Omega h_{1_n}|u_{1_n}|^2 < +\infty =$$

Thanks to the **Remark**3.9, we have:$\sqrt{h_{1_n}}|u|^2$ in $L^2(0,T;L^2(\Omega))$.
As $(\mathbf{m}_n)_n$ and $(h_{1_n})_n$ converge , the sequence of $\sqrt{h_{1_n}}u_{1_n}$ converges to $\sqrt{h_1}u_1$.
Moreover, for all M positive, $(\sqrt{h_{1_n}}u_{1_n}1_{|u_{1_n}|\leq M})_n$ converges to $\sqrt{h_1}u1_{|u_1|\leq M}$.
Finally, let us consider the following norm:

$$\int_\Omega \left|\sqrt{h_{1_n}}u_{1_n} - \sqrt{h_1}u_1\right|^2 \leq \int_\Omega \left(\left|\sqrt{h_{1_n}}u_{1_n}1_{|u_{1_n}|\leq M} - \sqrt{h_1}u1_{|u_1|\leq M}\right|\right.$$

$$\left. + \left|\sqrt{h_{1_n}}u_{1_n}1_{|u_{1_n}|>M}\right| + \left|\sqrt{h}u_1 1_{|u_1|>M}\right|\right)^2 \leq$$

$$3\int_\Omega \left|\sqrt{h_{1_n}}u_{1_n}1_{|u_{1_n}|\leq M} - \sqrt{h_1}u_1 1_{|u_1|\leq M}\right|^2 + 3\int_\Omega \left|\sqrt{h_{1_n}}u_{1_n}1_{|u_{1_n}|>M}\right|^2$$

$$+3\int_\Omega \left|\sqrt{h_1}u_1 1_{|u_1|>M}\right|^2.$$

Since $(\sqrt{h_n}u_n)_n$ is in $L^\infty(0,T;L^p(\Omega))$, $(\sqrt{h_{1_n}}u_{1_n}1_{|u_{1_n}|\leq M})_n$ is bounded in this space. So, as we have seen previously, the first integral tends to zero. Let us study the other two terms:

$$\int_\Omega \left|\sqrt{h_{1_n}}u_{1_n}1_{|u_{1_n}|>M}\right|^2 \leq \frac{1}{M^2}\int_\Omega h_{1_n}|u_{1_n}|^4 \leq \frac{k}{M^2} \text{ and}$$

$$\int_\Omega \left|\sqrt{h_1}u_1 1_{|u_1|>M}\right|^2 \leq \frac{1}{M^2}\int_\Omega h_1|u_1|^4 \leq \frac{k'}{M^2}$$

for all $M > 0$. When M tends to the infinity, our two integrals tend to zero.
Then

$$(\sqrt{h_{1_n}}u_{1_n})_n \text{converges strongly to } \sqrt{h_1}u_1 \text{ in } L^2(0,T;L^2(\Omega)).$$

This ends the proof of Theorem 3.10.

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