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Approximating positive solutions of nonlinear BVPs of ordinary second order hybrid differential equations

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Abstract. In this paper we prove the existence and approximation of solution for a nonlinear two point boundary value problem of ordinary second order hybrid differential equations with Dirichlet boundary conditions via construction of an algorithm. The nonlinearity present on the right hand side of the differential equation is assumed to be Carathoèodory and the proof is based on a Dhage iteration method based on a hybrid fxed point theorem of Dhage (2014) in an ordered Banach algebra.

AMS Subject Classifications: 34A45, 34B15, 47H07, 47H10.

Keywords: Nonlinear boundary value problems, Hybrid differential equation, Dhage iteration method, Existence and approximation theorem.

Contents

1. Introduction

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{R}_+$ the set of all nonnegative reals. Given a closed and bounded interval $J = [a, b] \subset \mathbb{R}$, $a < b$, consider the nonlinear two point hybrid boundary value problem (in short HBVP) of ordinary hybrid differential equation,

$$\left.\begin{aligned} -\left(\frac{x(t)}{f(t,x(t))}\right)'' &= g(t,x(t)) \quad \text{a.e. } t \in J, \\ x(a) = &0 = x(b), \end{aligned}\right\} \tag{1.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is continuous and $f : J \times \mathbb{R} \to \mathbb{R}$ is a Carathèodory function.

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When $f \equiv 1$ on $I \times \mathbb{R}$, the HBVP (1.1) reduces to the well-known nonlinear two point BVP

$$\left.\begin{aligned} -x''(t) &= g(t, x(t)) \quad \text{a.e. } t \in J, \\ x(a) &= 0 = x(b), \end{aligned}\right\} \tag{1.2}$$

which is studied earlier extensively in the literature (see Bailey *et al.* [1]).

Definition 1.1. *A function $x \in AC^1(J, \mathbb{R})$ is said to be a* lower solution *of the HBVP (1.1) if*

$$\left.\begin{aligned} -\left(\frac{x(t)}{f(t, x(t))}\right)'' &\leq g(t, x(t)) \quad \text{a.e. } t \in J, \\ x(a) \leq \ 0 \ &\geq x(b), \end{aligned}\right\} \tag{1.3}$$

where, $AC^1(J, \mathbb{R})$ is the space of functions $x \in C(J, \mathbb{R})$ whose first derivative exists and is absolutely continuous on I. Similarly, $x \in AC^1(J, \mathbb{R})$ is called an upper solution *of (1.1) on J if the reversed inequalities hold in (1.3). If equalities hold in (1.3), we say that x is a* solution *of (1.1) on J.*

Notice that the differential equation (1.1) is a hybrid nonlinear differential equation with a quadratic perturbation of second type. The details of classification of different types of perturbations of the differential are given in Dhage [4]. The existence of the solution to the problem (1.1) may be proved by using hybrid fixed point theorems of Dhage in a Banach algebra as did in Dhage [2, 3], Dhage and Dhage [9], Dhage and Dhage [11] and Dhage and Imdad [13]. The existence and approximation result for the PBVP and the BVP (1.2) is already proved respectively in Dhage and Dhage [10] and Dhage [7] via a new Dhage iteration method developed in [5, 6]. In the present paper, we shall extend above Dhage iteration method to the HBVP (1.1) and study the existence and approximation of positive solutions of under certain hybrid conditions on the nonlinearities f and g from algebra, analysis and topology.

2. Auxiliary Results

We need the following definitions in what follows.

Definition 2.1 (Dhage [2–5])**.** *An upper-semicontinuous and nondecreasing real function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the condition $\psi(0) = 0$ is called a $\mathcal{D}$-function on $\mathbb{R}_+$. The class of all $\mathcal{D}$-functions is denoted by $\mathfrak{D}$.*

A few examples of the $\mathcal{D}$ functions on $\mathbb{R}_+$ appear in Dhage and Dhage [10] and references therein.

Definition 2.2. *A function $\beta : J \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory if*

(i) the map $t \mapsto \beta(t, x)$ is measurable for each $x \in \mathbb{R}$, and

(ii) the map $x \mapsto \beta(t, x)$ is continuous for each $t \in J$.

The following lemma is often used in the study of nonlinear differential equations (see Dhage [3], Dhage and Imdad [13] and references therein).

Lemma 2.3 (Carathéodory)**.** *Let $\beta : J \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Then the map $(t, x) \mapsto \beta(t, x)$ is jointly measurable. In particular the map $t \mapsto \beta(t, x(t))$ is measurable on J for each $x \in C(J, \mathbb{R})$.*

We need the following hypotheses in the sequel.

(H_1) f defines a continuous bounded function $f : J \times \mathbb{R} \to \mathbb{R}_+ \setminus \{0\}$ with bound M_f.

(H$_2$) There exists a $\mathcal{D}$-function $\psi_f \in \mathfrak{D}$ such that

$$0 \le f(t,x) - f(t,y) \le \psi_f\big(x-y\big)$$

for all $\in J$ and $x, y \in \mathbb{R}$ with $x \ge y$. Moreover, $\dfrac{(b-a)^2}{8} M_g \psi_f(r) < r, \;\; r > 0$.

(H$_3$) The function g is Carathéodory on $J \times \mathbb{R}$ into $\mathbb{R}_+$.

(H$_4$) g is bounded on $J \times \mathbb{R}$ with bound M_g.

(H$_5$) $g(t,x)$ is nondecreasing in x for each $t \in J$.

(LS) The HBVP (1.1) and (1.3) has a lower solution $u \in AC^1(J,\mathbb{R})$.

(US) The HBVP (1.1) and (1.3) has an upper solution $v \in AC^1(J,\mathbb{R})$.

Lemma 2.4. *Given any function $h \in L^1(J,\mathbb{R})$, the HBVP*

$$\left.\begin{aligned} -\left(\frac{x(t)}{f(t,x(t))}\right)'' &= h(t) \quad a.e. \;\; t \in J, \\ x(a) = &0 = x(b), \end{aligned}\right\} \tag{2.1}$$

is equivalent to the quadratic hybrid integral equation (in short QHIE)

$$x(t) = \big[f(t,x(t))\big]\left(\int_a^b G(t,s)h(s)\,ds\right), \;\; t \in J, \tag{2.2}$$

where $G(t,s)$ is the Green's function associated with the homogeneous boundary value problem

$$\left.\begin{aligned} -x''(t) = 0 \;\; a.e. \;\; t \in J, \\ x(a) = 0 = x(b). \end{aligned}\right\} \tag{2.3}$$

Notice that the function x given by (2.2) belongs to the class $C(J,\mathbb{R})$. Clearly, $G(t,s)$ is continuous and nonnegative on $J \times J$ and satisfies the inequalities

$$0 \le G(t,s) \le \frac{b-a}{4} \quad \text{and} \quad \int_a^b G(t,s)\,ds \le \frac{(b-a)^2}{8}.$$

The proof of our main result will be based on the **Dhage monotone iteration principle** or **Dhage monotone iteration method** contained in a applicable hybrid fixed point theorem in the partially ordered Banach algebras. A non-empty closed convex subset K of the Banach algebra E is called a cone if it satisfies i) $K + K \subset K$, ii) $\lambda K \subseteq K$ for $\lambda > 0$ and iii) $\{-K\} \cap K = \{0\}$. We define a partial order $\preceq$ in E by the relation $x \preceq y \iff y - x \in K$. The cone K is called positive if iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplicative composition in E. In what follows we assume that the cone K in a partially ordered Banach algebra (E,K) is always positive. Then the following results are known in the literature.

Lemma 2.5 (Dhage [8])**.** *Every ordered Banach space (E,K) is regular.*

Lemma 2.6 (Dhage [8])**.** *Every partially compact subset S of an ordered Banach space (E,K) is a Janhavi set in E.*

Theorem 2.7 (Dhage [5, 6])**.** *Let $\big(E,K,\|\cdot\|\big)$ be a regular partially ordered complete normed linear algebra and let every chain C in E be a Janhavi set. Suppose that $\mathcal{A},\mathcal{B} : E \to K$ are two monotone nondecreasing operators such that* *(a) $\mathcal{A}$ is partially bounded and partial $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function $\psi_{\mathcal{A}}$,*

(b) $\mathcal{B}$ is partially continuous and uniformly partially compact,

(c) $M_{\mathcal{B}}\,\psi_{\mathcal{A}}(r) < r$, $r > 0$, where $M_{\mathcal{B}} = \sup\{\|\mathcal{B}(C)\| : C$ is a chain in $E\}$, and

(d) there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0\,\mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0\,\mathcal{B}x_0$.

Then the hybrid operator equation $\mathcal{A}x\,\mathcal{B}x = x$ has a solution x^* in K and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\,\mathcal{B}x_n$ converges monotonically to x^*.

The details of Dhage monotone iteration principle or method and related definitions of Janhavi set and uniformly partially compact operator along with some applications may be found in Dhage [5–8] and the references therein.

3. Existence and Approximation Result

Let $C_+(J,\mathbb{R})$ denote the space of all nonnegative-valued functions of $C(J,\mathbb{R})$. We assume that the space $C(J,\mathbb{R})$ is endowed with the norm $\|\cdot\|$ and the multiplication "$\cdot$" defined by

$$\|x\| = \max_{t\in J}|x(t)| \quad \text{and } (x\cdot y)(t) = x(t)y(t)\ \ t \in J. \tag{3.1}$$

We define a partial order $\preceq$ in E with the help of the cone K in E defined by

$$K = \big\{x \in E \mid x(t) \geq 0 \ \text{ for all } \ t \in J\big\} = C_+(J,\mathbb{R}), \tag{3.2}$$

which is obviously a positive cone in $C(J,\mathbb{R})$. Thus, we have $x \preceq y \iff y - x \in K$.

Clearly, $C(J,\mathbb{R})$ is a partially ordered Banach algebra with respect to above supremum norm, multiplication and the partially order relation in $C(J,\mathbb{R})$. A solution ξ^* of the HBVP (1.1) is *positive* if it is in the class of function space $C_+(J,\mathbb{R})$.

Theorem 3.1. *Suppose that hypotheses (H$_1$)-(H$_5$) and (LS) hold. Then the BVP (1.1) has a positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$\left.\begin{aligned} x_0(t) &= u(t),\ \ t \in J,\\ x_{n+1}(t) &= \big[f(t,x_n(t))\big]\left(\int_a^b G(t,s)g(s,x_n(t))\,ds\right),\ \ t\in J, \end{aligned}\right\} \tag{3.3}$$

converges monotone nondecreasingly to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then, in view of Lemmas 2.5 and 2.6, E is regular and every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\preceq$ so that every compact chain C is a Janhavi set in E.

Now by Lemma 2.3, the BVP (1.1) is equivalent to the QHIE

$$x(t) = \big[f(t,x(t))\big]\left(\int_a^b G(t,s)g(s,x(t))\,ds\right),\ t \in J. \tag{3.4}$$

Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = f(t,x(t)),\ \ t \in J, \tag{3.5}$$

and

$$\mathcal{B}x(t) = \int_a^b G(t,s)g(s,x(t))\,ds, \;\; t \in J. \tag{3.6}$$

From hypotheses (H_1) and (H_3), it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A},\mathcal{B} \,:\, E \to K$. Now the QHIE (3.4) is equivalent to the quadratic hybrid operator equation

$$\mathcal{A}x(t)\,\mathcal{B}x(t) = x(t), \; t \in J. \tag{3.7}$$

Now, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.7 in a series of following steps.

Step I: *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing operators on E.*

Let $x, y \in E$ be such that $x \succeq y$. Then, from the hypothesis (H_2) it follows that

$$\mathcal{A}x(t) = f(t,x(t)) \geq f(t,y(t)) = \mathcal{A}x(t)$$

for all $t \in J$. Hence $\mathcal{A}x \succeq \mathcal{A}(y)$ and that $\mathcal{A}$ is nondecreasing on E. Similarly, we have by hypothesis (H_5),

$$\mathcal{B}x(t) = \int_{t_0}^{t_1} G(t,s)g(s,x(s))d \geq \int_{t_0}^{t_1} G(t,s)g(s,y(s))\,ds = \mathcal{B}y(t)$$

for all $t \in I$. This implies that $\mathcal{B}x \succeq \mathcal{B}y$ whenever $x \succeq y$. Thus, $\mathcal{B}$ is also nondecreasing operator on E.

Step II: *Next we show that $\mathcal{A}$ is partially bounded and partial $\mathcal{D}$- Lipschitz on E.*

Now, for any $x \in E$, one has

$$\|\mathcal{A}x\| = \sup_{t\in J} |f(t,x(t))| \leq M_f$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{A}x\| \leq M_f$ for all $x \in E$ and so $\mathcal{A}$ is bounded and so partially bounded on E. Nxt let $x, y \in E$ be such that $x \succeq y$. Then, by hypothesis (H_2),

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| \leq \varphi_f(|x(t) - y(t)|) \leq \varphi_f(\|x - y\|)$$

for all $t \in J$. Taking the supremum over t, we get

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \varphi_f(\|x - y\|)$$

for all $x, y \in E$, $x \succeq y$. This shows that $\mathcal{A}$ is a partial $\mathcal{D}$-Lipschitz on E with $\mathcal{D}$-function φ_f.

Step III: *$\mathcal{B}$ is a partially continuous and partially compact on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C such that $x_n \to x$ as $n \to \infty$. Since the f is continuous, by dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \int_a^b G(t,s)g(s,x_n(s))\,ds \\ &= \int_a^b G(t,s)\left[\lim_{n\to\infty} g(s,x_n(s))\right]\,ds = \mathcal{B}x(t),\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges to $\mathcal{B}x$ pointwise on J. Next, we show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Now for any $t_1, t_2 \in J$, one obtains

$$|\mathcal{B}x_n(t_1) - \mathcal{B}x_n(t_2)| \leq M_g \int_a^b |G(t_1,s) - G(t_2,s)|\,ds \tag{3.8}$$

Since the function $t \to G(t,s)$ is continuous on compact Jit is uniformly continuous there. Consequently the function $t \to G(t,s)$ is uniformly continuous on J. Therefore, we have

$$|G(t_1,s) - G(t_2,s)| \to 0 \quad \text{as } t_1 \to t_2$$

uniformly on J. As a result, we have that

$$|\mathcal{B}x_n(t_1) - \mathcal{B}x_n(t_2)| \to 0 \quad \text{as } t_1 \to t_2,$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform and that $\mathcal{B}$ is a partially continuous operator on E into itself.

Next, we show that $\mathcal{B}$ is a uniformly partially compact operator on E. Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathcal{B}x$. By hypothesis (H_2)

$$|y(t)| = |\mathcal{B}x(t)| \leq \int_a^b G(t,s)|g(s,x(s))|\,ds \leq \frac{(b-a)^2}{8} M_g,$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\| = \|\mathcal{B}x\| \leq \dfrac{(b-a)^2}{8} M_g$, for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is a uniformly bounded subset of E. Next, proceeding with the arguments that given in Step II it can be shown that

$$\big|y(t_2) - y(t_1)\big| = |\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $y \in \mathcal{B}(C)$. This shows that $\mathcal{B}(C)$ is an equicontinuous subset of E. Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous subset of functions in E and hence it is compact in view of Arzelá-Ascoli theorem. Consequently $\mathcal{B}$ is a uniformly partially compact operator on E into itself.

Step IV: *$\mathcal{A}$ and $\mathcal{B}$ satisfy the growth inequality $M_B\,\psi_A(r) < r$, $r > 0$.*

Now, it can be shown $\|\mathcal{B}(C)\| \leq \dfrac{(b-a)^2}{8} M_g = M_{\mathcal{B}}$ for all chain C in E. Therefore, we obtain

$$M_{\mathcal{B}}\psi_{\mathcal{A}}(r) = \frac{(b-a)^2}{8} M_g \psi_f(r) < r$$

for all $r > 0$ and so the hypothesis (c) of Theorem 2.7 is satisfied.

Step VI: *The function u satisfies the operator inequality $u \preceq \mathcal{A}u\,\mathcal{B}u$.*

By hypothesis (LS), the HBVP (1.1) has a lower solution u defined on J. Then, we have

$$\left.\begin{aligned} -\left(\frac{u(t)}{f(t,u(t))}\right)'' &\leq g(t,u(t)) \quad \text{a.e. } t \in J,\\ \frac{u(a)}{f(a,u(a))} \leq 0 &\geq \frac{u(b)}{f(b,u(b))}. \end{aligned}\right\} \tag{3.9}$$

By using this, the maximum principle [15] and the definitions of the operators $\mathcal{A}$ and $\mathcal{B}$, it can be shown that the function $u \in C(J,\mathbb{R})$ satisfies the relation $u \preceq \mathcal{A}u\,\mathcal{B}u$ on J.

Thus, $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.7 and so the quadratic hybrid operator equation $\mathcal{A}x\,\mathcal{B}x = x$ has a positive solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\,\mathcal{B}x_n$ with initial term $x_0 = u$ converges monotone nondecreasingly to x^*. Therefore, the QHIE (3.4) and consequently the HBVP (1.1) has a positive solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) with $x_0 = u$, converges monotone nondecreasingly to x^*. This completes the proof. ■

Remark 3.2. The conclusion of Theorem 3.1 also remains true if we replace the hypothesis (LS) with (US). The proof of Theorem 3.1 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications. In this case the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (3.3) with $x_0(t) = v(t),\ t \in [0,T]$, converges monotone nonincrasingly to the solution x^* of he HIVP (1.1) on J. Again, the existence and approximation result, Theorem 3.1 includes similar result for the positive solution of the HBVP (1.2) as a special case.

Remark 3.3. We note that if the HBVP (1.1) has a lower solution $u \in AC^1(J,\mathbb{R})$ as well as an upper solution $v \in AC^1(J,\mathbb{R})$ such that $u \preceq v$, then under the given conditions of Theorem 3.1 it has corresponding solutions x_* and y^* and these solutions satisfy the inequality

$$u = x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_* \preceq y^* \preceq y_n \preceq \cdots \preceq y_1 \preceq y_0 = v.$$

Hence x_* and y^* are respectively the minimal and maximal impulsive solutions of the HBVP (1.1) in the vector segment $[u,v]$ of the Banach space $E = C(J,\mathbb{R})$, where the vector segment $[u,v]$ is a set of elements in $C(J,\mathbb{R})$ defined by

$$[u,v] = \{x \in C(J,\mathbb{R}) \mid u \preceq x \preceq v\}.$$

This is because of the order cone K defined by (3.2) is a closed convex subset of $C(J,\mathbb{R})$. However, we have not used any property of the cone K in the main existence results of this paper. A few details concerning the order relation by the order cones and the Janhavi sets in an ordered Banach space are given in Dhage [8].

4. An Example

Example 4.1. Given a closed interval $J = [-1,1]$ in $\mathbb{R}$, consider the nonlinear HBVP

$$\left.\begin{aligned} -\left(\frac{x(t)}{f(t,x(t))}\right)'' &= \tanh x(t) + 1 \quad \text{a.e. } t \in J,\\ x(-1) &= 0 = x(1), \end{aligned}\right\} \tag{4.1}$$

where the function $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is defined by

$$f(t,x) = \begin{cases} 1, & x \le 0,\\ 1 + \dfrac{x}{1+x}, & x > 0. \end{cases}$$

Then the function f satisfies the hypotheses (H$_1$)-(H$_2$) with $M_f = 2$ and $\psi_f(r) = \dfrac{r}{1+\xi^2}$, $0 \le \xi \le r$. Here $g(t,x) = \tanh x + 1$ and satisfies the hypotheses (H$_3$)-(H$_5$) with $M_g = 2$. Now the HBVP (4.1) is equivalent to the QHIE

$$x(t) = \big[f(t,x(t))\big]\left(\int_{-1}^{1} k(t,s)\big[\tanh x(s) + 1\big]\,ds\right),\quad t \in [-1,1],$$

where k is a Green's function associated with the homogeneous BVP

$$-x''(t) = 0,\ \ t \in [-1,1], \quad x(-1) = 0 = x(1), \tag{4.2}$$

defined by

$$k(t,s) = \begin{cases} \dfrac{(1-t)(1+s)}{2}, & -1 \le s \le t \le 1,\\[2ex] \dfrac{(1+t)(1-s)}{2}, & -1 \le t \le s \le 1, \end{cases} \tag{4.3}$$

which is continuous and nonnegative on $J \times J$. It can be verified that the function $u \in C(J, \mathbb{R})$ defined by $u(t) = -\int_{-1}^{1} k(t,s)\,ds$ and $v(t) = 4\int_{-1}^{1} k(t,s)\,ds$ are respectively the lower and upper solutions of the QHBVP (4.1) on $[-1,1]$. Hence, by an application of Theorem 3.1, the HBVP (4.1) has a positive solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$x_0(t) = -\int_{-1}^{1} k(t,s)\,ds \;\; t \in [-1,1],$$

$$x_{n+1}(t) = \big[f(t,x_n(t))\big]\left(\int_{-1}^{1} k(t,s)\big[\tanh x_n(s) + 1\big]\,ds\right), \;\; t \in [-1,1],$$

converges monotone nondecreasingly to x^*. Similarly, the sequence $\{y_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$y_0(t) = 4\int_{-1}^{1} k(t,s)\,ds, \;\; t \in [-1,1],$$

$$y_{n+1}(t) = \big[f(t,y_n(t))\big]\left(\int_{-1}^{1} k(t,s)\big[\tanh y_n(s) + 1\big]\,ds\right), \;\; t \in [-1,1],$$

converges monotone nonincreasingly to the positive solution y^* of the QHBVP (4.1) on $[-1,1]$. .

5. The Conclusion

Finally in the conclusion, we mention that the existence and approximation results for the BVP (1.1) and (1.2) on J may also be obtained by using other iteration methods already known in the literature. In case of well-known Picard iteration method, the nonlinearity f is required to satisfy a certain so called strong Lipschitz condition whereas in our Theorem 3.1, it is not a requirement. Similarly, in case of monotone iterative technique for the BVP (1.1) and (1.2), we need to have the existence of both comparable lower as well as upper solutions along with a cumbersome comparison result for getting theoretic approximation of the solution (see [14] and references therein). However, here in the present approach of this paper we get rid of above stringent conditions and still obtain the existence of and approximation of solution in an easy straight forward way. Again, in the case of existence result via generalized iteration method developed by Heikkilä and Lakshmikantham [14] (see also Dhage and Heikkilä [12] and references therein), we also need the existence of both comparable upper as well as lower solutions together with some other conditions such as integrability of the nonlinearity f, notwithstanding it does not yield any algorithm for the solution. Furthermore, the conclusion of the upper and lower solutions method is a by-product of our monotone iteration method as mentioned in Remark 3.3. Therefore, in view of above observations, we conclude that our Dhage iteration method of this paper is an elegant, relatively better and more powerful than all the above mentioned frequently used iteration methods for nonlinear problems because it provides the additional information of algorithm along with the monotonic characterization of the convergence of the sequence of iterations to the approximate solution of the BVP (1.1) and (1.2) defined on J under weaker conditions.

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Toeplitz properties of ω-order preserving partial contraction mapping

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Abstract. In this paper, spectral mapping theorem for the point spectrum on infinitesimal generator of a C_0-semigroup was further investigated. Toeplitz properties of semigroup considering ω-order preserving partial contraction mapping ($\omega - OCP_n$) as a semigroup of linear operator was established to obtained new results. We also consider $A \in \omega - OCP_n$ which is the infinitesimal generator of a C_0-semigroup using the Spectral Mapping Theorem (SMT) to obtain the relationships between the spectrum of A and the spectrum of each of the operators $\{T(t),\ t \geq 0\}$.

AMS Subject Classifications: 06F15, 06F05, 20M05.

Keywords: Toeplitz matrix, Spectrum, $\omega - OCP_n$, C_0-semigroup.

Contents

1. Introduction and Background

The emphasis of spectral theory in functional analysis is important because it studies the structure of a linear operator on the basis of its spectral properties such as the location of the spectrum, the behaviour of the resolvent and the asymptotics of its eigenvalues. It is an inclusive term for theories extending the eigenvector and eigenvalue theory of a single square matrix to a much broader theory of the structure of operators in a variety of mathematical spaces. Suppose X is Banach space, $X_n \subseteq X$ is a finite set, $(T(t))_{t\geq 0}$ the C_0-semigroup, $\omega - OCP_n$ the ω-order preserving partial contraction mapping, M_m be a matrix, $L(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of A and $A \in \omega - OCP_n$ is a generator of C_0-semigroup and its also Toeplitz matrix. This paper consist of results of Toeplitz ω-preserving partial contraction mapping generating a spectral mapping theorem. Balakrishnan [1], established fractional powers of closed operators. Banach [2], introduced the concept of Banach spaces. Bojanczyk *et al.* [3], obtained some results on stability of the Bareiss and related Toeplitz factorization algorithms. Böttcher and Grudsky [4], deduced some results on Teoplitz matrices, asymptotics linear algebra and functional analysis. Engel and Nagel

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[5], obtained one-parameter semigroup for linear evolution equations. Greiner *et al.* [6], showed some results on the spectral bond generator of semigroup of positive operators. Hasegawa [7], introduced some results on the convergence of resolvents of operators. Neerven [8], established the asymptotic behavior of semigroup of linear operator. Pazy [9], presented semigroup of linear operators and applications to partial differential equations. Rauf and Akinyele [10], obtained ω-order-preserving partial contraction mapping and established its properties, also in [11], Rauf *et al.* deduced some results of stability and spectra properties on semigroup of linear operator. Slemrod [12], explained asymptotic behavior of C_0-semigroup as determined by the spectrum of the generator. Vrabie [13], proved some results of C_0-semigroup and its applications. Yosida [14], established and proved some results on differentiability and representation of one-parameter semigroup of linear operators.

2. Preliminaries

Definition 2.1. *(C_0-Semigroup) [13] A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.*

Definition 2.2. *(ω-OCP_n) [10] A transformation $\alpha \in P_n$ is called ω-order-preserving partial contraction mapping if $\forall x, y \in$Domα : $x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.*

Definition 2.3. *(Resolvent Set) [5] We define the resolvent set of A denoted by $\rho(A)$ set of all $\lambda \in \mathbb{C}$ such that λI - A is one-to-one with range equal to X*

Definition 2.4. *(Spectrum) [5] The spectrum of A denoted by $\sigma(A)$ is defined as the complement of the resolvent set.*

Definition 2.5. *(Toeplitz matrix) [4] Toeplitz matrix is a matrix in which each descending diagonal from left to right is constant for any $n \times n$ and for any $m \times n$ matrices.*

Example 1
2×2 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{I} \\ e^{t} & e^{2t} \end{pmatrix}.$$

Example 2
3×3 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{2t} & e^{3t} \\ e^{2t} & e^{2t} & e^{2t} \\ e^{t} & e^{2t} & e^{2t} \end{pmatrix}.$$

Example 3
3×3 matrix $[M_m(\mathbb{C})]$, we have

for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.
Suppose we have

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & e^{3t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Example 4
Let X be the Banach space of Continuous function on [0,1] which are equal to zero at $x = 1$ with the supremum norm. Define

$$(T(t)f)(x) = \begin{cases} f(x+t) & if \;\; x+t \leq 1 \\ 0 & if \;\; x+t > 1 \end{cases}$$

$T(t)$ is obviously a C_0-Semigroup of Contractions on X. Its infinitesimal generator $A \in \omega$-OCP_n is given by

$$D(A) = \{f : f \in C'([0,1]) \cap X_1 f' \in X\}$$

and

$$Af = f' \quad for \quad f \in D(A).$$

one checks easily that for every $\lambda \in \mathbb{C}$ and $g \in X$ the equation $\lambda f - f' = g$ has a unique solution $f \in X$ given by

$$f(t) = \int_t^1 e^{\lambda(t-s)} g(s) ds.$$

Therefore $\sigma(A) = \phi$. on the other hand, since for every $t \geq 0$, $T(t)$ is a bounded linear operator, $\sigma(T(t)) \neq \phi$ for all $t \geq 0$ and the relation $\sigma(T(t)) = exp\{t\sigma(A)\}$ does not hold for any $t \geq 0$.

Theorem 2.6. *(Hille-Yoshida) [11] A linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if*

i. A is densely defined and closed; and

ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$, we have

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{1}{\lambda}. \tag{2.1}$$

3. Main Results

This section presents results of spectral mapping theorem for point spectrum generated by Toeplitz ω-OCP_n:

Theorem 3.1. *Let $T(t)_{t\geq 0}$ be a C_0-semigroup on a Banach space X, with generator $A \in \omega - OCP_n$ which is Toeplitz. Then we have the spectral inclusion relation*

$$\sigma(T(t)) > \exp(t\sigma(A)), \quad \forall t \geq 0.$$

Proof. Firstly we need to show that A is a Toeplitz matrix. Assume b is a trigonometric polynomial of the form

$$\varphi(t) = \sum_{j=-r}^{r} b_j t^j$$

$t \in T(t)$, and let X and Y be infinite matrices of all entries of which are sero outside the upper left $r \times r$ block, that is

$$P_r X P_r = X, \quad P_r Y P_r = Y.$$

without loss of generality assume that $r \geq 1$. Put

$$A_n = T_n(a) + P_n X P_n + \omega_n Y \omega_n,$$

where $A \in \omega - OCP_n$. Obviously, A_n is a band matrix with at most $2r+1$ non-zero diagonals.So, let

$$M = \max(\|T(a) + X\|, \|T(\hat{a} + Y\|), \quad M_0 = \|T(a)\|.$$

Since

$$\|T(a)\| = \|T(a)\| = \|a\|_\infty,$$

then we have

$$\|T(a) + X\| \geq \|T(a)\| = \|T(a)\|,$$

$$\|T(\tilde{a}) + Y\| \geq \|T(\tilde{a})\| = \|T(\tilde{a})\| = \|T(a)\|$$

we always have $M \geq M_0$, so that $\|A_n\| \to M$ as $n \to \infty$.

It is easy to see that

$$\int_0^t T(s)_x ds \in D(A)$$

or all $t \geq 0$. In fact, a direct application of the definition of the generator shows that

$$A\left(\int_0^t T(s)xds\right) = T(t)x - x, \quad \forall x \in X. \tag{3.1}$$

$$A\left(\int_0^t T(s)xds\right) = \int_0^t T(s)Axdx. \tag{3.2}$$

By applying (3.1) and (3.2) to the semigroup $T(t) - \lambda := \{e^{-\lambda t}T(t)\}_{t\geq 0}$ generated by $A - \lambda$, for all $\lambda \in \mathbb{C}$ and $t \geq 0$ we have

$$(\lambda - A)\int_0^t e^{\lambda(t-s)}T(s)xds = (e^{\lambda t} - T(t))x, \quad \forall x \in X$$

and $A \in \omega - OCP_n$, so that

$$\int_0^t e^{\lambda(t-s)}T(s)(\lambda - A)xds = (e^{\lambda t} - T(t))x \quad \forall x \in D(A) \tag{3.3}$$

and $A \in \omega - OCP_n$.

Suppose $e^{\lambda t} \in \varphi(T(t))$ for some $\lambda \in \mathbb{C}$ and $t \geq 0$, and denote the inverse of $e^{\lambda t} - T(t)$ by $K_{\lambda t}$. Since $K_{\lambda t}$ commutes with $T(t)$ and hence also with A, then we have

$$(\lambda - A)\int_0^t e^{\lambda(t-s)}T(s)K_{\lambda,tx}ds = x, \quad \forall x \in X$$

and $A \in \omega - OCP_n$, so that

$$\int_0^t e^{\lambda(t-s)} T(s) K_{\lambda,t}(\lambda - A)xds = x, \quad \forall x \in D(A)$$

and $A \in \omega - OCP_n$.

This shows that the bounded operator B_λ defined by

$$B_\lambda x := \int_0^t e^{\lambda(t-s)} T(s) K_{\lambda,t} xds$$

is a two-sided inverse of $\lambda - A$. It follows that $\lambda \in \varphi(A)$ is in the spectral inclusion relation which achieved the proof.

■

Theorem 3.2. *Assume $T(t)_{t\geq 0}$ is a semigroup of linear operator on a Banach space X, with generator $A \in \omega - OCP_n$ which is Toeplitz. Then*

$$\sigma_p(T(t)) \setminus \{0\} \exp(t\sigma_p(A)), \quad \forall t \geq 0.$$

Proof. Suppose $\lambda \in \sigma_p(A)$ and $x \in D(A)$ is an eigenvector corresponding to λ, the identity (3.3) shows that $T(t)x = e^{\lambda t}x$, that is $e^{\lambda t}$ is an eigenvalue of $T(t)$ with eigenvector x. This proves the inclusion $\supset$.

The inclusion $\subset$ is proved as follows. The case $t = 0$ being trivial, we fix $t > 0$. If $\lambda \in \sigma_p(T(t)) \setminus \{0\}$, then $\lambda = e^{\mu t}$ for some $\mu \in \mathbb{C}$. If x is an eigenvector, then

$$T(t)_x = e^{\mu t} x$$

implies that the map

$$s \mapsto e^{-\mu s} T(s)x$$

is a periodic with period t.

Since this map is not identically zero, the uniqueness theorem for the Fourier transform implies that at least one of its Fourier coefficients is non-zero. Thus, there exists an integer $k \in \mathbb{Z}$ such that

$$x_k := \frac{1}{t} \int_0^t e^{-(2\pi ik/t)s} (e^{\mu s} T(s)x)ds \neq 0.$$

we shall show that $\mu_k := \mu + 2\pi ik/t$ is an eigenvalue of A with eigenvector x_k.

By the t-periodicity of $s \mapsto e^{-\mu s}T(s)x$, for all $Rev > \omega_0(T(t))$, we have

$$\begin{aligned} R(v, A)x &= \int_0^\infty e^{-vs} T(s)xds \\ &= \sum_{n=0}^\infty \int_{nt}^{(n+1)t} e^{-vs} T(s)xds \\ &= \sum_{n=0}^\infty \int_0^t e^{-vs} T(s)(e^{-vnt} T(nt)x)ds \\ &= \sum_{n=0}^\infty e^{(\mu - v)nt} \int_0^t e^{-vs} T(s)xds \\ &= \frac{1}{1 - e^{(\mu - v)t}} \int_0^t e^{-vs} T(s)xds. \end{aligned} \tag{3.4}$$

Since the integral on the right hand side is an entire function, this shows that the map $v \mapsto R(v,A)x$ admits a holomorphic continuation to $\mathbb{C}\setminus\{\mu+2\pi in/t : n\in\mathbb{Z}\}$. Denoting this extension by $F_x(\cdot)$, (3.4) and the definition of x_k we have

$$\lim_{v\to\mu k}(v-\mu k)F_x(v) = x_k.$$

Also, by (3.4) and the t-periodicity of $s \mapsto e^{-\mu s}T(s)x$,

$$\begin{aligned}\lim_{v\to\mu k}(\mu-A)((v-\mu k)F_x(v)) &= \lim_{v\to\mu k}(\mu-A)((v-\mu k)F_x(v))\\ &= \lim_{v\to\mu k}\frac{v-\mu k}{1-e^{(\mu-v)t}}\left((I-e^{-vt}T(t))+(\mu k-v)\int_0^t e^{-vs}T(s)xds\right)\\ &= \frac{1}{t}(0+0)=0.\end{aligned}$$

From the closedness of A , it follows that $x_k \in D(A)$, $A \in \omega - OCP_n$ and $(\mu k - A)_{xk} = 0$.

The spectral mapping theorem also holds for the residual spectrum. This follows from a duality argument for which we need the following definitions.

Since $T(t)$ is a C_0-semigroup on X, then we define

$$X^{\odot} := \{x^* \in X^* : \lim_{t\to 0}\|T^*(t)_x - x^*\| = 0\},$$

where $T^*(t) := (T(T))^*$ is the adjoint operator. It is easy to see that $X^{\odot}$ is a closed $T^*(t)$ - invariant subspace of X^*, and the restriction $T^{\odot}$ of T^* to $X^{\odot}$ is a C_0-semigroup on $X^{\odot}$. We denote its generator by $A^{\odot} \in \omega - OCP_n$.

We claim that $\sigma_p(A^*) = \sigma_p(A^{\odot})$, where $A^* \in \omega - OCP_n$ is the adjoint of the generator A of $T(t)$, and $\sigma_p(T^*(t)) = \sigma_p(T^{\odot}(t))$, $t \geq 0$ and $A \in \omega - OCP_n$.

We start with the first of these assertion. For all $x^* \in D(A^*)$, $x \in X$ and $A \in \omega - OCP_n$, we have

$$\left.\begin{aligned}\langle T^*(t)x^* - x^*, x\rangle &= \langle x^*, T(t)x - x\rangle\\ &= \langle A^*x^*, \textstyle\int_0^t T(t)xdt\rangle\\ &= \textstyle\int_0^t\langle A^*x^*, T(t)x\rangle ds\\ &= \textstyle\int_0^t\langle T^*(t)A^*x^*, x\rangle dt.\end{aligned}\right\} \tag{3.5}$$

Therefore,

$$|\langle T^*(t)x^* - x^*, x\rangle| \leq t\|x\|\|A^*x^*\| \sup_{0\leq s\leq t}\|T(s)\|.$$

By taking the supremum over all $x \in X$ of norm ≤ 1, it follows that

$$\lim_{t\to 0}\|T^*(t)x^* - x^*\| = 0,$$

that is $x^* \in X^{\odot}$. This proves that $D(A^*) \subset X^{\odot}$.

Now assume that $A^*x^* = \lambda x^*$ for some $x^* \in D(A^*)$ and $A^* \in \omega - OCP_n$. Then $x^* \in X^{\odot}$ and (3.5) shows that

$$\left\langle \frac{1}{t}(T^{\odot}(t)x^* - x^*) - \lambda x^*, x\right\rangle = \frac{\lambda}{t}\int \langle T^{\odot}(t)x^* - x^*, x\rangle dt$$

and therefore,

$$\left\|\frac{1}{t}(T^{\odot}(t)x^* - x^*) - \lambda x^*\right\| \leq |\lambda|\|x^*\| \sup_{0\leq s\leq t}\|T(s)x - x\|.$$

Letting $t \to 0$, this shows that $x^* \in D(A^{\odot})$ and

$$A^{\odot}x^* = \lambda x^*,$$

so $\lambda \in \sigma_p(A^{\odot})$.

Conversely, if $\lambda \in \sigma_p(A^{\odot})$ and $A^{\odot}x^{\odot} = \lambda x^{\odot}$, for some $x^{\odot} \in D(A^{\odot})$ and $A^{\odot} \in \omega - OCP_n$, then for all $x \in D(A)$ we have

$$\begin{aligned}\langle x^{\odot}, Ax\rangle &= \lim_{t\to 0}\frac{1}{t}\langle x^{\odot}, T(t)x - x\rangle \\ &= \lim_{t\to 0}\frac{1}{t}\langle T^{\odot}(t)x^{\odot} - x^{\odot}, x\rangle = \langle A^{\odot}x^{\odot}, x\rangle \\ &= \lambda\langle x^{\odot}, x\rangle.\end{aligned}$$

This shows that $x^{\odot} \in D(A^*)$ and $A^*x^{\odot} = \lambda x^{\odot}$, so that $\lambda \in \sigma_p(A^*)$.

Next we prove that

$$\sigma_p(T^*(t)) = \sigma_p(T^{\odot}(t)) \quad for\ all\ t \geq 0.$$

Clearly, we have the inclusion

$$\sigma_p(T^{\odot}(t)) \subset \sigma_p(T^*(t)).$$

Since $T^{\odot}(t)$ is a restriction of $T^*(t)$.

Conversely, if $T^*(t) = \lambda x^*$ for some non-zero $x^* \in X^*$, then for all $\mu \in \varphi(A^*) = \varphi(A)$ we have $R(\mu, A^*)x^* \in D(A^*) \subset X^{\odot}$ and

$$T^{\odot}(t)R(\mu, A^*)x^* = R(\mu, A^*)T^*(t)x^* = \lambda R(\mu, A^*)x^*.$$

Hence, $R(\mu, A^*)x^*$ is an eigenvector of $T^{\odot}(t)$ with eigenvector λ. Hence, the proof is complete.

■

Theorem 3.3. *Assume $T(t)$ is a C_0-semigroup on a Banach space X, with generator $A \in \omega - OCP_n$ which is Toeplitz. Then,*

$$\left.\begin{aligned}&(i)\sigma_r(T(t)) \setminus \{0\} = \exp(t\sigma_r(A)).\\ &(ii) Suppose \quad A \in \omega - OCP_n \quad is\ a\ closed\ linear\ operator\ on \quad X, and \sigma(A) = \sigma_r(A) \cup \sigma_a(A).\end{aligned}\right\} \quad (3.6)$$

Proof. By (3.6) above, we have

$$\sigma_r(T(t)) = \sigma_p(T^*(t)) = \sigma_p(T^{\odot}(t))$$

and

$$\sigma_r(A) = \sigma_p(A^*) = \sigma_p(A^{\odot}).$$

It now follows from Theorem 3.2 applied to the C_0-semigroup $T^{\odot}(t)$, which proves (i).

To proof (ii), assume that $\lambda \in \sigma(A) \setminus \sigma_r(A)$. Then $\lambda - A$ has dense range. If $\lambda - A$ is not injective, then $\lambda \in \sigma_p(A) \subset \sigma_a(A)$ and we are done. Suppose therefore that $\lambda - A$ is injective.

Assume for the moment that there exists a constant $c > 0$ such that

$$\|(\lambda - A)x\| \geq c\|x\| \quad \forall x \in D(A) \quad and \quad A \in \omega - OCP_n.$$

Then the range of $\lambda - A$ is closed. Indeed, if $y_n \to y$ with

$$y_n = (\lambda - A)x_n$$

then

$$\|x_n - x_m\| \leq c^{-1}\|(\lambda - A)(x_n - x_m)\| = \|y_n - y_m\|$$

so the sequence (x_n) is Cauchy, with limit x. The closedness of A implies that $x \in D(A)$, $A \in \omega - OCP_n$ and $(\lambda - A)x = y$, proving that y belongs to the range of $\lambda - A$. Thus, the range of $\lambda - A$ is closed. Since it is also dense, it foloows that it is of X. Since $\lambda - A$ is injective, the inverse $R_\lambda := (\lambda - A)^{-1}$ is well-defined as a closed linear operator on X whose domain is all of X. Hence R_λ is bounded by closed graph theorem. Thus, $\lambda - A$ is invertible, a contraction. It follows that a constant $c > 0$ as above does not exists. But then there is a sequence x_n of norm one vector, $x_n \in D(A)$ for all n, such that

$$\lim_{n\to\infty} (\lambda - A)x_n = 0,$$

which proves that $\lambda \in \sigma_a(A)$.

■

Theorem 3.4. *Let $A \in \omega - OCP_n$ Toeplitz be a closed linear operator on a Banach space X. Then the topological boundary $Q_\sigma(A)$ of the spectrum $\sigma(A)$ is contained in the approximate point spectrum $\sigma_a(A)$.*

Proof. Let $\lambda \in Q_\sigma(A)$ be fixed and let $(\lambda_n) \subset \varphi(A)$ be a sequence such that $\lambda_n \to \lambda$. It follows from uniform boundedness theorem and suppose $A \in \omega - OCP_n$ is a closed linear operator on X. Then for all $\lambda \in \varphi(A)$, we have

$$\|R(\lambda, A)\| \geq \frac{1}{dist(\lambda, \sigma(A))}.$$

Then there exists an $x \in X$ such that

$$\lim_{n\to\infty} \|R(\lambda_n, A)x\| \to \infty.$$

Assume

$$x_n := \|R(\lambda_n, A)x\|^{-1} R(\lambda_n, A)x.$$

Then

$$\|x_n\| = 1$$

and

$$\lim_{n\to\infty} \|Ax_n - \lambda x_n\| = \lim_{n\to\infty} \|R(\lambda_n, A)x\|^{-1}. \cdot \|(\lambda_n - \lambda)R(\lambda_n, A)x - x\| = 0.$$

Hence, the proof is complete. ■

4. Conclusion

In this paper, it has been established that Toeplitz ω-order preserving partial contraction mapping (ω-OCP_n) generates results on spectral mapping theorem for point spectrum.

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Some results on compact fuzzy strong b-metric spaces

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Abstract. In this paper, the concept of compactness on fuzzy strong b-metric space is introduced. On the other hand some basic results are developed on compactness, completeness and totally boundedness.

AMS Subject Classifications: 54A40, 03E72.

Keywords: Compact fuzzy strong b-metric space, $\alpha - \epsilon-$net, $\alpha-$totally bounded set.

Contents

1. Introduction

Several authors generalized metric spaces and fuzzy metric spaces (for reference please see [1,3,4,5,7,9]) in different ways and studied various topological properties on such spaces (please see [2,6,8,10]).
In this paper, we have considered fuzzy strong b-metric space introduced by T. Oner[7] and explore some new concepts such as compactness, totally boundedness to develop some basic results on such spaces.
The organization of the paper is as follows:
In Section 2, some preliminary results are given to be used in this paper. In Section 3, an idea of compact fuzzy strong b-metric space is introduced. Definitions of closed and bounded sets are given and some basic results are studied. The concept of $\alpha - \epsilon-$net and $\alpha-$totally bounded set is introduced and some fundamental results are developed in Section 4.

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2. Preliminaries

In this section, some preliminary results are given which are used in this paper.

Definition 2.1. *([3]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions;*
1) $*$ is associative and commutative,
2) $*$ is continuous,
3) $a * 1 = a \quad \forall a \in [0,1]$,
4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0,1]$.

Definition 2.2. *([9]) An ordered triple (X, D, K) is called strong b-metric space, D is called strong b-metric on X if X is a nonempty set, $K \geq 1$ is a given real number and*
$D : X \times X \to [0, \infty)$ satisfies the following conditions $\forall x, y, z \in X$
1) $D(x, y) = 0$ iff $x = y$,
2) $D(x, y) = D(y, x)$,
3) $D(x, z) \leq D(x, y) + KD(y, z)$.

Definition 2.3 (7)**.** *Let X be a nonempty set, $K > 1$, $*$ is a continuous t-norm and M be a fuzzy set on $X \times X \times (0, \infty)$ such that $\forall x, y, z \in X$ and $t, s > 0$,*
1) $M(x, y, t) > 0$,
2) $M(x, y, t) = 1$ iff $x = y$,
3) $M(x, y, t) = M(y, x, t)$,
4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + Ks)$,
5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.
Then M is called a fuzzy strong b-metric on X and $(X, M, *, K)$ is called fuzzy strong b-metric space.

3. Compact fuzzy strong b-metric space

In this section some definitions are given and basic results are studied.

Definition 3.1. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space and $A \subset X$. A is said to be compact if every sequence in A has a convergent subsequence which converges to some point in A.*

Theorem 3.2. *Every compact fuzzy strong b-metric space is complete if $1 < K < 2$.*

Proof. Let $(X, M, *, K)$ be a compact fuzzy strong b-metric space.
Let $\{x_n\}$ be a Cauchy sequence in X.
Let r and t be arbitrary real numbers such that $r \in (0,1)$ and $t > 0$. Then $\exists r_0 \in (0,1)$ such that $(1 - r_0) * (1 - r_0) * (1 - r_0) > 1 - r$. (Since $*$ is a continuous t-norm)
Since $\{x_n\}$ is a Cauchy sequence, thus for $r_0 \in (0,1)$ and $t > 0$, there exists a natural number n_0 such that

$$M(x_n, x_{n_0}, \frac{t}{3}) > 1 - r_0 \quad \forall n \geq n_0. \tag{3.1}$$

Since X is compact, $\exists$ a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ which converges to some $x \in X$.
Thus, for $\frac{2t}{3K^2} - \frac{t}{3K} (> 0)$ and $r_0 \in (0,1)$, $\exists m \in N$ such that

$$M(x_{k_m}, x, \frac{2t}{3K^2} - \frac{t}{3K}) > 1 - r_0 \quad \forall m \geq n_0. \tag{3.2}$$

Since $k_m \geq m \geq n_0$, we have from (3.1), we have

$$M(x_{k_m}, x_{n_0}, \frac{t}{3}) > 1 - r_0. \tag{3.3}$$

Now for $n \geq n_0$, from (3.1), (3.2) and (3.3), we get

$$\begin{aligned} M(x_n, x, t) &= M(x_n, x, \frac{t}{3} + K.\frac{2t}{3K}) \\ &\geq M(x_n, x_{n_0}, \frac{t}{3}) * M(x_{n_0}, x, \frac{2t}{3K}) \\ &= M(x_n, x_{n_0}, \frac{t}{3}) * M(x_{n_0}, x, \frac{t}{3} + K(\frac{2t}{3K^2} - \frac{t}{3K})) \\ &\geq M(x_n, x_{n_0}, \frac{t}{3}) * M(x_{n_0}, x_{K_m}, \frac{t}{3}) * M(x_{K_m}, x, \frac{2t}{3K^2} - \frac{t}{3K}) \\ &> (1 - r_0) * (1 - r_0) * (1 - r_0). \end{aligned}$$

Thus for $t > 0$ and $r \in (0, 1)$ we have
$M(x_n, x, t) > 1 - r \quad \forall n \geq n_0$
$\Rightarrow \lim_{n\to\infty} x_n = x.$
$\Rightarrow X$ is complete. ■

Note 3.1. Converse of the result may not be true. We justify it by the following example.
Example 3.1. Let $X = R$. Define $M_b(x, y, t) = \frac{t}{t+D(x,y)}$ for $t > 0$
and $x, y \in X$ where $D(x, y) = |x - y| \quad \forall x, y \in X$.
By using Example 2.2[7] it is enough to prove that (X, D, K) is a strong b-metric space to show that $(X, M_D, *, K)$ is a fuzzy strong b-metric space induced by D where $*=$ product t-norm.
Solution . First we show that (X, D, K) is a strong b-metric space.
1. $D(x, y) = |x - y| = 0$ iff $x = y$
2. $D(x, y) = |x - y| = |y - x| = D(y, x)$
3. $D(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| \leq |x - y| + K|y - z|, K > 1$
$\therefore D(x, z) \leq D(x, y) + KD(y, z) \quad \forall x, y, z \in X.$
Thus (X, D, K) is a strong b-metric space.
So, $(X, M_D, *, K)$ is a fuzzy strong b-metric space.
Next we show that $(X, M_D, *, K)$ is complete.
Suppose $\{x_n\}$ is a Cauchy sequence in X.
We choose $\epsilon = \frac{tr}{1-r} (> 0)$ arbitrarily where $t > 0, r \in (0, 1)$.
Now for $t > 0$ and $r \in (0, 1)$, there exists n_0,
such that $M_D(x_n, x_m, t) = \frac{t}{t+|x_n - x_m|} > 1 - r, \quad \forall n, m \geq n_0$.
$\Rightarrow |x_n - x_m| < t(\frac{1}{1-r} - 1) = \frac{tr}{1-r} = \epsilon \quad \forall n, m \geq n_0.$
$\Rightarrow |x_n - x_m| < \epsilon \quad \forall n, m \geq n_0.$
So $\{x_n\}$ is a Cauchy sequence in R. Since R is complete, there exists $x \in R$ such that $x_n \to x$.
Now, $M_D(x_n, x, t) = \frac{t}{t+|x_n - x|} \quad \forall t > 0.$

$$\Rightarrow \lim_{n\to\infty} M_D(x_n, x, t) = \frac{t}{t + \lim_{n\to\infty} |x_n - x|} \quad \forall t > 0.$$

$$\Rightarrow \lim_{n\to\infty} M_D(x_n, x, t) = \frac{t}{t + 0} = 1 \quad \forall t > 0.$$

$\Rightarrow \lim_{n\to\infty} M_D(x_n, x, t) = 1 \quad \forall t > 0.$
Thus $x_n \to x$, for some $x \in X$.
So, $(X, M_D, *, K)$ is complete.
If possible suppose that $(X, M_D, *, K)$ is compact.
Let $\{x_n\}$ be a sequence in X such that $x_n = n \quad \forall n$.
Since X is compact, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $y_n \to y$, for some $y \in X$.
Now, $M_D(y_n, y, t) = \frac{t}{t+|y_n - y|} \quad \forall t > 0.$

$\lim\limits_{n\to\infty} M_D(y_n, y, t) = \dfrac{t}{t + \lim\limits_{n\to\infty}|y_n - y|} \quad \forall t > 0.$
$\Rightarrow 1 = \frac{t}{t+\lim\limits_{n\to\infty}|y_n - y|}$
$\Rightarrow \lim\limits_{n\to\infty}|y_n - y| + t = t$
$\Rightarrow \lim\limits_{n\to\infty}|y_n - y| = 0$
$\Rightarrow y_n \to y$, for some $y \in R$.
Which is a contradiction since the sequence of all natural numbers has no convergent sequence in R. w.r.t. usual metric.
Thus $(X, M_D, *, K)$ is not compact.

Definition 3.3. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space. A subset A of X is said to be bounded if* $\exists t > 0$, $r \in (0,1)$
such that $M(x,y,t) > 1 - r \quad \forall x, y \in A$.

Definition 3.4. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space. A subset F of X is said to be closed if for any sequence $\{x_n\}$ in F such that $x_n \to x$ implies $x \in F$.*
i.e. $\lim\limits_{n\to\infty} M(x_n, x, t) = 1 \quad \forall t > 0$ *implies* $x \in F$.

Proposition 3.5. *Every compact subset of a fuzzy strong b-metric space is closed and bounded.*

Proof. Let $(X, M, *, K)$ be a fuzzy strong b-metric space and A be a subset of X.
If possible suppose that A is not closed. So $\exists$ a sequence $\{x_n\}$ in A such that $x_n \to x$ but $x \notin$ A.
Since A is compact, so $\exists$ a subsequence $\{x_{n_K}\}$ of $\{x_n\}$ which converges to some point in A.
Since $x_n \to x$ thus $\{x_{n_K}\} \to x$ and hence $x \in A$.
Which is a contradiction. Thus A is closed.
Now we show that A is bounded.
If possible suppose that A is unbounded. Fix $x_0 \in A$.
Choose a sequence $\{\alpha_n\} \in (0,1) \quad \forall n$ such that $\alpha_n \to 1$ as $n \to \infty$.
Thus for a given $t > 0$, for each n, $\exists x_n \in A$ such that
$M(x_0, x_n, t) \leq 1 - \alpha_n$.
Now we obtain a sequence $\{x_n\}$ in A. Since A is compact, $\exists$ a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ which converges to some point $x \in A$.
Now we have $M(x_0, x_{n_l}, t) \leq 1 - \alpha_{n_l}$
We have, $1 - \alpha_{n_l} \geq M(x_0, x_{n_l}, t)$
$= M(x_0, x_{n_l}, \frac{t}{2} + \frac{Kt}{2K})$
$\geq M(x_0, x, \frac{t}{2}) * M(x, x_{n_l}, \frac{t}{2K})$
$\Rightarrow \lim\limits_{n\to\infty}(1 - \alpha_{n_l}) \geq \lim\limits_{n\to\infty} M(x_0, x, \dfrac{t}{2}) * \lim\limits_{n\to\infty} M(x, x_{n_l}, \dfrac{t}{2K})$
$\Rightarrow \quad 0 \geq M(x_0, x, \frac{t}{2}) * 1 = M(x_0, x, \frac{t}{2})$
$\Rightarrow M(x_0, x, \frac{t}{2}) = 0$ which contradict the condition (3.1). ∎

Note 3.2. Converse of the above result may not be true. We justify it by the following example.
Example 3.2. Let $X = l_2$.
Define $D(x,y) = (\sum\limits_{i=1}^{\infty}|x_i - y_i|^2)^{\frac{1}{2}}$ where $x = (x_1, x_2, x_3, ...)$ and $y = (y_1, y_2, y_3, ...)$
Then it is easy to verify that (X, D) is a strong b-metric space for $K \geq 1$
Again define $M_b(x,y,t) = \frac{t}{t+D(x,y)} \quad \forall t \in (0, \infty)$.
Then by using Example 2.2[7], it follows that $(X, M_b, *, K)$ is a fuzzy strong b-metric space w.r.t. the t-norm $*$=product.

Choose $A = \{(1,0,0,....),(0,1,0,...),(0,0,1,0,...),....\}$ subset of l_2.
For $x, y \in A$ with $x \neq y$ we get $M_b(x,y,t) = \frac{t}{t+\sqrt{2}}$.
Take $t = \sqrt{2}+1$ and $\alpha = \frac{1}{2}$.
Then $\forall x, y (x \neq y) \in A$ we get,
$M_b(x,y,\sqrt{2}+1) = \frac{\sqrt{2}+1}{\sqrt{2}+1+\sqrt{2}}$
Now, $\frac{\sqrt{2}+1}{\sqrt{2}+1+\sqrt{2}} - \frac{1}{2} = \frac{2\sqrt{2}+2-2\sqrt{2}-1}{2(2\sqrt{2}+1)}$
$= \frac{1}{2(2\sqrt{2}+1)} > 0$
Thus $M_b(x,y,\sqrt{2}+1) > 1-\alpha = 1-\frac{1}{2} \qquad \forall x, y (x \neq y) \in A$
Also for $x = y$, $M_b(x,y,\sqrt{2}+1) = 1 > 1-\frac{1}{2}$.
Thus A is bounded.
On the other hand if we consider the sequence $\{x_n\}$ in A where $x_n = (0,0,0,...,1(n^{th} place),0,...)$.
Clearly A is closed and since neither the sequence $\{x_n\}$ nor its any subsequence converges to some element in A, so A is not compact.

Proposition 3.6. *Every finite subset in a fuzzy strong b-metric space is bounded.*

Proof. Let $(X, M, *, K)$ be a fuzzy strong b-metric space and A be a finite subset of X containing n elements $x_1, x_2, ...x_n$.
Choose $t_0 > 0$ fixed. Let $\min_{i,j} M(x_i, x_j, t_0) = \beta \quad i, j = 1, 2, ...n$.
Clearly $\beta \in (0,1)$.
Choose $\alpha \in (0,1)$ such that $\min_{i,j} M(x_i, x_j, t_0) > 1-\alpha$
$\Rightarrow M(x_i, x_j, t_0) > 1-\alpha \qquad \forall x_i, x_j \in A$.
$\Rightarrow A$ is bounded. ■

4. Totally bounded set in fuzzy strong b-metric space

In this section the concept of $\alpha-$totally bounded set is introduced and some fundamental results on $\alpha-$ totally bounded sets are developed.

Definition 4.1. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space and $A \subset X$ and $\alpha \in (0,1)$ be given. Let $\epsilon > 0$ be a positive number. A set $B \subset X$ is said to be an $\alpha - \epsilon$-net for the set A if for any $x \in A$, $\exists y \in B$ such that*
$M(x, y, \frac{\epsilon}{K}) > 1-\alpha$.
B may be finite or infinite.

Definition 4.2. *A set A in a fuzzy strong b-metric space $(X, M, *, K)$ is said to be α-totally bounded for a given $\alpha \in (0,1)$, if for any $\epsilon > 0$, there exists a finite $\alpha - \epsilon$-net for the set A.*

Theorem 4.3. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space and $A \subset X$ be α-totally bounded for some $\alpha \in (0,1)$. Then A is bounded.*

Proof. Since A is α-totally bounded, so for each $\epsilon > 0$, there exists a finite $\alpha - \epsilon$-net B for the set A.
Choose $\epsilon_0 > 0$. Then for each $x \in A$, there exists $y \in B$ such that $M(x, y, \frac{\epsilon_0}{K}) > 1-\alpha$.
Since B is finite thus B is bounded. (by Proposition 3.6).
So $\exists \epsilon_1 > 0$ and $\alpha_0 \in (0,1)$ such that
$M(y_1, y_2, \epsilon_1) > 1-\alpha_0 \quad \forall y_1, y_2 \in B$.

Now, for arbitrary $x_1, x_2 \in A$ we have,

$$\begin{aligned} M(x_1, x_2, \epsilon_1 + 2\epsilon_0) &= M(x_1, x_2, \epsilon_1 + K.\frac{\epsilon_0}{K} + K.\frac{\epsilon_0}{K}) \\ &\geq M(x_1, y_2, \epsilon_1 + K\frac{\epsilon_0}{K}) * M(y_2, x_2, \frac{\epsilon_0}{K}) \\ &\geq M(x_1, y_1, \frac{\epsilon_0}{K}) * M(y_1, y_2, \epsilon_1) * M(x_2, y_2, \epsilon_0). \end{aligned} \tag{4.1}$$

Now $M(x_1, y_1, \frac{\epsilon_0}{K}) > 1 - \alpha$, $M(y_1, y_2, \epsilon_1) > 1 - \alpha_0$ and $M(x_2, y_2, \frac{\epsilon_0}{K}) > 1 - \alpha$.

Choose $\beta \in (0, 1)$ such that (since $*$ is continuous).
$(1 - \alpha) * (1 - \alpha_0) * (1 - \alpha) > 1 - \beta$.
From (4.1), we get
$M(x_1, x_2, \epsilon_1 + 2\epsilon_0) \geq (1 - \alpha) * (1 - \alpha_0) * (1 - \alpha) > 1 - \beta$.
$\Rightarrow M(x_1, x_2, \epsilon_1 + 2\epsilon_0) > 1 - \beta. \quad \forall x_1, x_2 \in A$.
$\Rightarrow$ A is bounded.

■

Note 4.1. The converse of the theorem is not true. We can prove it by the following example.

Example 4.2. Let $X = l_2$. Define $D(x, y) = (\sum_{i=1}^{\infty} |x_i - y_i|^2)^{\frac{1}{2}}$ where $x = (x_1, x_2, x_3, ...)$ and $y = (y_1, y_2, y_3, ...)$.
Then it is easy to verify that (X, D) is a strong b-metric space for $K \geq 1$
Again define $M_b(x, y, t) = \frac{t}{t+D(x,y)} \quad \forall t > 0, \forall x, y \in X$.

Then by using Example 2.2[7], it follows that $(X, M_b, *, K)$ is a fuzzy strong b-metric space w.r.t. the t-norm $*$=product.
Consider $A = \{(1, 0, 0, ...), (0, 1, 0...), (0, 0, 1, 0, ...), ..\}$. Then $A \subset X$. It is proved that A is bounded (by previous Example 3.2).
Now, we show that there is no $\alpha - \epsilon-$net for A. Choose $\epsilon = \frac{\sqrt{2}}{(1+K)}, \alpha = 1 - \frac{1}{\sqrt{2}}$ and if possible suppose that N is a finite $\alpha - \epsilon$-net for A. Then for $x_i, x_j, (i \neq j)\epsilon A$, there exist y_i, y_j from N such that $M_b(x_i, y_i, \epsilon) > 1 - \alpha$ and $M_b(x_j, y_j, \epsilon) > 1 - \alpha$.
Now, $M_b(x_i, x_j, \epsilon + K\epsilon) \geq M_b(x_i, y_i, \epsilon).M_b(x_j, y_j, \epsilon)$
$> (1 - \alpha).(1 - \alpha)$
$= (1 - \alpha)^2$
$\Rightarrow \frac{(1+K)\epsilon}{(1+K)\epsilon+\sqrt{2}} > (1 - \alpha)^2$
$\Rightarrow \frac{\sqrt{2}}{2\sqrt{2}} > (1 - \alpha)^2$
$\Rightarrow \frac{1}{2} > \frac{1}{2}$.
Which is a contradiction. So, A is not $\alpha - \epsilon-$bounded.

Definition 4.4. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space and $\alpha \in (0, 1)$.*
(i) A sequence $\{x_n\}$ is said to be α-convergent and converges to x if
$\lim_{n\to\infty} M(x_n, x, t) > 1 - \alpha \quad \forall t > 0$.
(ii) A sequence $\{x_n\}$ in X is said to be α-Cauchy sequence if
$\lim_{m,n\to\infty} M(x_n, x_m, t) > 1 - \alpha \quad \forall t > 0$.
(iii) A subset A of X is said to be α-compact if every sequence in A has an α-convergent subsequence converges to some element in A.
If the converging point belongs to X not to A then we say that A is α-compact in X.

Definition 4.5. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space and $A(\subset X)$ be a nonempty subset of X. Then $\alpha-$ diameter of A is defined as*
$\alpha - \delta(A) = \bigvee_{x,y \in A} \bigwedge \{t > 0 : M(x, y, t) > 1 - \alpha\}, \quad 0 < \alpha < 1.$

Theorem 4.6. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space and $A \subset X$.*
(1) if A is compact then A is α-totally bounded $\forall \alpha \in (0, 1)$.
(2) If X is α-complete and A is α-totally bounded $\forall \alpha \in (0, 1)$ then A is α-compact in X $\forall \alpha \in (0, 1)$ w.r.t. the t-norm $* = \min$.

Proof. (1) We assume that A is compact. Choose $\alpha \in (0, 1)$ and $\epsilon > 0$ be arbitrary. Let x_1 be an arbitrary element of X.
If $M(x, x_1, \frac{\epsilon}{K}) > 1 - \alpha \quad \forall x \in A$, then a finite $\alpha - \epsilon-$net B exists for A. i.e. $B = \{x_1\}$.
If not, $\exists$ a point $x_2 \in A$ such that $M(x_1, x_2, \frac{\epsilon}{K}) \leq 1 - \alpha$. If for every point $x \in A$ either $M(x, x_1, \frac{\epsilon}{K}) > 1 - \alpha$ or $M(x, x_2, \frac{\epsilon}{K}) > 1 - \alpha$ then a finite ϵ-net B exists for A.
i.e. $B = \{x_1, x_2\}$.
If, however, this is not true, then there exists $x_3 \in A$ such that $M(x_3, x_1, \frac{\epsilon}{K}) \leq 1-\alpha$ and $M(x_3, x_2, \frac{\epsilon}{K}) \leq 1-\alpha$.
Then a finite $\alpha - \epsilon-$net $B = \{x_1, x_2, x_3\}$ exists for A.
Continuing in this way, we obtain points $x_1, x_2, ..., x_n$; $x_1 \in X$ and $x_i \in A$, $2 \leq i \leq n$ for which
$M(x_i, x_j, \frac{\epsilon}{K}) \leq 1 - \alpha \quad$ for $i \neq j$.
There are two cases may arise.
Case I. The procedure stops after k th step.
Then we obtain points $x_1, x_2, ..., x_k$ such that for every $x \in A$ at least one of the inequalities
$M(x_i, x, \frac{\epsilon}{K}) > 1 - \alpha, \ \ i = 1, 2, ..., k$ holds and then $B = \{x_1, x_2, ...x_k\}$ is a finite $\alpha - \epsilon-$net for A and here A is α-totally bounded.
Case II. The procedure continues indefinitely.
Then we obtain an infinite sequence $\{x_n\}$, $x_1 \in X$ and $x_i \in A$ for $i > 1$ such that
$M(x_i, x_j, \frac{\epsilon}{K}) \leq 1 - \alpha \quad$ for $i \neq j$.
If possible suppose there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to x.
Now $M(x_{n_k}, x, \frac{\epsilon}{2K}) * M(x, x_{n_{k+1}}, \frac{\epsilon}{2K^2}) \leq M(x_{n_k}, x_{n_{k+1}}, \frac{\epsilon}{K}) \leq 1 - \alpha$
$\Rightarrow \lim_{k\to\infty} M(x_{n_k}, x, \frac{\epsilon}{2K}) * \lim_{k\to\infty} M(x, x_{n_{k+1}}, \frac{\epsilon}{2K^2}) \leq 1 - \alpha$
$\Rightarrow 1 * 1 \leq 1 - \alpha$
$\Rightarrow 1 \leq 1 - \alpha$ which is a contradiction.
Thus Case II does not arise.
Hence A is α-totally bounded. Since $\alpha \in (0, 1)$ is arbitrary thus A is α-totally bounded $\forall \alpha \in (0, 1)$.
2. We assume that X is α-complete and α-totally bounded for each $\alpha \in (0, 1)$.
So for every $\epsilon > 0$ and each $\alpha \in (0, 1)$, there exists a finite $\alpha - \epsilon-$net for A. Let $\alpha \in (0, 1)$ be given. We choose a sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ and $\epsilon_n > 0$ $\forall n$ and $\epsilon_{n+1} < \epsilon_n$ and construct for each $n = 1, 2, ...$ a finite $\alpha - \epsilon_n-$ net
$[x_1^{(n)}, x_2^{(n)}, ..., x_{k_n}^{(n)}]$ for the set A. Let $T = \{x_n\}$ be an arbitrary sequence of elements from A. Without loss of generality we may assume that $x_i \neq x_j$ if $i \neq j$ and T is the infinite set with elements x_n.
Around every point of the $\alpha - \epsilon_1$-net $[x_1^{(1)}, x_2^{(1)}, ..., x_{k_1}^{(1)}]$, we construct closed balls with radius ϵ_1. It is clear that each element of $\{x_n\}$ belongs to one or more of these balls.
Since the number of balls is finite, there exists at least one ball containing an infinite subset $T_1 \subset T$ (say $B[x_1^{(1)}, \alpha, \epsilon_1]$).
Now we show that $\alpha - \delta(T_1) \leq 2\frac{\epsilon_1}{K}$.
Let $x, y \in T_1$. Then $M(x, x_i^{(1)}, \frac{\epsilon_1}{K}) > 1 - \alpha$ and $M(y, x_i^{(1)}, \frac{\epsilon_1}{K}) > 1 - \alpha$ $(1 \leq i \leq k_1)$.
Now $M(x, y, 2\epsilon_1) = M(x, y, \epsilon_1 + K \cdot \frac{\epsilon_1}{K}) \geq M(x, x_i^{(1)}, \epsilon_1) * M(y, x_i^{(1)}, \frac{\epsilon_1}{K})$
$\qquad \geq M(x, x_i^{(1)}, \frac{\epsilon_1}{K}) * M(y, x_i^{(1)}, \frac{\epsilon_1}{K})$

$> (1-\alpha) * (1-\alpha) = 1-\alpha.$
$\Rightarrow \bigwedge\{t > 0 : M(x,y,t) > 1-\alpha\} \leq 2\epsilon_1$
$\Rightarrow \bigvee_{x,y\in T_1} \bigwedge\{t > 0 : M(x,y,t) > 1-\alpha\} \leq 2\epsilon_1$
$\Rightarrow \alpha - \delta(T_1) \leq 2\epsilon_1.$
Next, around every point of the $\alpha - \epsilon_2$-net $[x_1^{(2)}, x_2^{(2)}, ..., x_{k_2}^{(2)}]$
we construct closed sphere with radius ϵ_2.
By the same argument as above, there exists an infinite subset $T_2 \subset T_1$ and
$\alpha - \delta(T_2) \leq 2\epsilon_2.$
Continuing in this process, we obtain a sequence of infinite subsets $T \supset T_1 \supset T_2 \supset ... \supset T_n \supset ...$ where $\alpha - \delta(T_n) \leq 2\epsilon_n \ \ \forall n.$
We now choose a point $x_{p_1} \in T_1$, a point $x_{p_2} \in T_2$ different from x_{p_1}, a point $x_{p_3} \in T_3$ different from x_{p_1} and x_{p_2} and so on.
We have $x_{p_n} \in T_n$, $x_{p_m} \in T_m$ and for $n > m$, $T_n \subset T_m$.
Thus for $n > m$, $x_{p_n}, x_{p_m} \in T_m$.
So $\bigwedge\{t > 0 : M(x_{p_n}, x_{p_m}, t) > 1-\alpha\} \leq \alpha - \delta(T_m) \leq 2\epsilon_m.$
$\Rightarrow \lim_{n,m\to\infty} \bigwedge\{t > 0 : M(x_{p_n}, x_{p_m}, t) > 1-\alpha\} = 0$
Thus for a given $\epsilon > 0$, there exists a natural number say n_0 such that
$\bigwedge\{t > 0 : M(x_{p_n}, x_{p_m}, t) > 1-\alpha\} < \epsilon \ \ \forall m,n \geq n_0.$
$\Rightarrow M(x_{p_n}, x_{p_m}, \epsilon) > 1-\alpha \ \ \forall m,n \geq n_0.$
$\Rightarrow \lim_{m,n\to\infty} M(x_{p_n}, x_{p_m}, \epsilon) \geq 1-\alpha$
Since $\epsilon > 0$ is arbitrary, thus
$\Rightarrow \lim_{m,n\to\infty} M(x_{p_n}, x_{p_m}, t) \geq 1-\alpha \ \ \forall t > 0$
Choose $\beta \in (0,1)$ such that $1-\alpha > 1-\beta$.
So $\lim_{m,n\to\infty} M(x_{p_n}, x_{p_m}, t) > 1-\beta.$
Thus $\{x_{p_n}\}$ is a β-Cauchy sequence in A and hence in X. Since X is β-complete, thus there exists $x \in X$ such that
$\lim_{n\to\infty} M(x_{p_n}, x_{p_m}, t) > 1-\beta \ \ \forall t > 0.$
Hence A is β-compact in X.
Since $\alpha \in (0,1)$ is arbitrary, thus $\beta \in (0,1)$ is also arbitrary and hence the proof is complete. ■

Definition 4.7. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space and $A \subset X$.*
The closure of A is denoted by $\bar{A}$ and is defined by $\bar{A} = A \cup A'$ where A' denotes the derived set of A.

Proposition 4.8. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space and $A \subset X$. For $x \in \bar{A}$, for each $\epsilon > 0$ and $\alpha \in (0,1)$, there exists $y \in A$ such that*
$M(x,y,\epsilon) > 1-\alpha$.

Proof. Let $x \in \bar{A}$. So $x \in A \cup A'$.
Case I . $x \in A$. Then we choose $y = x$ and we have
$M(x,y,\epsilon) = M(x,x,\epsilon) = 1 > 1-\alpha$ for each $\epsilon > 0$ and $\alpha \in (0,1)$.
Case II . $x\ notin$ A and $x \in A'$.
Thus for each $\epsilon > 0$ and $\alpha \in (0,1)$, there exists $y \in A$ such that
$y \in B(x,\epsilon,\alpha)$.
i.e. $M(x,y,\epsilon) > 1-\alpha$. ■

Proposition 4.9. *Let $(X, M, *, K)$ be a fuzzy strong b-metric space and $A \subset X$. If A is compact then $\bar{A}$ is compact.*

Proof. Let $\{y_n\}$ be a sequence in $\bar{A}$.
Choose $\epsilon > 0$ be arbitrary and $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\alpha_n \to 0$ as $n \to \infty$.
Now by Proposition 4.8, for each y_n, there exists $x_n \in A$ such that
$M(x_n, y_n, \frac{\epsilon}{2}) > 1 - \alpha_n$......(i)
Thus we obtain a sequence $\{x_n\}$ in A. Since A is compact, thus there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ which converges to some point $x \in A$.
So $\lim\limits_{r\to\infty} M(x_{n_r}, x, t) = 1 \quad \forall t > 0$
i.e. $\lim\limits_{r\to\infty} M(x_{n_r}, x, \dfrac{\epsilon}{2K}) = 1$.........(ii)
Now $M(y_{n_r}, x, \epsilon) = M(y_{n_r}, x, \frac{\epsilon}{2} + K.\frac{\epsilon}{2K})$
$\qquad \geq M(y_{n_r}, x_{n_r}, \frac{\epsilon}{2}) * M(x_{n_r}, x, \frac{\epsilon}{2K})$
$\Rightarrow \lim\limits_{r\to\infty} M(y_{n_r}, x, \epsilon) \geq \lim\limits_{r\to\infty} M(y_{n_r}, x_{n_r}, \dfrac{\epsilon}{2}) * \lim\limits_{r\to\infty} M(x_{n_r}, x, \dfrac{\epsilon}{2K}) = 1$........(iii)
From (i) we get $M(x_{n_r}, y_{n_r}, \frac{\epsilon}{2}) > 1 - \alpha_{n_r}$
$\Rightarrow \lim\limits_{r\to\infty} M(x_{n_r}, y_{n_r}, \dfrac{\epsilon}{2}) \geq 1 - \lim\limits_{r\to\infty} \alpha_{n_r} = 1$
$\Rightarrow \lim\limits_{r\to\infty} M(x_{n_r}, y_{n_r}, \dfrac{\epsilon}{2}) = 1$...........(iv)
Using (ii) and (iv), from (iii) we have
$\lim\limits_{r\to\infty} M(y_{n_r}, x, \epsilon) \geq 1 * 1 = 1$
$\Rightarrow \lim\limits_{r\to\infty} M(y_{n_r}, x, \epsilon) = 1$
Since $\epsilon > 0$ is arbitrary, thus $\lim\limits_{r\to\infty} M(y_{n_r}, x, t) = 1 \quad \forall t > 0$.
Thus the subsequence $\{y_{n_r}\}$ of $\{y_n\}$ converges to x. Hence $\bar{A}$ is compact. ■

Note 4.1. Converse of the result is not true. We justify it by the following example.
Example 4.1. Let $X = R$. Define $M(x, y, t) = e^{-\frac{D(x,y)}{t}} \quad \forall t > 0; \forall x, y \in X$.
We write $D(x, y) = |x - y| \ \ \forall x, y \in X$. Then it is verified that (X, D, K) is a strong b-metric space (by previous Example 3.2).
Now, we shall prove that $(X, M, *, K)$ is a fuzzy strong b-metric space. Where $*$ is the product t-norm and $K > 1$.
1. $M(x, y, t) = e^{-\frac{D(x,y)}{t}} > 0 \ \ \forall x, y \in X$ and $\forall t > 0$.
2. $M(x, y, t) = 1 \ \ \forall x, y \in X$ and $\forall t > 0$.
$\Leftrightarrow e^{-\frac{D(x,y)}{t}} = 1 = e^0$
$\Leftrightarrow -\frac{D(x,y)}{t} = 0 \ \ \forall t > 0$.
$\Leftrightarrow D(x, y) = 0$
$\Leftrightarrow x = y$.
3. $M(x, y, t) = e^{-\frac{D(x,y)}{t}} = e^{-\frac{D(y,x)}{t}} \quad \forall t > 0$.
$\qquad = M(y, x, t) \ \ \forall x, y \in X$
4. Now, $\forall x, y, z \in X$,
$D(x, z) \leq D(x, y) + KD(y, z) \quad K > 1$.
$\frac{D(x,z)}{t+KS} \leq \frac{D(x,y)+KD(y,z)}{t+KS}; \ \ t, s > 0$.
$e^{\frac{D(x,z)}{t+KS}} \leq e^{\frac{D(x,y)+KD(y,z)}{t+KS}}$
$e^{\frac{D(x,z)}{t+KS}} \leq e^{\frac{D(x,y)}{t+KS}} \cdot e^{\frac{D(y,z)}{\frac{t}{K}+s}}$
$\qquad \leq e^{\frac{D(x,y)}{t}} \cdot e^{\frac{D(y,z)}{s}}$
$e^{\frac{D(x,z)}{t+KS}} \leq e^{(\frac{D(x,y)}{t} + \frac{D(y,z)}{t})}$
$e^{-\frac{D(x,z)}{t+KS}} \geq e^{-(\frac{D(x,y)}{t} + \frac{D(y,z)}{t})}$
$e^{-\frac{D(x,z)}{t+KS}} \geq e^{-\frac{D(x,y)}{t}} \cdot e^{-\frac{D(y,z)}{s}}$
$\therefore M(x, z, t + Ks) \geq M(x, y, t) \cdot M(y, z, s)$
5. This is clear that $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous. Thus $(X, M, \cdot, K)$ is a fuzzy strong b-metric

space.
Let $A = (0,1)$. Then $\bar{A} = [0,1]$.
Firstly, we will show that A is not compact in X. If possible suppose that A is compact. Let $\{x_n\}$ be a sequence in A where $x_n = \frac{1}{n+1} \quad \forall n \geq 1$.
Let $\{x_{k_n}\}$ be a sequence in A such that $x_{k_n} \to y$ for some $y \in A$.
$M(x_{k_n}, y, t) = e^{-\frac{D(x_{k_n}, y)}{t}} \quad \forall t > 0.$

$\lim\limits_{n\to\infty} M(x_{k_n}, y, t) = \lim\limits_{n\to\infty} e^{-\frac{D(x_{k_n}, y)}{t}} = e^{-\lim\limits_{n\to\infty} e^{\frac{D(x_{k_n}, y)}{t}}}$
$\Rightarrow e^0 = 1 = e^{-\lim\limits_{n\to\infty} e^{\frac{D(x_{k_n}, y)}{t}}} \quad \forall t > 0.$
$\Rightarrow \lim\limits_{n\to\infty} \dfrac{D(x_{k_n}, y)}{t} = 0 \quad \forall t > 0.$
$\Rightarrow \lim\limits_{n\to\infty} D(x_{k_n}, y) = 0$
$\Rightarrow \lim\limits_{n\to\infty} |x_{k_n} - y| = 0$
$\Rightarrow y = 0.$
$\Rightarrow y \notin$ A.
Which is a contradiction.
So, A is not complete.
Now we prove that $\bar{A} = [0,1]$ is compact.
By Hine-Borel theorem, $\bar{A} = [0,1]$ is compact in R w.r.t. usual norm given by $\|x\| = |x| \quad \forall x \in R$.
Let $\{x_n\}$ be a sequence in $\bar{A}$. So,there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ which converges in some point $x \in \bar{A}$.
i.e. $|x_{n_r} - x| \to 0$ as $r \to \infty$ and $x \in \bar{A}$.
i.e. $D(x_{n_r}, x) \to 0$ as $r \to \infty$ and $x \in \bar{A}$.
Now $M(x_{n_r}, x, t) = e^{-\frac{D(x_{n_r}, x)}{t}}$

$\Rightarrow \lim\limits_{r\to\infty} M(x_{n_r}, x, t) = \lim\limits_{r\to\infty} e^{-\frac{D(x_{n_r}, x)}{t}} = e^{-\lim\limits_{r\to\infty} e^{\frac{D(x_{n_r}, x)}{t}}}.$
Since $D(x_{n_r}, x) \to 0$ as $r \to \infty$, from above we have,
$\Rightarrow \lim\limits_{r\to\infty} M(x_{n_r}, x, t) = 1 \quad \forall t > 0.$
$\Rightarrow x_{n_r} \to x$ in $(X, M, *, K)$.
Since $\{x_n\}$ is an arbitrary sequence in $\bar{A}$, thus $\bar{A}$ is a compact subset in $(X, M, *, K)$.

5. Conclusion

The concept of fuzzy strong b-metric space is relatively a new idea by modifying the triangle inequality in fuzzy setting. In this paper, we explore an idea of compactness and totally boundedness on fuzzy strong b-metric spaces and establish some basic results. We think that the researchers will be enriched with serendipitous findings by this research work.

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Global stability for reaction-diffusion SIR model with general incidence function

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Abstract. The aim of this work is to study the dynamics of a reaction-diffusion SIR epidemic model with a nonlinear general incidence function. The local stability of the disease-free equilibrium is obtained via characteristic equations. The global existence, positivity and boundedness of solutions for reaction-diffusion system with homogeneous Neumann boundary conditions are proved. We mainly use the technique of Lyapunov functional to establish the global stability of both disease-free and endemic equilibria. Numerical simulations are presented to illustrate our theoretical results by using a suitable discretization of the model.

AMS Subject Classifications: 65L03, 65L03, 34D20, 34D23.

Keywords: SIR epidemic models, HBV model, immune, general incidence function, global stability, Lyapunov functional, reaction-diffusion.

Contents

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1. Introduction

Reaction-diffusion equations model a variety of physical and biological phenomena. These equations describe how the concentration or density distributed in space varies under the influence of two processes: local interactions of species, and diffusion which causes the spread of species in space. In population dynamics, diffusion terms correspond to a random motion of individuals and reaction terms describe their reproduction.
Recently, reaction-diffusion equations have been used by many authors in epidemiology as well as virology. Wang and Wang [11] proposed a mathematical model to simulate the hepatitis B virus (HBV) infection with spatial dependance. They introduced the random mobility of viruses into the basic model proposed by Nowak et al [7] and they assume that the motion of virus follows a Fickian diffusion, that is to say, the population flux of the virus is proportional to the concentration gradient and the proportionality constantis taken to be negative. They also neglected the mobility of susceptible cells and infected cells. Wang et al [12] introduced into [11] an intracellular time delay between the infection of a cell and the production of new virus particles. They considered the initial conditions in a one-dimensional interval with Neumann boundary conditions. The authors neglected the diffusion by assuming that the space is homogeneous in order to establish the global stability of equilibrium solutions. When the space is inhomogeneous, the effects of diffusion and intracellular time delay are obtained by computer simulations. Xu and Ma [14] introduced the saturation response response to the model [12], and obtained sufficient conditions for the global stability of the infected steady rate. In [13], Shaoli et al proposed a diffused HBV model with CTL immune response and nonlinear incidence for the control of viral infections. They Showed that the free diffusion of the virus has no effect on the global stability of such HBV infection problem with Neumann homogeneous boundary conditions. In their work, Yang et al [16] considered the SIR epidemic model with time delay, nonlinear incidence rate was also presented and studied by Xu and Ma [15]. They introduced spacial diffusion in these models and assumed that the three diffusion coefficients are equal in order to prove the existence of traveling waves solutions for the models. They discussed the local stability of a disease-free steady state and an endemic steady state to these models under homogeneous Neumann boundary conditions. Hattal et al [6] take account of the term $e^{-\mu t}$ the probability of surviving from $t-\tau$ to time t in their diffusion model.
Our work is derived from the following SIR epidemic model with a general incidence function described by

$$\begin{cases} \dot{S} = B - \mu_1 S - f(S,I), \\ \dot{I} = f(S,I) - (\mu_2 + \gamma)I, \\ \dot{R} = \gamma I - \mu_3 R, \end{cases} \tag{1.1}$$

where S, I and R are susceptible, infectious, and recovered classes, respectively. B is the recruitment rate of the population, μ_1 is the natural death rate of the population, μ_2 is the death rate due to disease, γ is the recovery rate of the infective individuals. $f(S,I)$ is the rate of transmission.

On the other hand, the spatial content of the environment has been ignored in the model (1.1). However, due to the large mobility of people within a country or even worldwide, spatially uniform models are not sufficient to give a realistic picture of disease diffusion. For this reason, the spatial effects cannot be neglected in studying the spread of epidemics.
This paper is organized as follows. The global existence, positivity and boundedness of solutions is described in Section 2. In Section 3, we give the Qualitative analysis of the spatial model (2.2) in which we determine the local and the global stability of the models. In addition, we present the numerical simulation to illustrate our results in Section 4. Finally, the conclusion of this paper is given in Section 5.

2. Presentation of the model

We consider the following SIR epidemic model with general incidence function and spatial diffusion:

$$\begin{cases} \dfrac{\partial S}{\partial t}(x,t) = d_S \Delta S(x,t) + B - \mu_1 S(x,t) - f(S(x,t), I(x,t)), \\ \\ \dfrac{\partial I}{\partial t}(x,t) = d_I \Delta I(x,t) + f(S(x,t), I(x,t)) - (\mu_2 + \gamma) I(x,t), \\ \\ \dfrac{\partial R}{\partial t}(x,t) = d_R \Delta R(x,t) + \gamma I(x,t) - \mu_3 R(x,t), \end{cases} \tag{2.1}$$

where $S(x,t)$, $I(x,t)$, and $R(x,t)$ represent the numbers of susceptible, infected, and removed individuals at location x and time t, respectively. The positive constants d_S, d_I, and d_R denote the corresponding diffusion rates for these three classes of individuals.
The aim of this work is to investigate the global dynamics of the reaction-diffusion system (2.1). Note that R does not appear in the first two equations; this allows us to study the system

$$\begin{cases} \dfrac{\partial S}{\partial t}(x,t) = d_S \Delta S(x,t) + B - \mu_1 S(x,t) - f(S(x,t), I(x,t)), \\ \\ \dfrac{\partial I}{\partial t}(x,t) = d_I \Delta I(x,t) + f(S(x,t), I(x,t)) - (\mu_2 + \gamma) I(x,t), \end{cases} \tag{2.2}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, \quad \text{on} \quad \partial\Omega \times (0, +\infty), \tag{2.3}$$

and initial conditions

$$S(x,0) = \psi_1(x) \geq 0, \;\; I(x,0) = \psi_2(x) \geq 0, \quad x \in \bar{\Omega}. \tag{2.4}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. $\dfrac{\partial S}{\partial \nu}$ and $\dfrac{\partial S}{\partial \nu}$ are , respectively, the normal derivatives of S and I on $\partial\Omega$.

Remark 2.1. *The Neumann condition is used to ensure the mobility of people in bounded domain* Ω*.*

Let us put

$$f(S,I) = g(S,I)I$$

The incidence function $f(S,I)$ is assumed to be continuously differentiable in the interior of $\mathbb{R}^2_+$ and satisfies the following hypotheses:

H1: $f(0,I) = f(S,0) = 0$ for all $S \geq 0 \;\; I \geq 0$.

H2: $\dfrac{\partial f}{\partial S}(S,I) > 0$ for all $S > 0$ and $I > 0$.

H3: $\dfrac{\partial f}{\partial I}(S,I) > 0$ for all $S > 0$ and $I > 0$.

H4: $f(S,I) \geq f_2(S^0, 0)I$ for all $S > 0$ and $I > 0$.

Let us denote by f_1 and f_2 the partial derivatives of f with respect to the first and to the second variable.

3. Global existence, positivity and boundedness of solutions

n this section, we establish the global existence, positivity, and boundedness of solutions of problem (2.2)-(2.4). Hence, the population should remain nonnegative and bounded.

Proposition 3.1. *For any given data satisfying the condition (2.4), there exists a unique solution of problem (2.2)-(2.4) defined on* $[0,+\infty[$ *and this solution remains nonnegative and bounded for all* $t \geq 0$.

Proof. The system (2.2)-(2.4) can be written abstractly in the Banach space $X = C(\bar{\Omega}) \times C(\Omega)$ of the form:

$$\begin{aligned} u'(t) &= Au(t) + G(u(t)) \\ u(0) &= u_0 \in X, \end{aligned} \tag{3.1}$$

where $u = \begin{pmatrix} S \\ I \end{pmatrix}$, $u_0 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, $Au(t) = \begin{pmatrix} d_S \Delta S \\ d_I \Delta I \end{pmatrix}$ and

$$G(u(t)) = \begin{pmatrix} B - \mu_1 S - f(S,I) \\ f(S,I) - (\mu_2 + \gamma)I \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}. \tag{3.2}$$

G is locally Lipchitz in X. From [8], we deduce that system (3.1) admits a unique local solution on $[0, T_{max}]$, where T_{max} is the maximal existence time for solution of system (3.1).
In addition, system (2.2) can be written in the form:

$$\begin{aligned} \frac{\partial S}{\partial t} - d_S \Delta S &= G_1(S,I) \\ \frac{\partial I}{\partial t} - d_I \Delta I &= G_2(S,I). \end{aligned} \tag{3.3}$$

The functions $G_1(S,I)$ and $G_2(S,I)$ are continuously differentiable and satisfy $G_1(0,I) = B \geq 0$ and $G_2(S,0) = f(S,0) \geq 0$ for all $S, I \geq 0$. Since initial data of system (2.2) are nonnegatives, we deduce the positivity of the local solution (see [10]).
Now, we show the boundedness of solutions. So from (2.2)-(2.4) we have

$$\begin{aligned} \frac{\partial S}{\partial t} - d_S \Delta S &\leq B - \mu_1 S, \\ \frac{\partial S}{\partial \nu} &= 0, \\ S(x,0) = \psi_1(x) &\leq \|\psi_1\|_\infty = \max_{x \in \bar{\Omega}} \psi_1(x). \end{aligned} \tag{3.4}$$

By the comparison principle [9], we have $S(x,t) \leq S_1(t)$,
where $S_1(t) = \psi_1(x) e^{-\mu_1 t} + \dfrac{B}{\mu_1}(1 - e^{-\mu_1 t})$ is the solution of the problem

$$\begin{aligned} \frac{dS_1}{dt} &= B - \mu_1 S_1, \\ S_1(0) &= \|\psi\|_\infty. \end{aligned} \tag{3.5}$$

Since $S_1(t) \leq \max\{\dfrac{B}{\mu_1}, \|\psi_1\|_\infty\}$ for $t \in [0, \infty)$, we have that

$$S(x,t) \leq \max\{\frac{B}{\mu_1}, \|\psi_1\|_\infty\}, \forall (x,t) \in \bar{\Omega} \times [0, T_{max}). \tag{3.6}$$

From Theorem 2.1 given by Alikakos in [1], to establish the L^∞ uniform boundedness of $I(x,t)$, it is sufficient to show the L^1 uniform boundedness of $I(x,t)$.
Since

$$\frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0$$

and

$$\frac{\partial}{\partial t}(S+I) - \Delta(d_S S + d_I I) \leq B - \mu_1(S+I),$$

we get

$$\frac{\partial}{\partial t}\Big(\int_\Omega (S+I)dx\Big) \leq mes(\Omega)B - \mu_1\Big(\int_\Omega (S+I)dx\Big). \tag{3.7}$$

Hence,

$$\int_\Omega (S+I)dx \leq mes(\Omega)\max\{\frac{B}{\mu_1}, \|\psi_1+\psi_2\|_\infty\}, \tag{3.8}$$

which implies that

$$\sup_{t\geq 0}\int_\Omega I(x,t)dx \leq K = mes(\Omega)\max\{\frac{B}{\mu_1}, \|\psi_1+\psi_2\|_\infty\}.$$

Using the Theorem 3.1 of [1], we deduce that there exists a positive constant K^* that depends on K and $\|\psi_1 + \psi_2\|_\infty$ such that

$$\sup \|I(.,t)\|_\infty \leq K^*. \tag{3.9}$$

From the above, we have proved that $S(x,t)$ and $I(x,t)$ are L^∞ bounded on $\bar{\Omega}\times[0,T_{\max})$.
Therefore, it follows from the standard theory for semilinear parabolic systems(see [4]) that $T_{\max} = +\infty$. This completes the proof of the proposition. ■

4. Qualitative analysis of the spacial model

Considering (1.1), we get that the basic reproduction number of disease in the absence of spatial dependence is given by

$$R_0 = \frac{f_2(S^0,0)}{\mu_2+\gamma}. \tag{4.1}$$

This describes the average number of secondary infections produced by a single infectious individual during the entire infectious period. It is not hard to show that the system (2.2) is always a disease-free equlibrium of the form $E_0 = (\frac{B}{\mu_1}, 0)$ when $R_0 \leq 1$. Further, if $R_0 > 1$, the system (2.2) has an endemic stationary state $E^* = (S^*, I^*)$.

4.1. Local stability of the equilibria

The purpose of this part is to determine the local stability for reaction-diffusion equations (2.2)-(2.4) by applying the method of Hattaf presented in [5].
First, we linearize the dynamical system (2.2) around arbitrary spatially homogeneous fixed point $\bar{E}(\bar{S},\bar{I})$ for a small space and time dependent fluctuations and expand them in Fourier space. For this, let

$$\begin{aligned} S(\overrightarrow{x},t) &\sim \bar{S}e^{\lambda t}e^{i\overrightarrow{k}.\overrightarrow{x}}, \\ I(\overrightarrow{x},t) &\sim \bar{I}e^{\lambda t}e^{i\overrightarrow{k}.\overrightarrow{x}}, \end{aligned} \tag{4.2}$$

where $\overrightarrow{x} = (x, y) \in \mathbb{R}^2$ and $\overrightarrow{k}.\overrightarrow{k} :=< \overrightarrow{k}, \overrightarrow{k} >:= k^2$. $\overrightarrow{k}$ and λ are the wavenumber vector and frequency, respectively. Then, we can obtain the corresponding characteristic equation as follows:

$$\det(J - k^2 D - \lambda I_2) = 0, \tag{4.3}$$

where I_2 is the identity matrix, $D = \text{diag}(d_S, d_I)$ is the diffusion matrix, and J is the Jacobian matrix of (2.2) without diffusion ($d_S = d_I = 0$) at $\bar{E}$ which is given by

$$J = \begin{pmatrix} -\mu_1 - f_1(\bar{S}, \bar{I}) & -f_2(\bar{S}, \bar{I}) \\ f_1(\bar{S}, \bar{I}) & f_2(\bar{S}, \bar{I}) - (\mu_2 + \gamma) \end{pmatrix} \tag{4.4}$$

The characterization of the local stability of disease-free equilibrium E_0 is given by the following result.

Theorem 4.1. *The disease-free equilibrium E_0 is locally asymptotically stable if $R_0 \leq 1$ and it is unstable if $R_0 > 1$.*

Proof. Evaluating (4.3) at E_0, we have the following equation,

$$\begin{aligned} &(-\mu_1 - f_1(S^0, 0) - k^2 d_S - \lambda)(f_2(S^0, 0) - (\mu_2 + \gamma) - k^2 d_I - \lambda) \\ &+ f_1(S^0, 0) f_2(S^0, 0) = 0. \end{aligned} \tag{4.5}$$

By developping (4.5), we get

$$\begin{aligned} &\lambda^2 + (\mu_1 + f_1(S^0, 0) - f_2(S^0, 0) + \mu_2 + \gamma + k^2 d_S + k^2 d_I)\lambda - \mu_1 f_2(S^0, 0) \\ &+ \mu_1(\mu_2 + \gamma) + f_1(S^0, 0)(\mu_2 + \gamma) + \mu_1 k^2 d_I + f_1(S^0, 0) k^2 d_I + k^4 d_I d_S \\ &+ k^2 d_S(\mu_2 + \gamma - f_2(S^0, 0)) = 0. \end{aligned} \tag{4.6}$$

Since $R_0 \leq 1$, we obtain

$$-f_2(S^0, 0) + \mu_2 + \gamma > 0.$$

Therefore, by the Routh-Hurwitz criterion all the roots of equation (4.5) have a negative real parts. This shows that equilibrium E_0 is locally asymptotically stable. ■

Next, we focus on the local stability of the endemic equilibrium E^*.

Theorem 4.2. *The endemic equilibrium E^* is locally asymptotically stable if $R_0 > 1$.*

Proof. Evaluating (4.3) at $E^*(S^*, I^*)$, we get

$$\begin{aligned} &\lambda^2 + (\mu_1 + f_1(S^*, I^*) - f_2(S^*, I^*) + \mu_2 + \gamma + k^2 d_S + k^2 d_I)\lambda - \mu_1 f_2(S^*, I^*) \\ &+ \mu_1(\mu_2 + \gamma) + f_1(S^*, I^*)(\mu_2 + \gamma) + \mu_1 k^2 d_I + f_1(S^*, I^*) k^2 d_I + k^4 d_I d_S \\ &+ k^2 d_S(\mu_2 + \gamma - f_2(S^*, I^*)) = 0. \end{aligned} \tag{4.7}$$

By using **H3**, we conclude that E^* is locally asymptotically stable. ■

4.2. Global stability of the equilibria

The purpose of this subsection is to determine the global stability for reaction-diffusion equations (2.2)-(2.4) by constructing Lyapunov functionals proposed in [2] and applying the method of Hattaf presented in [3].

Theorem 4.3. *If $R_0 \leq 1$, the disease-free equilibrium E_0 of (2.2)-(2.4) is globally asymptotically stable for all diffusion coefficients.*

We define

$$g(S,I)=\frac{f(S,I)}{I}.$$

Proof. Consider the following Lyapunov functional

$$V_1(t)=S(t)-S^0-\int_{S^0}^{S(t)}\frac{g(S^0,0)}{g(X,0)}dX+I,$$

where $S^0=\dfrac{B}{\mu_1}$. Calculating the time derivative of V along the positive solution of system (1.1), we get

$$\begin{aligned}\dot V_1(t)&=(1-\frac{g(S^0,0)}{g(S,0)})\dot S+\dot I\\&=(1-\frac{g(S^0,0)}{g(S,0)})(B-\mu_1 S)+\frac{g(S^0,0)}{g(S,0)})f(S,I)-(\mu_2+\gamma)I\\&=\mu_1 S^0(1-\frac{g(S^0,0)}{g(S,0)})(1-\frac{S}{S^0})+(\mu_2+\gamma)I\bigg(\frac{g(S^0,0)}{g(S,0)}g(S,I)\frac{1}{f_2(S^0,0)}R_0-1\bigg).\end{aligned}$$

By using $g(S,I)=\dfrac{f(S,I)}{I}$ which implies $g(S^0,0)=f_2(S^0,0)$, so we have

$$\dot V_1(t)\le\mu_1 S^0(1-\frac{g(S^0,0)}{g(S,0)})(1-\frac{S}{S^0})+(\mu_2+\gamma)I\bigg(\frac{g(S,I)}{g(S,0)}R_0-1\bigg).$$

By **H4**, we get

$$\dot V_1(t)\le\mu_1 S^0(1-\frac{g(S^0,0)}{g(S,0)})(1-\frac{S}{S^0})+(\mu_2+\gamma)I(R_0-1).$$

By using **H2** we obtain the following inequalities

$$1-\frac{g(S^0,0)}{g(S,0)}\ge 0\quad\text{for}\quad S\ge S^0,$$

$$1-\frac{g(S^0,0)}{g(S,0)}<0\quad\text{for}\quad S<S^0.$$

Thus, we have

$$(1-\frac{S}{S^0})(1-\frac{g(S^0,0)}{g(S,0)})\le 0,$$

then,

$$\frac{dV_1}{dt}\le 0$$

Now, From [3], we construct the Lyapunov functional for system (2.2) at E_0 as follows

$$W_1=\int_\Omega V_1(S(x,t),I(x,t))dx$$

Calculating the time derivative of W_1 along the solution of system (2.2)-(2.4), we have

$$\begin{aligned}\frac{dW_1}{dt} &= \int_\Omega \Big\{\mu_1 S^0(1-\frac{g(S^0,0)}{g(S,0)})(1-\frac{S}{S^0}) \\ &+ (\mu_2+\gamma)I\Big(\frac{g(S^0,0)}{g(S,0)}\frac{f(S,I)}{I}\frac{1}{f_2(S^0,0)}R_0-1\Big)\Big\}dx \\ &- d_S g(S^0,0)\int_\Omega \frac{g_1(S,0)}{(g(S,0))^2} \mid \nabla S\mid^2 dx \\ &\leq \int_\Omega \Big\{\mu_1 S^0(1-\frac{g(S^0,0)}{g(S,0)})(1-\frac{S}{S^0}) + (\mu_2+\gamma)I(R_0-1)\Big\}dx \\ &- d_S g(S^0,0)\int_\Omega \frac{g_1(S,0)}{(g(S,0))^2} \mid \nabla S\mid^2 dx.\end{aligned} \tag{4.8}$$

Since $R_0 \leq 1$, we have

$$\frac{dW_1}{dt} \leq 0.$$

Thus, the disease-free equilibrium E_0 is stable, and

$$\frac{dW_1}{dt} = 0,$$

if and only if $S = S_0$ and $I(R_0-1) = 0$. ■

Theorem 4.4. *If $R_0 > 1$, the endemic equilibrium E^* of (2.2)-(2.4) is globally asymptotically stable for all diffusion coefficients.*

Proof. Consider the following Lyapunov functional

$$V_2(t) = S(t) - S^* - \int_{S^*}^{S(t)} \frac{g(S^*,I^*)}{g(X,I^*)}dX + I^*\Phi(\frac{I(t)}{I^*}),$$

where $\Phi(x) = x - 1 - \ln x$, $x \in \mathbb{R}_+^*$. Obviously, $\Phi : \mathbb{R}_+^* \longrightarrow \mathbb{R}^+$ attains its global minimum at $x = 1$ and $\Phi(1) = 0$.

The function $\psi : x \longmapsto x - x^* - \int_{x^*}^{x} \frac{g(x^*,I^*)}{g(X,I^*)}dX$ has the global minimum at $x = x^*$ and $\psi(x^*) = 0$. Then, $\psi(x) \geq 0$ for any $x > 0$.

Hence, $V_2(t) \geq 0$ with equality holding if and only if $\frac{S(t)}{S^*} = \frac{I(t)}{I^*} = 1$ for all $t \geq 0$.

Finding the time derivative of $V_2(t)$ along the positive of system (2.1) gives

$$\begin{aligned}\frac{dV_2}{dt} &= (1-\frac{g(S^*,I^*)}{g(S,I^*)})\dot{S} + (1-\frac{I^*}{I})\dot{I} \\ &= (1-\frac{g(S^*,I^*)}{g(S,I^*)})(B-\mu_1 S - f(S,I)) + (1-\frac{I^*}{I})(f(S,I)-(\mu_2+\gamma)I)\end{aligned}$$

Note that $B = \mu_1 S^* + f(S^*,I^*)$ and $f(S^*,I^*) = (\mu_2+\gamma)I^*$.

Hence;

$$\begin{aligned}\frac{dV_2}{dt} &= \mu_1 S^*(1-\frac{g(S^*,I^*)}{g(S,I^*)})(1-\frac{S}{S^*}) \\ &+ f(S^*,I^*)\Big[(1-\frac{f(S^*,I^*)}{f(S,I^*)})((1-\frac{f(S,I)}{f(S^*,I^*)})+(1-\frac{I^*}{I})(\frac{f(S,I)}{f(S^*,I^*)}-\frac{I}{I^*})\Big] \\ &= \mu_1 S^*(1-\frac{S}{S^*})(1-\frac{g(S^*,I^*)}{g(S,I^*)})+f(S^*,I^*)\Big[2-\frac{g(S^*,I^*)}{g(S,I^*)}+\frac{I}{I^*}\frac{g(S,I)}{g(S,I^*)} \\ &- \frac{I}{I^*}-\frac{g(S,I)}{g(S^*,I^*)}\Big] \\ &= \mu_1 S^*(1-\frac{S}{S^*})(1-\frac{g(S^*,I^*)}{g(S,I^*)})+f(S^*,I^*)\Big[3-\frac{g(S^*,I^*)}{g(S,I^*)}-\frac{g(S,I)}{g(S^*,I^*)} \\ &- \frac{g(S,I^*)}{g(S,I)}+\Big(\frac{I}{I^*}\times\frac{g(S,I)}{g(S,I^*)}-\frac{I}{I^*}-1+\frac{g(S,I^*)}{g(S,I)}\Big)\Big]\end{aligned}$$

By adding and substracting $\ln\frac{g(S^*,I^*)}{g(S,I^*)}+\ln\frac{g(S,I)}{g(S^*,I^*)}+\ln\frac{g(S,I^*)}{g(S,I)}$,
we get,

$$\begin{aligned}\frac{dV_2}{dt} &= \mu_1 S^*(1-\frac{S}{S^*})(1-\frac{g(S^*,I^*)}{g(S,I^*)}) \\ &+ f(S^*,I^*)\Big[-\Phi\Big(\frac{g(S^*,I^*)}{g(S,I^*)}\Big)-\Phi\Big(\frac{g(S,I)}{g(S^*,I^*)}\Big)-\Phi\Big(\frac{g(S,I^*)}{g(S,I)}\Big) \\ &+ \Big(\frac{I}{I^*}\times\frac{g(S,I)}{g(S,I^*)}-\frac{I}{I^*}-1+\frac{g(S,I^*)}{g(S,I)}\Big)\Big] \\ &= \mu_1 S^*(1-\frac{S}{S^*})(1-\frac{g(S^*,I^*)}{g(S,I^*)}) \\ &+ f(S^*,I^*)\Big[-\Phi\Big(\frac{g(S^*,I^*)}{g(S,I^*)}\Big)-\Phi\Big(\frac{g(S,I)}{g(S^*,I^*)}\Big)-\Phi\Big(\frac{g(S,I^*)}{g(S,I)}\Big) \\ &+ \frac{1}{I^*g(S,I^*)g(S,I)}\Big(g(S,I)-g(S,I^*)\Big)\Big(g(S,I)I-g(S,I^*)I^*\Big)\Big]\end{aligned}$$

By using **H2**, we have the following trivial inequalities

$$1-\frac{g(S^*,I^*)}{g(S,I^*)}\geq 0 \quad \text{for} \quad S\geq S^*,$$

$$1-\frac{g(S^*,I^*)}{g(S,I^*)}< 0 \quad \text{for} \quad S< S^*.$$

Thus, we have

$$(1-\frac{S}{S^*})(1-\frac{g(S^*,I^*)}{g(S,I^*)})\leq 0$$

By using **H4**, we have $g(S,I)$ is monotonically decreasing for I and by **H3** $g(S,I)I$ is monotocally increasing for I, and so

$$\Big(g(S,I)-g(S,I^*)\Big)\Big(g(S,I)I-g(S,I^*)I^*\Big)\leq 0.$$

On the other hand, the function Φ is always positive.

Now, we construct the Lyapunov functional for system (2.2) at E^* as follows

$$W_2 = \int_\Omega V_2(S(x,t), I(x,t))dx$$

Calculating the time derivative of W_2 along the solution of system (2.2)-(2.4), we have

$$\begin{aligned}\frac{dW_2}{dt} &= \int_\Omega \Big\{\mu_1 S^*(1-\frac{S}{S^*})(1-\frac{g(S^*,I^*)}{g(S,I^*)}) \\ &+ f(S^*,I^*)\Big[-\Phi\Big(\frac{g(S^*,I^*)}{g(S,I^*)}\Big) - \Phi\Big(\frac{g(S,I)}{g(S^*,I^*)}\Big) - \Phi\Big(\frac{g(S,I^*)}{g(S,I)}\Big) \\ &+ \frac{1}{I^* g(S,I^*) g(S,I)}\Big(g(S,I)-g(S,I^*)\Big)\Big(g(S,I)I - g(S,I^*)I^*\Big)\Big]\Big\}dx \\ &- d_S g(S^*,I^*)\int_\Omega \frac{g_1(S,I^*)}{(g(S,I^*))^2}|\nabla S|^2 dx\end{aligned}$$

Since the function Φ is monotone on each side of 1 and is minimized at 1 and
$\Big(g(S,I)-g(S,I^*)\Big)\Big(g(S,I)I - g(S,I^*)I^*\Big) \leq 0,$
then

$$\frac{dW_2}{dt} \leq 0.$$

Thus, the endemic equilibrium E^* is globally asymptotically stable. ■

5. Numerical simulations

In this section, we present the numerical simulations to illustrate our theoretical results. To simplify, we consider system (2.2) under Neumann boundary conditions.

$$\frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, \quad t > 0, \quad x = 0, 1, \tag{5.1}$$

and initial conditions

$$S(x,0) = \begin{cases} 1.1x, & 0 \leq x < 0.5, \\ 1.1(1-x), 0,5 \leq x \leq 1, \end{cases} \qquad I(x,0) = \begin{cases} 0.5x, & 0 \leq x < 0.5, \\ 0.5(1-x), 0,5 \leq x \leq 1, \end{cases} \tag{5.2}$$

We choose the following data set of system (2.2): $d_S = 0.1,\ \ \gamma = 0.5,\ \ \mu_1 = 0.1,\ \ \beta = 0.2,\ \ d_I = 0.1,\ \ \mu_2 = 0.6,\ \ B = 0.5$. By calculation, we have $R_0 < 1$. In this case, system (2.2) has a disease-free equilibrium E_0. Hence, by Theorem 3.3, E_0 is globally asymptotically stable. Numerical simulation illustrates our result (see Figure 1).

In Figure 2, we choose $\beta = 0.8$ and do not change the other parameters values. By calculation, we have $R_0 > 1$ which satisfy Theorem 3.4; the system (2.2) has a unique endemic equilibrium E^*. Therefore, by Theorem 3.3, E^* is globally assymtotically stable. Numerical simulation illustrates well this result (see Figure 2).

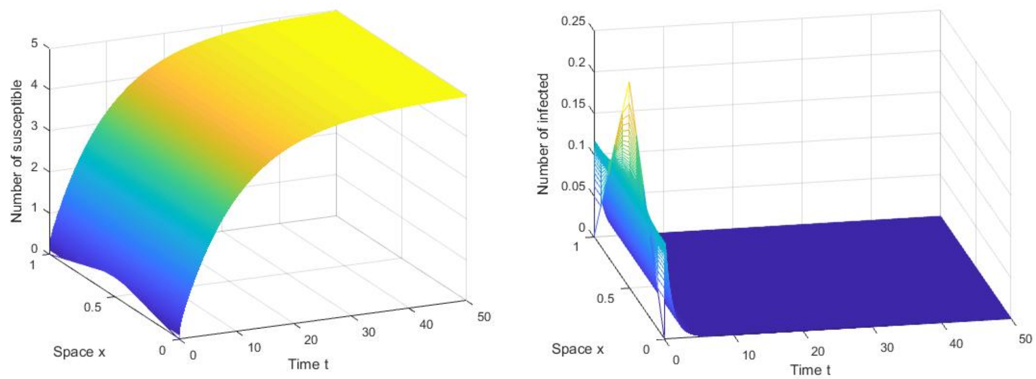

Figure 1: The temporal solution found by numerical of problem (2.2) with the Neumann boundary conditions (5.1) and initial conditions (5.2) when $R_0 \leq 1$.

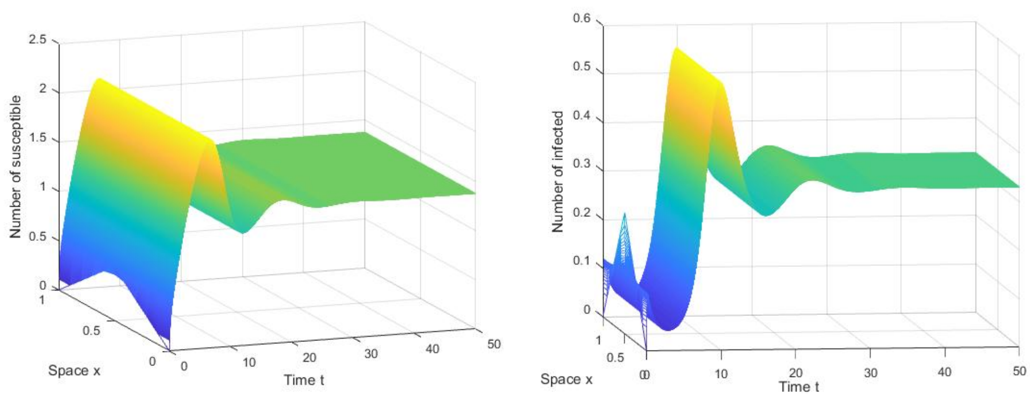

Figure 2: The temporal solution found by numerical of problem (2.2) with the Neumann boundary conditions (5.1) and initial conditions (5.2) when $R_0 > 1$.

6. Conclusion

In this paper, we investigated the dynamics of a reaction-diffusion epidemic model with general incidence function. The global dynamics of the model are completely determined by the basic reproduction number R_0. We proved that the disease-free equilibrium is globally asymptotically stable if $R_0 \leq 1$, which leads to the eradication of disease from population. When $R_0 > 1$ then disease-free equilibrium becomes unstable and a unique endemic equilibrium exists and is globally asymptotically stable, which means that the disease persists in the population.
From our theoretical and numerical results, we conclude that the spatial diffusion has no effect on the stability behavior of equilibria in the case of Neumann conditions and spatially constant coefficients.

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On pseudo valuation and pseudo almost valuation semidomains

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Abstract. In this paper, we characterize pseudo valuation semidomains and discuss some conditions which force a semidomain to be a pseudo valuation semidomain. Also, the notion of pseudo almost valuation semidomains is introduced and some results regarding pseudo almost valuation semidomains are investigated.

AMS Subject Classifications: 16Y60, 16Y99.

Keywords: Semidomains, prime ideals, pseudo valuation semidomains, pseudo almost valuation semidomains

Contents

1. Introduction

Several algebraists have done immense work in the field of valuation rings. Pseudo valuation domains are the rings which are closely related to valuation domains. In [12], Hedstrom introduced the notion of pseudo valuation domains and further studied by Anderson [2] and Badawi [3]. Semirings [8] are the generalization of elaborately studied algebraic structures such as rings, bounded distributive lattices and have significant applications in computer science, engineering and optimization theory (cf. [11, 13, 14]). The brief structure of semirings have been studied by various researchers (cf. [5–10, 16, 18]). The algebraic structure pseudo valuation semidomain is a generalization of the pseudo valuation domain.

In this paper, by a semiring S, we mean a nonempty set S on which operations of addition and multiplication have been defined such that the following conditions are satisfied:
(i) $(S,+)$ is a commutative monoid with identity element 0;
(ii) $(S,.)$ is a monoid with identity element 1;
(iii) Multiplication distributes over addition from either side;
(iv) $0s=0=s0$, for all $s\in S$;
(v) $1\neq 0$.

Moreover, a commutative semiring S is said to be a semidomain if S is multiplicatively cancellative semiring i.e., $xy=xz$ implies $y=z$ for all $x,y,z\in S$ with $x\neq 0$. Using techniques adapted from the ring theory, it is easy to show that a semidomain S can be embedded into a semifield, known as the semifield of fractions, denoted

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by $F(S)$ (see p. 22 in [7]). For instance, we denote every element of $F(S)$ by a/b, where $a \in S$ and $b \in S\backslash\{0\}$. Throughout this article, S represents a semidomain with semifield of fractions $F(S)$.

A semidomain S is a valuation semidomain if and only if its lattice of ideals is a chain [17, Theorem 2.4]. A proper ideal $\wp$ of a semiring S is said to be a prime ideal, if $xy \in \wp$ implies either $x \in \wp$ or $y \in \wp$. An ideal N of a semiring S is a maximal ideal if it is maximal among all proper ideals of S. If x and y are elements of a semiring S, then y divide x, written as $y|x$, if there exists $a \in S$ such that $x = ay$. This is equivalent to $(x) \subseteq (y)$ and for more on the ideals of semiring, one can refer to (cf. [15, 16, 19, 20]). A prime ideal $\wp$ of a semidomain S is called strongly prime if $x, y \in F(S)$ and $xy \in \wp$ infers that $x \in \wp$ or $y \in \wp$. If every prime ideal of a semidomain S is a strongly prime, then S is called a pseudo valuation semidomain. However, an oversemiring of a semidomain S is a semiring between S and $F(S)$. If I is an ideal of S, then we define the set $(I : I) = \{x \in F(S)|xI \subseteq I\}$. One can easily verify that $(I : I)$ is an oversemiring of S and I is an ideal of $(I : I)$.

The radical of an ideal I of a semiring S, denoted by $\sqrt{I}$, is the intersection of all prime ideals of S containing I. Furthermore, a semiring S is said to be quasi-local if it has only one maximal ideal. It was shown in Nasehpour [17, Proposition 3.5] that a semiring S is quasi-local if and only if $S\backslash U(S)$ is an ideal of S, where $U(S)$ is the set of all unit elements of S. In this paper, we prove some results of pseudo valuation semidomains and introduce a new closely related class of semidomains called pseudo almost valuation semidomains. The main objective of this paper is to obtain results analogous to Hedstrom's and Badawi's results.

The second section is devoted to the characterization of pseudo valuation semidomains. Furthermore, several comparable conditions which allow a semidomain to be a pseudo valuation semidomain are discussed. Also, the relationship between valuation, pseudo valuation and quasi-local semidomains are derived. In section 3, we introduce and characterize the notion of pseudo almost valuation and almost valuation semidomains. In addition, we study some properties of pseudo almost valuation semidomains by establishing the connection between pseudo almost valuation, almost valuation and quasi-local semidomains.

2. Properties of Pseudo valuation semidomains

In this section, we characterize the pseudo valuation semidomains. We start this section with the following examples of pseudo valuation semidomain.

Example 2.1. *Let $S = \{0, 1\}$, we define operations with the help of the following tables:*

$\oplus$	0	1
0	0	1
1	1	1

$\odot$	0	1
0	0	0
1	0	1

One can easily see that S is a semidomain with only one prime ideal $I = \{0\}$, which is a strongly prime ideal. Therefore, S is a pseudo valuation semidomain.

Example 2.2. *Consider $S = \{0, 1, 2, ..., p-1\}$, where p is a prime number with binary operations addition $\oplus$ and multiplication $\odot$ as follows:*

$$\left\{ \begin{array}{c} a \oplus b = a + b \text{ if } a + b \leq p-1, \text{ otherwise, } a \oplus b \equiv a + b (mod p); \\ a \odot b = ab \text{ if } ab \leq p-1, \text{ otherwise, } a \odot b \equiv ab (mod p), \end{array} \right\}$$

for all $a, b \in S$. Clearly, S is a pseudo valuation semidomain.

The proofs of the next two lemmas are quite easy, so we omit the proofs.

Lemma 2.3. *Let S be a semidomain. Then the following statements hold:*
(i) If $\wp$ is a strongly prime ideal of S and I is an ideal of S, then $\wp$ and I are comparable.
(ii) If S is a pseudo valuation semidomain, then S is quasi-local.

Lemma 2.4. *If $\wp$ is a prime ideal of S, then $\wp$ is a strongly prime ideal if and only if for every $a \in F(S)\backslash S$, $a^{-1}\wp \subseteq \wp$.*

Theorem 2.5. *A semidomain S is a pseudo valuation semidomain if and only if for a maximal ideal N of S, $a^{-1}N \subseteq N$, for every $a \in F(S)\backslash S$.*

Proof. Let S be a pseudo valuation semidomain with a maximal ideal N. Then by definition of S, N is a strongly prime ideal and by the above lemma, $a^{-1}N \subseteq N$, for every $a \in F(S)\backslash S$.

Conversely, assume that N is a maximal ideal of S such that for every $a \in F(S)\backslash S$, $a^{-1}N \subseteq N$. First, we will show that S is quasi-local. For this, let $N^{'}$ be a maximal ideal of S such that $N \neq N^{'}$. By using Lemma 2.3 and Lemma 2.4, we get that N and $N^{'}$ are comparable, which is a contradiction. Thus S is quasi-local with maximal ideal N. Now, let $\wp$ be any prime ideal of S. To show S is a pseudo valuation semidomain, it suffices to prove that $\wp$ is a strongly prime ideal of S. Again by using Lemma 2.3, $\wp$ and N are comparable. Since N is a maximal ideal, so $\wp \subseteq N$. Further, by using the hypothesis, we have $a^{-1}\wp \subseteq N$, for every $a \in F(S)\backslash S$. For $p \in \wp$, we obtain $a^{-1}pa^{-1} \in N$ which infers that $(a^{-1}p)(a^{-1}p) \in \wp$. This leads to $a^{-1}p \in \wp$, since $\wp$ is a prime ideal. By Lemma 2.4, $\wp$ is a strongly prime ideal of S. Hence, S is a pseudo valuation semidomain. ■

In light of the proof of Theorem 2.5, we have the following corollary.

Corollary 2.6. *Let $\wp$ be a strongly prime ideal of S. If U is a prime ideal of S and $U \subseteq \wp$, then U is a strongly prime ideal of S.*

Proposition 2.7. *If N is a maximal ideal of S such that xN and N are comparable for each $x \in F(S)$, then S is quasi-local.*

Proof. Let N be a maximal ideal of S such that xN and N are comparable for each $x \in F(S)$. Suppose that $N^{'}$ is a maximal ideal of S with $N^{'} \neq N$. Choose $a \in N\backslash N^{'}$ and $b \in N^{'}\backslash N$. By the hypothesis, either $ab^{-1}N \subseteq N$ or $N \subseteq ab^{-1}N$. If $ab^{-1}N \subseteq N$, then $aN \subseteq bN \subseteq N^{'}$. As $N^{'}$ is a prime ideal and $a \in N$, so $a^2 \in aN \subseteq N^{'}$ implies $a \in N^{'}$, which is not possible. Now, if $N \subseteq ab^{-1}N$, then $a^{-1}bN \subseteq N$. As $a \in N$, so $b = a^{-1}ba \in N$, which is a contradiction. Hence S is quasi-local. ■

By using the above result, one can easily prove

Proposition 2.8. *If N is a maximal ideal of S, then the following statements are equivalent:*
(i) aN and N are comparable, whenever $a \in F(S)$.
(ii) S is quasi-local and for each $a \in F(S)$, either $aN \subseteq S$ or $N \subseteq aS$.

Lemma 2.9. *If N is a maximal ideal of S such that for each pair I_1, I_2 of ideals of S, either $I_1 \subseteq I_2$ or $I_2N \subseteq I_1$, then S is quasi-local.*

Proof. Let N be a maximal ideal of S such that for each pair I_1, I_2 of ideals of S, either $I_1 \subseteq I_2$ or $I_2N \subseteq I_1$. Assume that M is a maximal ideal of S with $M \neq N$. By using the given hypothesis, for $I_1 = M$ and $I_2 = N$, either $M \subseteq N$ or $N^2 \subseteq M$. Since M and N are two distinct maximal ideals of S, so $M \subseteq N$, a contradiction. But $N^2 \subseteq M$ implies $N \subseteq M$, which is again a contradiction. Hence, S is quasi-local. ■

Theorem 2.10. *Let N be a maximal ideal of S. Then S is a pseudo valuation semidomain if and only if for each pair I_1, I_2 of ideals of S, either $I_1 \subseteq I_2$ or $I_2N \subseteq I_1$.*

Proof. Let S be a pseudo valuation semidomain and I_1, I_2 be any pair of ideals of S. Since S is a pseudo valuation semidomain, then N is a strongly prime ideal. By Lemma 2.3, I_1 and I_2 are comparable to N. Further, let $I_1 \nsubseteq I_2$ and $a \in I_1\backslash I_2$. Then for each $b \in I_2$, we get $a/b \notin S$. By using Lemma 2.4, we have $(a/b)^{-1}N \subseteq N$, which infers that $bN \subseteq aN \subseteq I_1N$. Thus, $NI_2 \subseteq NI_1 \subseteq I_1$. Conversely, assume that for each pair I_1, I_2 of ideals of S, either $I_1 \subseteq I_2$ or $I_2N \subseteq I_1$. By the above lemma, S is quasi-local with maximal ideal N. We have to show that for every $x \in F(S)\backslash S$, $x^{-1}N \subseteq N$. Now, let $a, b \in S$ with $a/b \notin S$. Then $(a) \nsubseteq (b)$. As $(a) \nsubseteq (b)$, then by the hypothesis $N(b) \subseteq (a)$. Further, $Nb \subseteq (a)$ leads to $Nba^{-1} \subseteq S$. If $N(b/a) = S$, then $N = S(a/b)$ and $a/b \in N \subset S$, which is a contradiction. So $Nba^{-1} \subseteq N$. Therefore, $x^{-1}N \subseteq N$, whenever $x \in F(S)\backslash S$ and by Theorem 2.5, S is a pseudo valuation semidomain. ■

In the forthcoming result, we prove that a semidomain S is a pseudo valuation semidomain if and only if for each nonunit $a \in S$, we have $b^{-1}a \in S$ for all $b \in F(S)\backslash S$.

Theorem 2.11. *A semidomain S is a pseudo valuation semidomain if and only if for each nonunit $a \in S$, we have $b^{-1}a \in S$ for all $b \in F(S)\backslash S$.*

Proof. Let S be a pseudo valuation semidomain and a be a nonunit of S. By Lemma 2.3, the semidomain S is quasi-local. As a is a nonunit, so a belongs to the maximal ideal, say N of S. Further, by using Lemma 2.4, we conclude that $b^{-1}a \in N \subset S$, for each $b \in F(S)\backslash S$. Thus, $b^{-1}a \in S$ for each nonunit $a \in S$ and $b \in F(S)\backslash S$.

Conversely, assume that $b^{-1}a \in S$ for each nonunit $a \in S$ and $b \in F(S)\backslash S$. It suffices to prove that each prime ideal $\wp$ of S is a strongly prime ideal. Let $x, y \in F(S)$ such that $xy \in \wp$. If $x, y \in S$, then $x \in \wp$ or $y \in \wp$. Suppose that $x \in F(S)\backslash S$, then by the hypothesis $y = x^{-1}xy \in S$. Now, we have to show that y is a nonunit of S. If possible, let y be a unit of S. Then $x = xyy^{-1} \in \wp$, which is a contradiction. Thus, y is a nonunit and $y^2 = x^{-1}yxy \in \wp$. Since $\wp$ is a prime ideal of S, so $y \in \wp$. Therefore, $\wp$ is a strongly prime ideal and hence S is a pseudo valuation semidomain. ■

We need the following lemma for the proof of subsequent results.

Lemma 2.12. *[17] For a semidomain S, the following statements are equivalent:*
(i) *S is a valuation semiring.*
(ii) *For any element $a \in F(S)$, either $a \in S$ or $a^{-1} \in S$.*
(iii) *For any ideals I, J of S, either $I \subseteq J$ or $J \subseteq I$.*
(iv) *For any elements $a, b \in S$, either $(a) \subseteq (b)$ or $(b) \subseteq (a)$.*

Proposition 2.13. *If S is a pseudo valuation semidomain with a nonzero principal prime ideal, then S is a valuation semidomain.*

Proof. Let $\wp = (p)$ be a nonzero principal prime ideal generated by some prime p of S. Assume that $\wp$ is nonmaximal, then there exists a nonunit element $a \in S\backslash\wp$. By Lemma 2.3, we have $\wp \subset (a)$. In particular, $p \in (a)$, which is not true, as $\wp$ is prime and $a \notin \wp$, a is a nonunit element of S. This concludes that $\wp$ is a maximal ideal of S. Let a, b be nonunits in S and $ab^{-1} \in F(S)\backslash S$. By Lemma 2.4, we have $ba^{-1}p \in \wp$ and $ba^{-1} = s$, for some $s \in S$. Therefore, $b = as$ gives that $(b) \subseteq (a)$. Hence, by Lemma 2.12, S is a valuation semidomain. ■

By using Lemma 2.12, one can easily prove

Proposition 2.14. *Every valuation semidomain is a pseudo valuation semidomain.*

Now, we will close this section with

Theorem 2.15. *Let S be a quasi-local semidomain with a maximal ideal N. Then the following statements are equivalent:*
(i) *S is a pseudo valuation semidomain.*
(ii) *$(N : N)$ is a valuation semidomain with maximal ideal N.*

Proof. $(i) \Rightarrow (ii)$ Suppose that S is a pseudo valuation semidomain and $a \in F(S)\backslash S$. Then, by Lemma 2.4, we have $a^{-1}N \subseteq N$. So $a^{-1} \in (N : N)$ and Lemma 2.12 infers that $(N : N)$ is a valuation semidomain. Now, N is an ideal of $(N : N)$, so it is sufficient to prove that N is a maximal ideal of $(N : N)$. For this, let $a \in (N : N)$ be a nonunit element. Suppose that $a \notin N$. Then, $a \notin S$. Therefore, by Lemma 2.4, we have $a^{-1}N \subseteq N$. This implies $a^{-1} \in (N : N)$, a contradiction. Thus, if a is a nonunit of $(N : N)$, then $a \in N$. Hence, N is a maximal ideal of $(N : N)$.

$(ii) \Rightarrow (i)$ Suppose that $(N : N)$ is a valuation semidomain with a maximal ideal N. Let $\wp$ be any prime ideal of S and $p \in \wp, a \in (N : N)$. As S is quasi-local with maximal ideal N and $p \in \wp$, so $p \in N$. Thus,

$ap \in N$ and $(ap)(ap) \in \wp$, as $apa \in N$. From $(ap)(ap) \in \wp$, we get $ap \in \wp$. Hence, $\wp$ is an ideal of $(N : N)$. To show that $\wp$ is a prime ideal of $(N : N)$, let $ab \in \wp$ with $a, b \in (N : N)$. If $a, b \in S$, then either $a \in \wp$ or $b \in \wp$. Suppose that $a \notin S$, so $a \notin N$ and $a^{-1} \in (N : N)$. As $\wp$ is an ideal of $(N : N)$, we have $b = a^{-1}ab \in \wp$. Thus, $\wp$ is a prime ideal of $(N : N)$. By Proposition 2.14, $(N : N)$ is a pseudo valuation semidomain. So $\wp$ is a strongly prime ideal. Therefore, every prime ideal of S is strongly prime and hence S is a pseudo valuation semidomain. ■

3. Pseudo almost valuation semidomains

In this section, we study pseudo almost valuation semidomains. Let us recall that a prime ideal $\wp$ of an integral domain R with quotient field K is called a pseudo strongly prime ideal in sense of Badawi [4] if, whenever $a, b \in K$ and $ab\wp \subseteq \wp$, then there is a positive integer $n \geq 1$ such that either $a^n \in R$ or $b^n\wp \subseteq \wp$. If every prime ideal of R is a pseudo strongly prime ideal, then R is called a pseudo almost valuation domain. This motivated us to give the definition of a pseudo almost valuation semidomain and its several characterizations. Throughout this section, $\ell(A)$ =$\{a \in F(S) \mid a^n \notin S$ for every $n \geq 1\}$, where $A \subseteq S$. We begin with

Definition 3.1. *A prime ideal $\wp$ of S is said to be a pseudo strongly prime if, whenever $a, b \in F(S)$ and $ab\wp \subseteq \wp$, then there is a positive integer $n \geqslant 1$ such that either $a^n \in S$ or $b^n\wp \subseteq \wp$. If every prime ideal of S is a pseudo strongly prime ideal, then S is a pseudo almost valuation semidomain.*

Proposition 3.2. *Every pseudo valuation semidonain is a pseudo almost valuation semidomain.*

Proof. Assume that S is a pseudo valuation semidomain and $\wp$ is any prime ideal of S. To show that S is a pseudo almost valuation semidomain, it suffices to show that $\wp$ is a pseudo strongly prime ideal. Let $a, b \in F(S)$ such that $ab\wp \subseteq \wp$. Suppose that $a \in \ell(S)$. Then, by Lemma 2.4, we have $a^{-n}(a^nb^n\wp) \subseteq \wp$ for every $n \geq 1$, since $a^nb^n\wp \subseteq \wp$ and $a^n \in F(S)\backslash S$. This concludes that $b^n\wp \subseteq \wp$. Therefore, $\wp$ is a pseudo strongly prime ideal of S and hence S is a pseudo almost valuation semidomain. ■

Lemma 3.3. *If $\wp$ is a prime ideal of S, then following statements are equivalent:*
(i) $\wp$ is a pseudo strongly prime ideal.
(ii) For every $a \in \ell(S)$, there is an $n \geq 1$ such that $a^{-n}\wp \subseteq \wp$.

Proof. $(i) \Rightarrow (ii)$ Assume that $\wp$ is a pseudo strongly prime ideal of S and $a \in \ell(S)$. Let $p \in \wp$, then $p = aa^{-1}p \in \wp$ which infers that $a^{-1}p \in \wp$. As $a^{-1}p \in a^{-1}\wp$, so $a^{-1}\wp \subseteq \wp$. Thus, there is an $n \geq 1$ such that $a^{-n}\wp \subseteq \wp$, since $a \in \ell(S)$.

$(ii) \Rightarrow (i)$ Assume that for every $a \in \ell(S)$, there is an $n \geq 1$ such that $a^{-n}\wp \subseteq \wp$. To see that $\wp$ is a pseudo strongly prime ideal, let $ab\wp \subseteq \wp$ with $a, b \in F(S)$. Assume that $a \in \ell(S)$, then by the hypothesis, there is an $n \geq 1$ such that $a^{-n}\wp \subseteq \wp$. As $ab\wp \subseteq \wp$, so $a^nb^n\wp \subseteq \wp$. This implies that $b^n\wp = a^{-n}(a^nb^n\wp) \subseteq \wp$, since $a^{-n}\wp \subseteq \wp$. Therefore, $\wp$ is a pseudo strongly prime ideal. ■

Proposition 3.4. *If $\wp_1$ and $\wp_2$ are pseudo strongly prime ideals of S, then $\wp_1$ and $\wp_2$ are comparable.*

Proof. Suppose that $\wp_1$ and $\wp_2$ are pseudo strongly prime ideals of S. Our claim is that $\wp_1$ and $\wp_2$ are comparable. If not, then there exists $a \in \wp_1\backslash\wp_2$ and $b \in \wp_2\backslash\wp_1$. As $a \notin \wp_2$, so $a/b \in \ell(S)$. By using Lemma 3.3, there is an $n \geq 1$ such that $(a/b)^{-n}\wp_1 \subseteq \wp_1$. Therefore, $(b)^n = (a/b)^{-n}a^n \in \wp_1$. As $\wp_1$ is prime, so $(b)^n \in \wp_1$ concludes that $b \in \wp_1$, which is a contradiction. Hence, $\wp_1$ and $\wp_2$ are comparable. ■

Next corollary is an immediate consequence of Proposition 3.4.

Corollary 3.5. *If S is a pseudo almost valuation semidomain, then the prime ideals of S are linearly ordered. In particular, S is quasi-local.*

Theorem 3.6. *Let N be a maximal ideal of S. Then S is a pseudo almost valuation semidomain if and only if N is a pseudo strongly prime ideal.*

Proof. Let S be a pseudo almost valuation semidomain. Then, N is a pseudo strongly prime ideal, as N is a maximal ideal of S. Conversely, let N be a pseudo strongly prime ideal of S. We first observe that S is quasi-local. If not, then there exists a maximal ideal $N^{'}$ of S such that $N \neq N^{'}$. By using Proposition 3.4, we get that N and $N^{'}$ are comparable, which is a contradiction. Therefore, S is quasi-local with maximal ideal N. We must show that each prime ideal $\wp$ of S is a pseudo strongly prime ideal. By Proposition 3.4, we have $\wp \subseteq N$. Assume that $a \in \ell(S)$. Then, by Lemma 3.3, there exists an $n \geq 1$ such that $a^{-n}\wp \subseteq N$. Now, $(a^{-n}p)a^{-n} \in N$ for $p \in \wp$. So $(a^{-n}p)(a^{-n}p) \in \wp$ which leads to $a^{-n}\wp \subseteq \wp$, as $\wp$ is a prime ideal. Further, by Lemma 3.3, we conclude that $\wp$ is a pseudo strongly prime ideal of S. Hence, S is a pseudo almost valuation semidomain. ■

Corollary 3.7. *Let $\wp_1$ be a pseudo strongly prime ideal of S and $\wp_2$ be a prime ideal of S such that $\wp_2 \subseteq \wp_1$. Then $\wp_2$ is a pseudo strongly prime ideal of S.*

Recall from [1] that an integral domain R is an almost valuation domain if for each pair $a, b \in R\backslash\{0\}$, there exists an integer $n \geq 1$(depending on a, b) with $a^n|b^n$ or $b^n|a^n$. Influenced by this concept, we give the following definition

Definition 3.8. *A semidomain S is said to be an almost valuation semidomain if for each pair $a, b \in S\backslash\{0\}$, there exists an integer $n \geq 1$ (depending on a, b) with $a^n|b^n$ or $b^n|a^n$.*

The upcoming proposition follows immediately by definition of almost valuation semidomain.

Proposition 3.9. *For a semidomain S, the following statements are equivalent:*
(i) S is an almost valuation semidomain.
(ii) For each $x \in F(S)\backslash\{0\}$, there exists an $n \geq 1$(depending on x) with x^n or $x^{-n} \in S$.

Lemma 3.10. *Every valuation semidomain is an almost valuation semidomain.*

Proof. Let S be a valuation semidomain. Then, by using Lemma 2.12, we get that for any element $a \in F(S)$, either $a \in S$ or $a^{-1} \in S$. Further, consider $a \neq 0$, we have $a \in S$ or $a^{-1} \in S$. Therefore, for $n \geq 1$, we conclude that $a^n \in S$ or $a^{-n} \in S$. By Proposition 3.9, S is an almost valuation semidomain. ■

Remark 3.11. *If I is a proper ideal of S and $D = (I : I)$, then $ID = I$.*

Theorem 3.12. *Let S be a quasi-local semidomain with maximal ideal N. Then S is a pseudo almost valuation semidomain if and only if $D = (N : N)$ is almost valuation semidomain with maximal ideal $\sqrt{ND}$ (the radical of ND in D).*

Proof. Let S be a pseudo almost valuation semidomain and let, $a \in \ell(D)$. Obviously, $a \in \ell(S)$. By Lemma 3.3, we get that there is an $n \geq 1$ such that $a^{-n}N \subseteq N$, since N is a pseudo strongly prime ideal and $a \in \ell(S)$. Therefore, $a^{-n} \in D$. By Proposition 3.9 and Remark 3.11, D is almost valuation semidomain and $ND = N$. Now, we have to show that $\sqrt{ND}$ is a maximal ideal of D. Let a be a nonunit element of D. If $a \notin \sqrt{ND}$, then $a \in \ell(S)$, as $ND = N$ and a is a nonunit of D. Further, by using Lemma 3.3, we conclude that there is an $n \geq 1$ such that $a^{-n}N \subseteq N$ which infers that $a^{-n} \in D$. This implies that a is a unit in D, a contradiction. Therefore, $a \in \sqrt{ND}$ and hence $\sqrt{ND}$ is a maximal ideal of D.

Conversely, let $D = (N : N)$ be an almost valuation semidomain with maximal ideal $\sqrt{ND}$. Let $a \in \ell(S)$, then $a \notin \sqrt{ND}$. If $a^n \in D$ for some $n \geq 1$, then a^n is unit of D which implies $a^{-n}N \subseteq N$. If $a \in \ell(D)$, then there is an $m \geq 1$ such that $a^{-m} \in D$, as D is almost valuation semidomain. Therefore, there is an $m \geq 1$ such that $a^{-m}N \subseteq N$. So, N is a pseudo strongly prime ideal. By Theorem 3.6, S is a pseudo almost valuation semidomain. ■

Proposition 3.13. *If S is a semidomain and for every $a, b \in S$, there is an $n \geq 1$ such that either $a^n | b^n$ or $b^n | a^n c$ for every nonunit $c \in S$, then the prime ideals of S are linearly ordered.*

Proof. Assume that S is a semidomain and for every $a, b \in S$, there is an $n \geq 1$ such that either $a^n | b^n$ or $b^n | a^n c$ for every nonunit $c \in S$. If a is a nonunit, then there is an $n \geq 1$ such that either $a^n | b^n$ or $b^n | a^{n+1}$. Let $\wp_1$ and $\wp_2$ be distinct prime ideals of S and $a \in \wp_1 \backslash \wp_2$. Therefore, for every $b \in \wp_2$, there is an $n \geq 1$ such that $a^n | b^n$ which leads to $b \in \wp_1$. So $\wp_2 \subseteq \wp_1$. Hence, the prime ideals of S are linearly ordered. ■

Theorem 3.14. *If S is a semidomain, then the following statements are equivalent:*
(i) S is a pseudo almost valuation semidomain.
(ii) For every $b \in \ell(S)$, there is an $n \geq 1$ such that $ab^{-n} \in S$ for every nonunit $a \in S$.

Proof. $(i) \Rightarrow (ii)$ Let S be a pseudo almost valuation semidomain and a be a nonunit of S. By Corollary 3.5, S is quasi-local. So a belongs to the maximal ideal, say N of S. Moreover, by Lemma 3.3, we get that for every $b \in \ell(S)$, there is an $n \geq 1$ such that $b^{-n}a \in N \subset S$. Therefore, for every $b \in \ell(S)$, there is an $n \geq 1$ such that $ab^{-n} \in S$ for every nonunit $a \in S$.

$(ii) \Rightarrow (i)$ Assume that for every $b \in \ell(S)$, there is an $n \geq 1$ such that $ab^{-n} \in S$ for every nonunit $a \in S$. To show that S is a pseudo almost valuation semidomain, we need only show that each prime ideal $\wp$ of S is a pseudo strongly prime ideal. Let $xy\wp \subseteq \wp$ with $x, y \in F(S)$. Assume that $x \in \ell(S)$. By using the hypothesis, there is an $n \geq 1$ such that $(x^n y^n p)x^{-n} = y^n p \in S$ for all $p \in \wp$, as $x^n y^n \wp \subseteq \wp$. Now, it suffices to show that $y^n p$ is a nonunit of S. If possible, let $y^n p$ be a unit element for $p \in \wp$. Then, $x^n = x^n y^n p (y^n p)^{-1} \in S$, which is a contradiction. Thus, $y^n p$ is a nonunit of S and $(y^n p)x^{-n} \in S$ for $p \in \wp$. Further, $(y^n p)^2 = (y^n p x^{-n})(x^n y^n p) \in \wp$ infers that $y^n p \in \wp$, as $\wp$ is a prime ideal. So $y^n \wp \subseteq \wp$. Therefore, $\wp$ is a pseudo strongly prime ideal. Hence, S is a pseudo almost valuation semidomain. ■

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On some fractional order differential equations with weighted conditions

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Abstract. In this paper, we study some Cauchy problems with weighted conditions of a fractional order differential equation . We study by using some fixed point Theorems the existence of at least one solution in the two spaces $C_{1-\kappa}(I)$ and $C(I)$, where $I = [0, \hbar]$.

AMS Subject Classifications: 26A33, 34A12, 47D09.

Keywords: Fractional derivatives and integrals, Existence of solution, Cauchy problem.

Contents

1. Introduction and Background

A Cauchy problem in mathematics asks for the solution of a partial differential equation that satisfies certain conditions that are given on a hypersurface in the domain. A Cauchy problem can be an initial value problem or a boundary value problem. Also, Cauchy problems are very natural in physics: The typical example is the solution of Newton's equation in classical mechanics, which is a second-order equation for the position of a particle. We know indeed that the motion of a particle is uniquely specified by its initial position and velocity.

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An important result about Cauchy problems for ordinary differential equations is the existence and uniqueness theorem, which states that, under mild assumptions, a Cauchy problem always admits a unique solution in a neighbourhood of the point x_0 where the initial conditions are given.

In this work, we study the two weighted Cauchy-type problems

$$\begin{cases} D^{\kappa}\, u(\varsigma) \;=\; f\Bigg(\varsigma, u, \int_0^{\varsigma} h(\varsigma, s, u(s))\, ds\Bigg), \;\; \varsigma \;>\; 0, \;\; 0 \;<\; \kappa \;<\; 1, \\ \\ \varsigma^{1-\kappa}\, u(\varsigma)|_{\varsigma=0} \;=\; b, \; b \;\in\; \Re \end{cases} \tag{1.1}$$

and

$$\begin{cases} D^{\kappa}\, u(\varsigma) \;=\; f\Bigg(\varsigma, u, \int_0^{\varsigma} h(\varsigma, s, u(s))\, ds\Bigg), \;\; \varsigma \;>\; 0, \;\; 0 \;<\; \kappa \;<\; 1, \\ \\ u(0) \;=\; 0. \end{cases} \tag{1.2}$$

The weighted Cauchy-type problems were studied in many papers see [1]-[7].

In [8], the author studied the existence of a solution of the weighted problem

$$\begin{cases} D^{\alpha}\, u(t) \;=\; f(t, u(t)) \;+\; \int_0^{t} \; g(t, s, u(s))\, ds, \;\; t \;>\; 0, \\ \\ t^{1-\alpha}\, u(t)|_{t=0} \;=\; b, \quad \text{where } 0 \;<\; \alpha \;<\; 1, \; b \;\in\; \Re, \end{cases} \tag{1.3}$$

in the space $C_{1-\alpha}(I)$, where the functions f and g satisfied the following conditions

(1) $t^{1-\alpha} f(t, u)$ is continuous on $R^{+} \times C^{0}_{1-\alpha}(\Re^{+})$ and

$$|f(t, u)| \leq t^{\mu} \varphi(t) |u|^{m_1}, \; \mu \geq 0, \; m_1 > 1,$$

(2) $s^{1-\alpha} g(t, s, u(s))$ is continuous on $D_{\Re^{+}} \times C^{0}_{1-\alpha}(\Re^{+})$ where

$$D_{\Re^{+}} = \{(t, s) \in \Re^{+} \times \Re^{+}, \; 0 \leq s \leq t\}$$

and

$$|g(t, s, u(s))| \leq (t-s)^{\beta-1} s^{\sigma} \psi(s) |u|^{m_2}, \; 0 < \beta < 1, \; \sigma \geq 0, \; m_2 > 1,$$

where $\varphi(t)$ and $\psi(t)$ are such that

(3) $\varphi(t)$ is continuous and $t^{\mu-(1-\alpha)m_1} \varphi(t)$ is continuous in case

$$\mu - (1-\alpha) m_1 < 0,$$

(4) $\psi(t)$ is continuous and $t^{\sigma-(1-\alpha)m_2} \psi(t)$ is continuous in case

$$\sigma - (1-\alpha) m_2 < 0.$$

Problem (1.3) is a special case of our problem (1.1), we will study the existence of at least one solution of problem (1.1) in the space $C_{1-\kappa}(I)$ under similar conditions of paper [8].

2. Preliminaries and Definitions

In this section, we state the definitions and theorems which will be used in our paper.

Let $L_1 = L_1[J]$ be the class of Lebesgue integrable functions on the interval J, $J = [0, \infty)$, with norm defined by

$$||f|| = \int_J | f(\varsigma) | \, d\varsigma, \ \ f \in L_1,$$

then we have the following definition for the fractional (arbitrary) order integration.

Definition 2.1. *The fractional (arbitrary) order integral of the function $f \in L_1[a, b]$ of order $\beta > 0$ is defined by (see [9] - [11])*

$$I_a^\beta f(\varsigma) = \int_a^\varsigma \frac{(\varsigma - s)^{\beta - 1}}{\Gamma(\beta)} f(s) \, ds,$$

or

$$I_a^\beta f(\varsigma) = \int_0^{\varsigma - a} \frac{u^{\beta - 1}}{\Gamma(\beta)} f(\varsigma - u) \, du.$$

When $a = 0$, we can write $I^\beta f(\varsigma) = I_0^\beta f(\varsigma) = f(\varsigma) \star \phi_\beta(\varsigma)$, where

$$\phi_\beta(\varsigma) := \begin{cases} \frac{\varsigma^{\beta - 1}}{\Gamma(\beta)}, & \varsigma > 0, \\ 0, & \varsigma \le 0, \end{cases}$$

and ϕ satisfies the property

$$\phi_{\beta_1}(\varsigma) \star \phi_{\beta_2}(\varsigma) = \phi_{\beta_1 + \beta_2}(\varsigma).$$

Also $\phi_\beta(\varsigma) \to \delta(\varsigma)$ as $\beta \to 0$, where $\delta(\varsigma)$ is the Dirac-delta function (see [5]).

For $\kappa, \beta \in R^+$, we have

(a) $I_a^\kappa : L_1 \to L_1$,

(b) $I^\kappa I^\beta f(t) = I^{\kappa + \beta} f(t)$.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\beta \in (0, 1)$ of a Lebesgue-measurable function $f : R^+ \to R$ is defined by (see [9] - [11])*

$$D_a^\beta f(\varsigma) = \frac{d}{d\varsigma} I_a^{1 - \beta} f(\varsigma) = \frac{1}{\Gamma(1 - \beta)} \frac{d}{d\varsigma} \int_a^\varsigma (\varsigma - s)^{-\beta} f(s) \, ds.$$

Theorem 2.3. ***(Schauder fixed point Theorem)****[12]*

Let W be a convex subset of a Banach space X, and $T : W \to W$ is compact, continuous map. Then T has at least one fixed point in W.

3. Main Results

Define the two spaces

$$C(I) := \{u : u(\varsigma) \ \text{ is continuous on } \ I = [0, \hbar], ||u|| = \max_{\varsigma \in I} |u(\varsigma)|\}$$

and

$$C_{1-\kappa}(I) = \{u : \varsigma^{1-\kappa}u(\varsigma) \text{ is continuous on } I \text{ with the weighted norm } ||u||_{1-\kappa} = ||\varsigma^{1-\kappa}u||\}.$$

Our paper will be divided into two parts, in the first part we will study the existence of a solution for problem (1.1) in the space $C_{1-\kappa}(I)$. And in the second part we will study the existence of a solution for problem (1.2) in the space $C(I)$.

3.1. Solution in $C_{1-\kappa}(I)$

Suppose that the two functions f and h satisfy the following conditions

(1*) for each $\varsigma \in I,\ f(\varsigma,\cdot,\cdot)$ is continuous,
for each $(u,v) \in \Re \times \Re,\ f(\cdot,u,v)$ is measurable, and

$$|f(\varsigma,u,v)| \leq \varsigma^{\mu}\varphi(\varsigma)|u|^{m_1} + |v|, \mu \geq 0, m_1 > 1,$$

(2*) for each $(\varsigma,s) \in I \times I,\ h(\varsigma,s,\cdot)$ is continuous,
for each $u \in \Re,\ h(\cdot,\cdot,u)$ is measurable, and

$$|h(\varsigma,s,u(s))| \leq (\varsigma - s)^{\beta-1}s^{\sigma}\psi(s)|u|^{m_2}, 0 < \beta < 1, \sigma \geq 0, m_2 > 1,$$

where $\varphi(\varsigma)$ and $\psi(\varsigma)$ are continuous functions.

3.1.1. Integral representation

In ([1]-[2]) the authors proved that the Cauchy problem (1.1) is equivalent to the nonlinear integral equation of fractional order

$$u(\varsigma) = b\,\varsigma^{\kappa-1} + \int_0^{\varsigma} \frac{(\varsigma - s)^{\kappa-1}}{\Gamma(\kappa)}\, f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))\, d\theta\right) ds. \tag{3.1}$$

Define the operator T by

$$Tu(\varsigma) = b\,\varsigma^{\kappa-1} + \int_0^{\varsigma} \frac{(\varsigma - s)^{\kappa-1}}{\Gamma(\kappa)}\, f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) ds.$$

It is clear that the fixed point of the operator T is the solution of the integral equation (3.1).

3.1.2. Existence of solution

Theorem 3.1. *Assume that assumptions (1*)-(2*) and (3)-(4) are satisfied, then the weighted Cauchy-type problems (1.1) has at least one solution $u \in C_{1-\kappa}(I)$.*

Proof. Define the set

$$S_r = \left\{u \in C_{1-\kappa}(I) : ||u - b\,\varsigma^{\kappa-1}||_{1-\kappa} \leq r\right\}.$$

Now,

$$||Tu - b\,\varsigma^{\kappa-1}||_{1-\kappa} = \max_{\varsigma\in I}\left|\varsigma^{1-\kappa}I^{\kappa}f\left(\varsigma, u(\varsigma), \int_0^{\varsigma} h(\varsigma,s,u(s))\, ds\right)\right|$$

$$\leq \max_{\varsigma\in I} \varsigma^{1-\kappa} I^{\kappa}\left| f\left(\varsigma, u(\varsigma), \int_0^{\varsigma} h(\varsigma, s, u(s))\, ds\right)\right|$$

$$\leq \max_{\varsigma\in I} \varsigma^{1-\kappa} I^{\kappa}\left(\varsigma^{\mu}\varphi(\varsigma)|u(\varsigma)|^{m_1} + \int_0^{\varsigma} |h(\varsigma, s, u(s))|\, ds\right)$$

$$\leq \max_{\varsigma\in I} \varsigma^{1-\kappa} I^{\kappa}\left(\varsigma^{\mu}\varphi(\varsigma)|u(\varsigma)|^{m_1} + \int_0^{\varsigma}(\varsigma - s)^{\beta-1} s^{\sigma}\psi(s)|u(s)|^{m_2}\, ds\right)$$

$$\leq \max_{\varsigma\in I} \varsigma^{1-\kappa} \int_0^{\varsigma} \frac{(\varsigma - s)^{\kappa-1}}{\Gamma(\kappa)} s^{\mu}\varphi(s)|u(s)|^{m_1} s^{(1-\kappa)m_1} s^{-(1-\kappa)m_1} ds$$

$$+ \max_{\varsigma\in I} \varsigma^{1-\kappa} \int_0^{\varsigma} \frac{(\varsigma - s)^{\kappa-1}}{\Gamma(\kappa)} \int_0^{s}(s-\theta)^{\beta-1}\theta^{\sigma}\psi(\theta)|u(\theta)|^{m_2}\, d\theta\, ds$$

$$\leq \max_{\varsigma\in I} \varsigma^{1-\kappa}||\varphi||\, ||u||_{1-\kappa}^{m_1} \int_0^{\varsigma} \frac{(\varsigma - s)^{\kappa-1}}{\Gamma(\kappa)} s^{\mu-(1-\kappa)m_1} ds$$

$$+ \max_{\varsigma\in I} \varsigma^{1-\kappa} \int_0^{\varsigma} \frac{(\varsigma - s)^{\kappa-1}}{\Gamma(\kappa)} \int_0^{s}(s-\theta)^{\beta-1}\theta^{\sigma}\psi(\theta)|u(\theta)|^{m_2}\theta^{(1-\kappa)m_2}\theta^{-(1-\kappa)m_2}\, d\theta\, ds$$

$$\leq \max_{\varsigma\in I} \varsigma^{1-\kappa}||\varphi||\, ||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)(m_1+1)+2)}\varsigma^{\mu-(1-\kappa)(m_1+1)+1}$$

$$+ \max_{\varsigma\in I} \varsigma^{1-\kappa}||\psi||\, ||u||_{1-\kappa}^{m_2} \int_0^{\varsigma} \frac{(\varsigma - s)^{\kappa-1}}{\Gamma(\kappa)} \int_0^{s}(s-\theta)^{\beta-1}\theta^{\sigma-(1-\kappa)m_2}\, d\theta\, ds$$

$$\leq ||\varphi||\, ||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)}\hbar^{\mu-(1-\kappa)m_1+1}$$

$$+ \max_{\varsigma\in I} \varsigma^{1-\kappa}||\psi||\, ||u||_{1-\kappa}^{m_2} \int_0^{\varsigma} \frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)} \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+1)} s^{\sigma-(1-\kappa)m_2+\beta}\, ds$$

$$\leq ||\varphi||\, ||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)}\hbar^{\mu-(1-\kappa)m_1+1}$$

$$+ \max_{\varsigma\in I} \varsigma^{1-\kappa}||\psi||||u||_{1-\kappa}^{m_2} \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+1)} \frac{\Gamma(\sigma-(1-\kappa)m_2+\beta+1)}{\Gamma(\sigma-(1-\kappa)m_2+\kappa+\beta+1)}\varsigma^{\sigma-(1-\kappa)m_2+\kappa+\beta}$$

$$\leq ||\varphi||\, ||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)}\hbar^{\mu-(1-\kappa)m_1+1}$$

$$+||\psi||\, ||u||_{1-\kappa}^{m_2} \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\kappa+\beta+1)}\hbar^{\sigma-(1-\kappa)m_2+\beta+1}.$$

If $u \in S_r$, then

$$||Tu - b\varsigma^{\kappa-1}||_{1-\kappa} \leq \frac{\Gamma(\mu-(1-\kappa)m_1+1)||\varphi||\, (r+|b|)^{m_1}}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)}\hbar^{\mu-(1-\kappa)m_1+1}$$

$$+\frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)||\psi||\, (r+|b|)^{m_2}}{\Gamma(\sigma-(1-\kappa)m_2+\kappa+\beta+1)}\hbar^{\sigma-(1-\kappa)m_2+\beta+1}$$

$$\leq C_1\, (r+|b|)^{m_1}\, \hbar^{\gamma} \;+\; C_2\, (r+|b|)^{m_2}\, \hbar^{\delta},$$

where

$$C_1 \;=\; \frac{\Gamma(\mu-(1-\kappa)m_1+1)||\varphi||}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)},$$

$$C_2 \;=\; \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)||\psi||}{\Gamma(\sigma-(1-\kappa)m_2+\kappa+\beta+1)},$$

$$\gamma \;=\; \mu-(1-\kappa)m_1+1 \;>\; 0$$

and

$$\delta \;=\; \sigma-(1-\kappa)m_2+\beta+1 \;>\; 0.$$

If we take $r=|b|$ and $\hbar$ very small, then

$$||Tu-b\varsigma^{\kappa-1}||_{1-\kappa} \;\leq\; r,$$

then $T(S_r)\subset S_r$.

Now, we prove that T is continuous on S_r. Indeed: let $u_1,u_2\in S_r$, then we get

$$||Tu_1-Tu_2||_{1-\kappa}=\max_{\varsigma\in I}\left|\varsigma^{1-\kappa}I^{\kappa}\left[f\left(\varsigma,u_1(\varsigma),\int_0^{\varsigma}h(\varsigma,s,u_1(s))ds\right)-f\left(\varsigma,u_2(\varsigma),\int_0^{\varsigma}h(\varsigma,s,u_2(s))\,ds\right)\right]\right|$$

$$\leq \max_{\varsigma\in I}\varsigma^{1-\kappa}\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}\left|f\left(s,u_1(s),\int_0^{s}h(s,\theta,u_1(\theta))d\theta\right)-f\left(s,u_2(s),\int_0^{s}h(s,\theta,u_2(\theta))\,d\theta\right)\right|ds.$$

From the continuity of f and h, we can deduce that for a given $\epsilon>0$ there exists a $\delta_1>0$ such that for all $(s,u_1,v_1),(s,u_2,v_2)\in I\times C_{1-\kappa}(I)\times C_{1-\kappa}(I)$, we have

$$s^{1-\kappa}\left|f\left(s,u_1(s),\int_0^{s}h(s,\theta,u_1(\theta))d\theta\right)-f\left(s,u_2(s),\int_0^{s}h(s,\theta,u_2(\theta))\,d\theta\right)\right|<\epsilon$$

provieded that $||u_1-u_2||_{1-\kappa}<\delta_1$.

To prove that $T(S_r)$ is equicontinuous, let $\tau_1, \tau_2 \in I, \tau_1 < \tau_2, |\tau_2 - \tau_1| < \delta$, then

$$\tau_2^{1-\kappa}Tu(\tau_2) - \tau_1^{1-\kappa}Tu(\tau_1) = {\tau_2}^{1-\kappa}\int_0^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\, f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) ds$$

$$-{\tau_1}^{1-\kappa}\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)}\, f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) ds$$

$$= {\tau_2}^{1-\kappa}\int_0^{\tau_1}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\, f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) ds$$

$$+{\tau_2}^{1-\kappa}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\, f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) ds$$

$$-{\tau_1}^{1-\kappa}\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)}\, f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) ds$$

$$\leq (\tau_2^{1-\kappa} - \tau_1^{1-\kappa})\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)} f(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta) ds$$

$$+{\tau_2}^{1-\kappa}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\, f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) ds,$$

$$|\tau_2^{1-\kappa}Tu(\tau_2) - \tau_1^{1-\kappa}T_1u(\tau_1)| \leq (\tau_2^{1-\kappa} - \tau_1^{1-\kappa})\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)}|f(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta)| ds$$

$$+{\tau_2}^{1-\kappa}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\left|f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right)\right| ds$$

$$\leq \left(\tau_2^{1-\kappa} - \tau_1^{1-\kappa}\right)\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)}\left(s^{\mu}\varphi(s)|u|^{m_1} + \int_0^s |h(s,\theta,u(\theta))|d\theta\right) ds$$

$$+{\tau_2}^{1-\kappa}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\left(s^{\mu}\varphi(s)|u(s)|^{m_1} + \int_0^s |h(s,\theta,u(\theta))|d\theta\right) ds$$

$$\leq \left(\tau_2^{1-\kappa} - \tau_1^{1-\kappa}\right)\Bigg[\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)} s^{\mu-(1-\kappa)m_1}\varphi(s)s^{(1-\kappa)m_1}|u(s)|^{m_1} ds$$

$$+\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)}\int_0^s (s-\theta)^{\beta-1}\theta^{\sigma-(1-\kappa)m_2}\psi(\theta)\theta^{(1-\kappa)m_2}|u(\theta)|^{m_2} d\theta\, ds\Bigg]$$

$$+{\tau_2}^{1-\kappa}\Bigg[\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)} s^{\mu-(1-\kappa)m_1}\varphi(s)s^{(1-\kappa)m_1}|u(s)|^{m_1} ds$$

$$+\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\int_0^s (s-\theta)^{\beta-1}\theta^{\sigma-(1-\kappa)m_2}\psi(\theta)\theta^{(1-\kappa)m_2}|u(\theta)|^{m_2} d\theta\, ds\Bigg]$$

$$\leq \left(\tau_2^{1-\kappa} - \tau_1^{1-\kappa}\right)\Bigg[||\varphi||\ ||u||_{1-\kappa}^{m_1}\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)} s^{\mu-(1-\kappa)m_1}\ ds$$

$$+||\psi||\ ||u||_{1-\kappa}^{m_2}\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)}\int_0^s (s-\theta)^{\beta-1}\theta^{\sigma-(1-\kappa)m_2} d\theta\ ds\Bigg]$$

$$+{\tau_2}^{1-\kappa}\Bigg[||\varphi||\ ||u||_{1-\kappa}^{m_1}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)} s^{\mu-(1-\kappa)m_1}\ ds$$

$$\leq \left(\tau_2^{1-\kappa} - \tau_1^{1-\kappa}\right)\Bigg[||\varphi||\ ||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)} \tau_1^{\mu-(1-\kappa)m_1+\kappa}$$

$$+||\psi||\ ||u||_{1-\kappa}^{m_2} \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+1)} \int_0^{\tau_1} \frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)} s^{\sigma-(1-\kappa)m_2+\beta} ds\Bigg]$$

$$+\tau_2{}^{1-\kappa}\Bigg[||\varphi||\ ||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)} (\tau_2-\tau_1)^{\mu-(1-\kappa)m_1+\kappa}$$

$$+||\psi||\ ||u||_{1-\kappa}^{m_2} \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+1)} \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)} s^{\sigma-(1-\kappa)m_2+\beta} ds\Bigg]$$

$$\leq \left(\tau_2^{1-\kappa} - \tau_1^{1-\kappa}\right)\Bigg[||\varphi||\ ||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)} \tau_1^{\mu-(1-\kappa)m_1+\kappa}$$

$$+||\psi||\ ||u||_{1-\kappa}^{m_2} \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+1)} \frac{\Gamma(\sigma-(1-\kappa)m_2+\beta+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+\kappa+1)} \tau_1{}^{\sigma-(1-\kappa)m_2+\beta+\kappa}\Bigg]$$

$$+\tau_2{}^{1-\kappa}\Bigg[||\varphi||||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)} (\tau_2-\tau_1)^{\mu-(1-\kappa)m_1+\kappa}$$

$$+||\psi||\ ||u||_{1-\kappa}^{m_2} \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+\kappa+1)} (\tau_2-\tau_1)^{\sigma-(1-\kappa)m_2+\beta+\kappa}\Bigg]$$

$$\leq \left(\tau_2^{1-\kappa} - \tau_1^{1-\kappa}\right)\Bigg[||\varphi||\ ||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)} \tau_1^{\mu-(1-\kappa)m_1+\kappa}$$

$$+||\psi||\ ||u||_{1-\kappa}^{m_2} \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+\kappa+1)} \tau_1{}^{\sigma-(1-\kappa)m_2+\beta+\kappa}\Bigg]$$

$$+\tau_2{}^{1-\kappa}\Bigg[||\varphi||\ ||u||_{1-\kappa}^{m_1} \frac{\Gamma(\mu-(1-\kappa)m_1+1)}{\Gamma(\mu-(1-\kappa)m_1+\kappa+1)} (\tau_2-\tau_1)^{\mu-(1-\kappa)m_1+\kappa}$$

$$+||\psi||\ ||u||_{1-\kappa}^{m_2} \frac{\Gamma(\beta)\Gamma(\sigma-(1-\kappa)m_2+1)}{\Gamma(\sigma-(1-\kappa)m_2+\beta+\kappa+1)} (\tau_2-\tau_1)^{\sigma-(1-\kappa)m_2+\beta+\kappa}\Bigg]$$

$$\leq \left(\tau_2^{1-\kappa} - \tau_1^{1-\kappa}\right)\Bigg[C_1\ ||u||_{1-\kappa}^{m_1}\ \tau_1^{\gamma+\kappa-1} + C_2\ ||u||_{1-\kappa}^{m_2} \tau_1{}^{\delta+\kappa-1}\Bigg]$$

$$+\tau_2{}^{1-\kappa}\Bigg[C_1\ ||u||_{1-\kappa}^{m_1}\ (\tau_2-\tau_1)^{\gamma+\kappa-1} + C_2\ ||u||_{1-\kappa}^{m_2}\ (\tau_2-\tau_1)^{\delta+\kappa-1}\Bigg].$$

Therefore TS_r is equi-continuous, by Arzela-Ascoli Theorem then TS_r is relatively compact. Therefore the conditions of Schauder fixed point Theorem are hold, which implies that T has a fixed point in S_r. Then the nonlinear integral equation (3.1) has at least one solution $u \in C_{1-\kappa}(I)$ and consequently the weighted Cauchy-type problem (1.1) has at least one solution $u \in C_{1-\kappa}(I)$.

3.2. Solution in $C(I)$

3.2.1. Integral representation

Lemma 3.2. *The Cauchy problem (1.2) is equivalent to the nonlinear integral equation of fractional order*

$$u(\varsigma) = \int_0^\varsigma \frac{(\varsigma - s)^{\kappa-1}}{\Gamma(\kappa)} f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))\, d\theta\right) ds. \tag{3.2}$$

Proof: Let $u(\varsigma)$ be a solution of

$$D^\kappa u(\varsigma) = \frac{d}{d\varsigma} I^{1-\kappa} u(\varsigma) = f\left(\varsigma, u(\varsigma), \int_0^\varsigma h(\varsigma, s, u(s))\, ds\right),$$

integrate both sides, we get

$$I^{1-\kappa}u(\varsigma) - I^{1-\kappa}u(\varsigma)|_{\varsigma=0} = I\, f\left(\varsigma, u(\varsigma), \int_0^\varsigma h(\varsigma, s, u(s))\, ds\right),$$

operating by I^κ on both sides of the last equation, we get

$$Iu(\varsigma) - I^\kappa\, C = I^{1+\kappa} f\left(\varsigma, u(\varsigma), \int_0^\varsigma h(\varsigma, s, u(s))\, ds\right),$$

differentiate both sides, we get

$$u(\varsigma) - C_1\, \varsigma^{\kappa-1} = I^\kappa f\left(\varsigma, u(\varsigma), \int_0^\varsigma h(\varsigma, s, u(s))\, ds\right),$$

from the initial condition, we find that $C_1 = 0$, then we get (3.2)
Define the operator F by

$$Fu(\varsigma) = \int_0^\varsigma \frac{(\varsigma - s)^{\kappa-1}}{\Gamma(\kappa)} f\left(s, u(s), \int_0^s h(s,\theta,u(\theta))d\theta\right) ds.$$

It is clear that the fixed point of the operator F is the solution of the integral equation (3.2).

3.2.2. Existence of solution

Theorem 3.3. *Assume that the assumptions (1*)-(2*) are satisfied. Then the weighted Cauchy-type problem (1.2) has at least one solution $u \in C(I)$.*

Proof. Define the set

$$B_r = \left\{u \in C(I) : ||u|| \le r_1\right\}.$$

Now,

$$||Fu|| = \max_{\varsigma\in I}\left|I^\kappa f\left(\varsigma, u(\varsigma), \int_0^\varsigma h(\varsigma, s, u(s))\, ds\right)\right|$$

$$\leq \max_{\varsigma\in I} I^{\kappa}\left|f\left(\varsigma, u(\varsigma), \int_0^{\varsigma} h(\varsigma,s,u(s))\, ds\right)\right|$$

$$\leq \max_{\varsigma\in I} I^{\kappa}\left(\varsigma^{\mu}\varphi(\varsigma)|u(\varsigma)|^{m_1} + \int_0^{\varsigma} |h(\varsigma,s,u(s))|\, ds\right)$$

$$\leq \max_{\varsigma\in I} I^{\kappa}\left(\varsigma^{\mu}\varphi(\varsigma)|u(\varsigma)|^{m_1} + \int_0^{\varsigma}(\varsigma-s)^{\beta-1}s^{\sigma}\psi(s)|u(s)|^{m_2}\, ds\right)$$

$$\leq \max_{\varsigma\in I} I^{\kappa}\varsigma^{\mu}\varphi(\varsigma)|u(\varsigma)|^{m_1} + \max_{\varsigma\in I}\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}\int_0^{s}(s-\theta)^{\beta-1}\theta^{\sigma}\psi(\theta)|u(\theta)|^{m_2}\, d\theta ds$$

$$\leq \max_{\varsigma\in I}\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}s^{\mu}\varphi(s)|u(s)|^{m_1}ds$$

$$+\max_{\varsigma\in I}||\psi||\,||u||^{m_2}\frac{\Gamma(\beta)\Gamma(\sigma+1)}{\Gamma(\beta+\sigma+1)}\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}s^{\beta+\sigma}ds$$

$$\leq \max_{\varsigma\in I}||\varphi||\,||u||^{m_1}\frac{\Gamma(\mu+1)}{\Gamma(\kappa+\mu+1)}\varsigma^{\kappa+\mu}$$

$$+\max_{\varsigma\in I}||\psi||\,||u||^{m_2}\frac{\Gamma(\beta)\Gamma(\sigma+1)}{\Gamma(\beta+\sigma+1)}\frac{\Gamma(\beta+\sigma+1)}{\Gamma(\kappa+\beta+\sigma+1)}\varsigma^{\kappa+\beta+\sigma}$$

$$\leq ||\varphi||\,||u||^{m_1}\frac{\Gamma(\mu+1)}{\Gamma(\kappa+\mu+1)}\hbar^{\kappa+\mu} + ||\psi||\,||u||^{m_2}\frac{\Gamma(\beta)\Gamma(\sigma+1)}{\Gamma(\kappa+\beta+\sigma+1))}\hbar^{\kappa+\beta+\sigma}.$$

If $u\in B_r$, then

$$||Fu|| \leq \frac{\Gamma(\mu+1)||\varphi||\, r_1^{m_1}}{\Gamma(\kappa+\mu+1)}\hbar^{\kappa+\mu} + \frac{\Gamma(\beta)\Gamma(\sigma+1)||\psi||\, r_1^{m_2}}{\Gamma(\kappa+\beta+\sigma+1)}\hbar^{\kappa+\beta+\sigma}.$$

If we take $\hbar$ very small, then

$$||Fu|| \;\leq\; r_1,$$

then $F(B_r)\subset B_r$.

From the continuity of f and h, we obtain that the operator F is continuous.

To prove that $F(B_r)$ is equicontinuous

Let $\tau_1,\tau_2\in[0,\hbar], \tau_1<\tau_2, |\tau_2-\tau_1|<\delta$, then

$$Fu(\tau_2)-Fu(\tau_1) = \int_0^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\, f\left(s,u(s),\int_0^{s}h(s,\theta,u(\theta))d\theta\right)\, ds$$

$$-\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)}\, f\left(s,u(s),\int_0^{s}h(s,\theta,u(\theta))d\theta\right)\, ds$$

$$=\int_0^{\tau_1}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\, f\left(s,u(s),\int_0^{s}h(s,\theta,u(\theta))d\theta\right)\, ds$$

$$+\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\ f\left(s,u(s),\int_0^s h(s,\theta,u(\theta))d\theta\right)\ ds$$

$$-\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)}\ f\left(s,u(s),\int_0^s h(s,\theta,u(\theta))d\theta\right)\ ds$$

$$\leq\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)}f\left(s,u(s),\int_0^s h(s,\theta,u(\theta))d\theta\right)ds$$

$$+\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\ f\left(s,u(s),\int_0^s h(s,\theta,u(\theta))d\theta\right)ds,$$

$$-\int_0^{\tau_1}\frac{(\tau_1-s)^{\kappa-1}}{\Gamma(\kappa)}\ f\left(s,u(s),\int_0^s h(s,\theta,u(\theta))d\theta\right)\ ds$$

$$\left|Fu(\tau_2)-Fu(\tau_1)\right|\leq\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\ \left|f\left(s,u(s),\int_0^s h(s,\theta,u(\theta))d\theta\right)\right|ds$$

$$\leq\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\left(s^{\mu}\varphi(s)|u(s)|^{m_1}+\int_0^s|h(s,\theta,u(\theta))|d\theta\right)ds$$

$$\leq\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}s^{\mu}\varphi(s)|u(s)|^{m_1}ds$$

$$+\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\int_0^s(s-\theta)^{\beta-1}\theta^{\sigma}\psi(\theta)|u(\theta)|^{m_2}d\theta ds$$

$$\leq||\varphi||\ ||u||^{m_1}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}s^{\mu}\ ds$$

$$+||\psi||\ ||u||^{m_2}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}\int_0^s(s-\theta)^{\beta-1}\theta^{\sigma}d\theta\ ds$$

$$\leq||\varphi||\ ||u||^{m_1}\frac{\Gamma(\mu+1)}{\Gamma(\mu+\kappa+1)}(\tau_2-\tau_1)^{\mu+\kappa}$$

$$+||\psi||\ ||u||^{m_2}\frac{\Gamma(\beta)\Gamma(\sigma+1)}{\Gamma(\sigma+\beta+1)}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\kappa-1}}{\Gamma(\kappa)}s^{\sigma+\beta}ds$$

$$\leq||\varphi||\ ||u||^{m_1}\frac{\Gamma(\mu+1)}{\Gamma(\mu+\kappa+1)}(\tau_2-\tau_1)^{\mu+\kappa}$$

$$+||\psi||\ ||u||^{m_2}\frac{\Gamma(\beta)\Gamma(\sigma+1)}{\Gamma(\sigma+\beta+\kappa+1)}(\tau_2-\tau_1)^{\sigma+\beta+\kappa}.$$

Therefore FB_r is equi-continuous, by Arzela-Ascoli Theorem then FB_r is relatively compact. Therefore the conditions of Schauder fixed point Theorem are hold, which implies that F has a fixed point in B_r. Then the nonlinear integral equation (3.2) has a solution $u\in C(I)$ and consequently from Lemma 3.2, we get that the weighted Cauchy-type problem (1.2) has a solution $u\in C(I)$.

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Nevanlinna theory for the upper half disc-I

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Abstract. In this paper, we prove the Poisson Integral theorem and Poisson-Jenson formula for the upper half disc and consequently introduce the proximity function, the counting function and the characteristic function of a meromorphic function in the upper half disc which are basic functions of Nevanlinna theory.

AMS Subject Classifications: 30D35

Keywords: Nevanlinna theory, Upper half disc, Meromorphic functions, Poisson integral formula, Characteristic functions, etc.

Contents

1. Preamble

Nevanlinna theory for meromorphic functions in the complex plane is about a century old and still is an emerging active area of research. This theory has wide range of applications including complex differential equations and value sharing of meromorphic functions etc. Nevanlinna theory for an angular domain is also developed by some authors like (see [4],[5],[6]) using Carleman's formula.
In this paper, we propose a similar theory for the upper-half plane. Our main tool here is the conformal self map of the upper half disc as defined in (2.1).

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2. Preliminaries

Let $\mathbb{C}$ denote the set of all complex numbers, $\overline{\mathbb{C}}$, the extended complex plane, $\mathbb{C}_+ = \{z : Imz > 0\}$ and $\overline{\mathbb{C}}_+ = \{z : Imz \geq 0\}$. Throughout, let $\mathbf{D} = \{|\xi| < R : Im\xi > 0\}$ and $\overline{\mathbf{D}} = \{|\xi| \leq R : Im\xi \geq 0\}$ be open and closed upper-half discs respectively.
In what follows, we see that conformal self map of upper-half disc

$$\Phi_z(\xi) = \frac{R(\xi - z)}{R^2 - \xi\overline{z}} \frac{R^2 - \xi z}{R(\xi - \overline{z})} \tag{2.1}$$

where $z, \xi \in \mathbf{D}$ plays a cardinal role in the development of the theory.

The analogous notations of Nevanlinna theory for meromorphic functions in the upper-half disc shall be introduced as and when required.

3. Main results

We now present the core result namely Poisson Integral formula for the upper-half disc.

Theorem 3.1. *Let $f(z)$ be analytic on the closed upper-half disc $\overline{\mathbf{D}} = \{|\xi| \leq R : Im\xi \geq 0\}$. For $z = re^{i\phi}$ $(0 < r < R)$ in $\mathbf{D}$, we have*

$$f(z) = \frac{1}{2\pi}\int_0^{\pi} f(Re^{i\theta})\left\{\frac{R^2 - |z|^2}{|\xi - z|^2} - \frac{R^2 - |z|^2}{|\xi - \overline{z}|^2}\right\}d\theta + \frac{1}{2\pi}\int_{-R}^{R} f(t)\left\{\frac{2r\sin\phi}{|z - t|^2} - \frac{2R^2 r\sin\phi}{|R^2 - zt|^2}\right\}dt \tag{3.1}$$

Proof. By the Cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} \frac{f(\xi)}{\xi - z}d\xi \tag{3.2}$$

If $z_1 = \frac{R^2}{\overline{z}} = \frac{R^2}{r}e^{i\phi}$, then $|z_1| = \frac{R^2}{r} > R$ and hence z_1 lies outside the upper-half circle $\{|\xi| = R : Im\xi \geq 0\}$.
Thus $\frac{f(\xi)}{\xi - z_1} = \frac{f(\xi)}{\xi - \frac{R^2}{\overline{z}}}$ is analytic on the closed upper-half disc $\{|\xi| \leq R : Im\xi \geq 0\}$.
In view of this and by the Cauchy's theorem, we have

$$\frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} \frac{f(\xi)}{\xi - \frac{R^2}{\overline{z}}}d\xi = 0 \tag{3.3}$$

For $z \in \mathbf{D}$, $\overline{z}$ lies outside the upper-half disc $\{|\xi| \leq R : Im\xi \geq 0\}$ and $\overline{z}_1 = \frac{R^2}{z}$ is also outside the upper-half disc $\{|\xi| \leq R : Im\xi \geq 0\}$.
In view of the above observations and by using the Cauchy's theorem again we have,

$$\frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} \frac{f(\xi)}{\xi - \overline{z}}d\xi = 0 \tag{3.4}$$

and

$$\frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} \frac{f(\xi)}{\xi - \frac{R^2}{z}}d\xi = 0. \tag{3.5}$$

Combining the above equations $(3.2) - (3.5)$ we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0}\frac{f(\xi)}{\xi - z}d\xi - \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0}\frac{f(\xi)}{\xi-\overline{z}}d\xi \\ &+ \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0}\frac{f(\xi)}{\xi-\frac{R^2}{z}}d\xi - \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0}\frac{f(\xi)}{\xi-\frac{R^2}{\overline{z}}}d\xi \\ &= \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} f(\xi)\left[\frac{1}{\xi-z} - \frac{1}{\xi-\overline{z}} + \frac{1}{\xi-\frac{R^2}{z}} - \frac{1}{\xi-\frac{R^2}{\overline{z}}}\right]d\xi \end{aligned}$$

$$= \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} f(\xi)\left[\frac{1}{\xi-z} - \frac{1}{\xi-\overline{z}} + \frac{z}{R^2-\xi z} - \frac{\overline{z}}{R^2-\xi\overline{z}}\right]d\xi \tag{3.6}$$

$$= \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} f(\xi)\left[\frac{R^2-|z|^2}{(\xi-z)(R^2-\xi\overline{z})} - \frac{R^2-|z|^2}{(\xi-\overline{z})(R^2-\xi z)}\right]d\xi \tag{3.7}$$

Equation (3.6) can also be rewritten as

$$f(z) = \frac{1}{2\pi i}\int_{|\xi|=R, Im\xi\geq 0} f(\xi)\left[\frac{(z-\overline{z})}{(\xi-z)(\xi-\overline{z})} - \frac{R^2(z-\overline{z})}{(R^2-\xi z)(R^2-\xi\overline{z})}\right]d\xi \tag{3.8}$$

For $\xi = \xi_1 + \xi_2$, where $\xi_1 = Re^{i\theta}$, $0 < \theta < \pi$ and $\xi_2 = t$, $-R < t < R$ and in view of (3.7) and $(3,8)$, we have

$$f(z) = \frac{1}{2\pi}\int_0^{\pi} f(Re^{i\theta})\left\{\frac{R^2-|z|^2}{|\xi-z|^2} - \frac{R^2-|z|^2}{|\xi-\overline{z}|^2}\right\}d\theta + \frac{1}{2\pi}\int_{-R}^{R} f(t)\left\{\frac{2r\sin\phi}{|z-t|^2} - \frac{2R^2 r\sin\phi}{|R^2-zt|^2}\right\}dt$$

■

The above Poisson Integral formula for the upper-half disc leads to the following result. This result plays a cardinal role in the development of the Nevanlinna theory for the upper-half disc.

Theorem 3.2. Poisson Jensen formula for upper-half disc
Let $f(z)$ be analytic in the closed upper-half disc $\overline{\mathbf{D}} = \{|\xi| \leq R : Im\xi \geq 0\}$ except for the poles $b_1, b_2, ..., b_n$ in $\mathbf{D}$ and $a_1, a_2, ..., a_m$ be zeros of $f(z)$ in $\mathbf{D}$. Then for any $z \neq a_m, b_n$ in $\mathbf{D}$, we have

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{\pi}\log\left|f(Re^{i\theta})\right|\left\{\frac{R^2-|z|^2}{|\xi-z|^2} - \frac{R^2-|z|^2}{|\xi-\overline{z}|^2}\right\}d\theta + \frac{1}{2\pi}\int_{-R}^{R}\log|f(t)|\left\{\frac{2r\sin\phi}{|z-t|^2} - \frac{2R^2 r\sin\phi}{|R^2-zt|^2}\right\}dt$$

$$- \sum_{|a_m|<R, Ima_m>0}\log\left|\frac{(R^2-\overline{a}_m z)}{R(z-a_m)}\frac{R(z-\overline{a}_m)}{(R^2-a_m z)}\right| + \sum_{|b_n|<R, Imb_n>0}\log\left|\frac{(R^2-\overline{b}_n z)}{R(z-b_n)}\frac{R(z-\overline{b}_n)}{(R^2-b_n z)}\right| \tag{3.9}$$

Proof. Set

$$g(z) = \frac{\prod_{|b_n|<R, Imb_n>0}\frac{(z-b_n)}{(z-\overline{b}_n)}}{\prod_{|a_m|<R, Ima_m>0}\frac{(z-a_m)}{(z-\overline{a}_m)}} f(z) \tag{3.10}$$

Then $g(z)$ is analytic in $\mathbf{D}$ having no zeros and poles in $\mathbf{D}$ and hence there exists an analytic branch $\log g(z)$ in $\mathbf{D}$. In view of the Poisson Integral formula for the upper-half disc i. e. by (3.1), we have

$$\log g(z) = \frac{1}{2\pi}\int_0^{\pi}\log g(Re^{i\theta})\left\{\frac{R^2-|z|^2}{|\xi-z|^2} - \frac{R^2-|z|^2}{|\xi-\overline{z}|^2}\right\}d\theta + \frac{1}{2\pi}\int_{-R}^{R}\log g(t)\left\{\frac{2r\sin\phi}{|z-t|^2} - \frac{2R^2 r\sin\phi}{|R^2-zt|^2}\right\}dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} \beta_1 \log g(Re^{i\theta})d\theta + \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log g(t)dt \tag{3.11}$$

where

$$\beta_1 = \frac{R^2 - |z|^2}{|\xi - z|^2} - \frac{R^2 - |z|^2}{|\xi - \overline{z}|^2}$$

and

$$\beta_2 = \frac{2r\sin\phi}{|z-t|^2} - \frac{2R^2 r\sin\phi}{|R^2 - zt|^2}$$

Taking real part on both sides of equation (3.11), we get

$$\log|g(z)| = \frac{1}{2\pi}\int_0^{\pi} \beta_1 \log\left|g(Re^{i\theta})\right| d\theta + \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log|g(t)|\, dt \tag{3.12}$$

By (3.10), we get

$$\log|g(z)| = \log|f(z)| + \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{z-b_n}{z-\overline{b}_n}\right| - \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{z-a_m}{z-\overline{a}_m}\right| \tag{3.13}$$

and

$$\log|g(t)| = \log|f(t)| + \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{t-b_n}{t-\overline{b}_n}\right| - \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{t-a_m}{t-\overline{a}_m}\right| \tag{3.14}$$

Since $|t-a_m| = |t-\overline{a}_m|$ and $|t-b_n| = \left|t-\overline{b}_n\right|$, (3.14) leads to

$$\log|g(t)| = \log|f(t)| \tag{3.15}$$

Using (3.13) and (3.15) in (3.12), we get

$$\log|f(z)| + \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{z-b_n}{z-\overline{b}_n}\right| - \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{z-a_m}{z-\overline{a}_m}\right|$$

$$= \frac{1}{2\pi}\int_0^{\pi} \beta_1 \left[\log\left|f(Re^{i\theta})\right| + \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{Re^{i\theta}-b_n}{Re^{i\theta}-\overline{b}_n}\right| - \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{Re^{i\theta}-a_m}{Re^{i\theta}-\overline{a}_m}\right|\right] d\theta$$

$$+ \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log|f(t)|\, dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} \beta_1 \log\left|f(Re^{i\theta})\right| d\theta + \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log|f(t)|\, dt + \sum_{|b_n|<R, Imb_n>0} \frac{1}{2\pi}\int_0^{\pi} \beta_1 \log\left|\frac{Re^{i\theta}-b_n}{Re^{i\theta}-\overline{b}_n}\right| d\theta$$

$$- \sum_{|a_m|<R, Ima_m>0} \frac{1}{2\pi}\int_0^{\pi} \beta_1 \log\left|\frac{Re^{i\theta}-a_m}{Re^{i\theta}-\overline{a}_m}\right| d\theta \tag{3.16}$$

Since

$$\left|\frac{Re^{i\theta}-a_m}{Re^{i\theta}-\overline{a}_m}\right| = \left|\frac{R-\overline{a}_m e^{i\theta}}{R-a_m e^{i\theta}}\right| \tag{3.17}$$

and

$$\left|\frac{Re^{i\theta}-b_n}{Re^{i\theta}-\overline{b}_n}\right| = \left|\frac{R-\overline{b}_n e^{i\theta}}{R-b_n e^{i\theta}}\right| \tag{3.18}$$

In view of (3.17) and (3.18), (3.16) becomes.

$$\log|f(z)| + \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{z-b_n}{z-\overline{b}_n}\right| - \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{z-a_m}{z-\overline{a}_m}\right|$$

$$= \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left|f(Re^{i\theta})\right| d\theta + \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log|f(t)|\, dt$$

$$+ \sum_{|b_n|<R, Imb_n>0} \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left|\frac{R-\overline{b}_n e^{i\theta}}{R-b_n e^{i\theta}}\right| d\theta - \sum_{|a_m|<R, Ima_m>0} \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left|\frac{R-\overline{a}_m e^{i\theta}}{R-a_m e^{i\theta}}\right| d\theta \qquad (3.19)$$

Since a_m' s are zeros of $f(z)$ in $\mathbf{D}$, we have $\frac{R-\frac{\overline{a}_m z}{R}}{R-\frac{a_m z}{R}} \neq 0$ in $\mathbf{D}$ and hence there exists an analytic branch of $\log\left(\frac{R-\frac{\overline{a}_m z}{R}}{R-\frac{a_m z}{R}}\right)$ in $\mathbf{D}$. By the Poisson Integral formula for the upper-half disc, we get

$$\log\left(\frac{R-\frac{\overline{a}_m z}{R}}{R-\frac{a_m z}{R}}\right) = \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left(\frac{R-\overline{a}_m e^{i\theta}}{R-a_m e^{i\theta}}\right) d\theta + \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log\left(\frac{R-\frac{\overline{a}_m t}{R}}{R-\frac{a_m t}{R}}\right) dt$$

Using $\left|R^2-\overline{a}_m t\right| = \left|R^2-a_m t\right|$ and taking real part part on both sides of above equation, we get

$$\log\left|\frac{R^2-\overline{a}_m z}{R}\frac{R}{R^2-a_m z}\right| = \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left|\frac{R-\overline{a}_m e^{i\theta}}{R-a_m e^{i\theta}}\right| d\theta \qquad (3.20)$$

Similarly,

$$\log\left|\frac{R^2-\overline{b}_n z}{R}\frac{R}{R^2-b_n z}\right| = \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left|\frac{R-\overline{b}_n e^{i\theta}}{R-b_n e^{i\theta}}\right| d\theta \qquad (3.21)$$

From equations (3.19), (3.20) and (3.21), we have

$$\log|f(z)| + \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{z-b_n}{z-\overline{b}_n}\right| - \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{z-a_m}{z-\overline{a}_m}\right|$$

$$= \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left|f(Re^{i\theta})\right| d\theta + \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log|f(t)|\, dt$$

$$+ \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{R^2-\overline{b}_n z}{R}\frac{R}{R^2-b_n z}\right|$$

$$- \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{R^2-\overline{a}_m z}{R}\frac{R}{R^2-a_m z}\right|$$

Hence

$$\log|f(z)| = \frac{1}{2\pi}\int_0^\pi \beta_1 \log\left|f(Re^{i\theta})\right| d\theta + \frac{1}{2\pi}\int_{-R}^{R} \beta_2 \log|f(t)|\, dt$$

$$- \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{R^2-\overline{a}_m z}{R(z-a_m)}\frac{R(z-\overline{a}_m)}{R^2-a_m z}\right| + \sum_{|b_m|<R, Imb_m>0} \log\left|\frac{R^2-\overline{b}_m z}{R(z-b_m)}\frac{R(z-\overline{b}_m)}{R^2-b_m z}\right|$$

Substituting the values of β_1 and β_2 in the above equation, we obtain the desired equality

$$\log|f(z)| = \frac{1}{2\pi}\int_0^\pi \log\left|f(Re^{i\theta})\right| \left\{\frac{R^2-|z|^2}{|\xi-z|^2} - \frac{R^2-|z|^2}{|\xi-\overline{z}|^2}\right\} d\theta + \frac{1}{2\pi}\int_{-R}^{R} \log|f(t)| \left\{\frac{2r\sin\phi}{|z-t|^2} - \frac{2R^2 r\sin\phi}{|R^2-zt|^2}\right\} dt$$

$$- \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{(R^2-\overline{a}_m z)}{R(z-a_m)}\frac{R(z-\overline{a}_m)}{(R^2-a_m z)}\right| + \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{(R^2-\overline{b}_n z)}{R(z-b_n)}\frac{R(z-\overline{b}_n)}{(R^2-b_n z)}\right|$$

■

4. Nevanlinna functions in the upper-half disc $\mathbb{D}$

Poisson-Jensen formula for upper-half disc enables us to define Nevanlinna functions in the upper-half disc $\mathbf{D}$: For a meromorphic function f in $\mathbf{D}$ and $a \in \mathbf{D}$, we define

Definition 4.1. ***Proximate function of*** $f - a$ ***in D***

$$m(\mathbf{D}, a, f) := \frac{1}{2\pi}\int_0^{\pi} \log^+ \left|f(Re^{i\theta})\right| \left\{\frac{R^2 - |a|^2}{|\xi - a|^2} - \frac{R^2 - |a|^2}{|\xi - \overline{a}|^2}\right\} d\theta + \frac{1}{2\pi}\int_{-R}^{R} \log^+ |f(t)| \left\{\frac{2r\sin\phi}{|a - t|^2} - \frac{2R^2 r\sin\phi}{|R^2 - at|^2}\right\} dt \tag{4.1}$$

where, log^+ *is the positive logarithmic function.*
Briefly we write this as

$$m(\mathbf{D}, a, f) = m_1(\mathbf{D}, a, f) + m_2(\mathbf{D}, a, f)$$

where,

$$m_1(\mathbf{D}, a, f) = \frac{1}{2\pi}\int_0^{\pi} \log^+ \left|f(Re^{i\theta})\right| \left\{\frac{R^2 - |a|^2}{|\xi - a|^2} - \frac{R^2 - |a|^2}{|\xi - \overline{a}|^2}\right\} d\theta,$$

$$m_2(\mathbf{D}, a, f) = \frac{1}{2\pi}\int_{-R}^{R} \log^+ |f(t)| \left\{\frac{2r\sin\phi}{|a - t|^2} - \frac{2R^2 r\sin\phi}{|R^2 - at|^2}\right\} dt$$

Definition 4.2. ***Counting function of*** f ***in D***

$$N(\mathbf{D}, f) = N(\mathbf{D}, \infty, f) = \sum_{|b_n|<R, Imb_n>0} \log\left|\frac{(R^2 - \overline{b}_n a)}{R(a - b_n)}\frac{R(a - \overline{b}_n)}{(R^2 - b_n a)}\right| \tag{4.2}$$

where b_n*'s are poles of* f *in* $\mathbf{D}$*, appearing according to their multiplicities.*
and

$$N(\mathbf{D}, a, f) = N(\mathbf{D}, \frac{1}{f - a}) = \sum_{|a_m|<R, Ima_m>0} \log\left|\frac{(R^2 - \overline{a}_m a)}{R(a - a_m)}\frac{R(a - \overline{a}_m)}{(R^2 - a_m a)}\right| \tag{4.3}$$

where a_m*'s are zeros of* $f - a$ *in* $\mathbf{D}$*, appearing according to their multiplicities.*
$\overline{N}(\mathbf{D}, f)$*,* $\overline{N}(\mathbf{D}, a, f)$ *denotes the distinct poles* f *and zeros of* $f - a$ *in* $\mathbf{D}$*, respectively.*

Definition 4.3. ***Characteristic function of*** $f - a$ ***in D***

$$T(\mathbf{D}, a, f) := m(\mathbf{D}, a, f) + N(\mathbf{D}, a, f) \tag{4.4}$$

5. Properties of Nevanlinna functions in D

As in the Nevanlinna theory for the whole complex plane, we have the following basic results in $(\mathbf{D})$
Let $f_i (i = 1, 2, ..., p)$ be p meromorphic functions in $\overline{\mathbf{D}}$, we have

$$m\left(\mathbf{D}, a, \sum_{i=1}^{p} f_i\right) \leq \sum_{i=1}^{p} m(\mathbf{D}, a, f_i) + \log p \tag{5.1}$$

$$m\left(\mathbf{D}, a, \prod_{i=1}^{p}\right) \leq \sum_{i=1}^{p} m\left(\mathbf{D}, a, f_i\right) \tag{5.2}$$

$$N\left(\mathbf{D}, a, \sum_{i=1}^{p} f_i\right) \leq \sum_{i=1}^{p} N(\mathbf{D}, a, f_i) \tag{5.3}$$

$$N\left(\mathbf{D}, a, \prod_{i=1}^{p}\right) \leq \sum_{i=1}^{p} N(\mathbf{D}, a, f_i) \tag{5.4}$$

$$T\left(\mathbf{D}, a, \sum_{i=1}^{p} f_i\right) \leq \sum_{i=1}^{p} T(\mathbf{D}, a, f_i) + \log p \tag{5.5}$$

$$T\left(\mathbf{D}, a, \prod_{i=1}^{p}\right) \leq \sum_{i=1}^{p} T(\mathbf{D}, a, f_i) \tag{5.6}$$

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Sequential Henstock integral for $L^p[0,1]$-interval valued functions

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Abstract. In this paper, we introduce the concept of Sequential Henstock integrals for $L^p[0,1]$- interval valued functions and discuss some of their properties.
AMS Subject Classifications: MSC 2010: 28B05, 28B10, 28B15, 46G10

Keywords: Sequential Henstock integral, Interval valued functions, guages, right and left endpoints.

Contents

1. Introduction and Background

Henstock integral of a function introduced in the mid-1950s by R. Henstock and J. Kursweil is a powerful generalisation of the Riemann integral, which can handled nowhere-continuous functions which gives a simpler and more satisfactory version of the fundamental theorem of calculus. Simply put, the Henstock integral includes the Riemann, Improper Riemann, Newton and Lebesgue integrals and is equivalent to the Denjoy and Perron integrals(see [1-9]). While the standard definition of the Henstock integral uses the $\varepsilon - \delta$ definition, then the Sequential Henstock integral was introduced, by employing sequences of guage functions. Many authors have worked on the application of the Henstock integral to functions taking real values and have made generalisations on a number of its' properties, see [1-16].
For instance, Cao [3] gave a generalization of the definition of the Henstock integral for Banach space-valued function, and then established some of its properties. Macalalag and Paluga [9] studied the Henstock-type integral for l_p-valued functions with $0 < p < 1$ and obtained its basic properties. The authors have studied the Sequential characterization of the Henstock integral and obtained equivalence results between Henstock and Certain Sequential Henstock Integrals when dealing with real-valued functions.(see [6]). Wu and Gong[15] introduced the notion of the Henstock (H) integral of interval valued functions and Fuzzy number-valued functions and obtained a number of properties. Hamid and Elmuiz[5] established the concept of the Henstock Stieltjes (HS) integrals of interval valued functions and Fuzzy number-valued functions and obtained some number of properties of these integrals. It is well known that the class of $L^p[0,1]$-valued functions with $0 < p < 1$ is a Banach Space with the norm denoted by $\|.\|_{L^p}$.
In this paper, we introduce the notion of Sequential Henstock integral for $L^p[0,1]$-interval valued functions with $0 < p < 1$, and investigate some of its basic properties.

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2. Main Results

Let $\mathbb{R}$ denote the set of real numbers, $F(X)$ as an interval valued function, F^-, the left endpoint, F^+ as right endpoint, $\{\delta_n(x)\}_{n=1}^{\infty}$, as set of gauge functions, P_n, as set of partitions of subintervals of a compact interval $[a,b]$, X, as non empty interval in $\mathbb{R}$ and $d(X) = X^+ - X^-$, as width of the interval X and $\ll$ as much more smaller and consider the integral of interval functions defined on the compact interval and ranging in a quasi-Banach $L^p[0,1]$-space which carries a quasi-norm denoted by $\|.\|_{L^p}$.

Let E a Lebesgue measurable set in any euclidean space, and q any positive number, we define $L^p(E)$ to be the class of all real valued Lebesgue measurable functions f on E for which $\int_E |f|^q < \infty$. As it's well known, whenever $q < 1$, this class of functions is a Banach Space with the norm $\|f\|_q = (\int_E |f|^q)^{\frac{1}{q}}$. When $0 < p < 1$, the function $\|f\|_p$ no longer satisfies the triangular inequality but only the weaker condition

$$\|f_1 + f_2\| \leq 2^{\gamma}(\|f_1\| + \|f_2\|)$$

where $\gamma = \dfrac{(1-p)}{p}$(see [3]).

Definition 2.1[10,12] A gauge on $[a,b]$ is a positive real-valued function $\delta : [a,b] \to \mathbb{R}^+$. This gauge is δ-fine if $[u_{i-1}, u_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$.

Definition 2.2[10,12] A sequence of tagged partition P_n of [a,b] is a finite collection of ordered pairs $P_n = \{(u_{(i-1)_n} u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ where $[u_{i-1}, u_i] \in [a,b]$, $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ and $a = u_0 < u_{i_1} <, ..., < u_{m_n} = b$.

Definition 2.3 [12] A function $f : [a,b] \to \mathbb{R}$ is Henstock integrable to α on $[a,b]$ if there exists a number $\alpha \in \mathbb{R}$ such that if $\varepsilon > 0$ there exists a function $\delta(x) > 0$ such that for $\delta(x)$-fine tagged partitions $P = \{(u_{i-1}\, u_i), t_i\}_{i=1}^{n}$, we have

$$|\sum_{i=1}^{n} f(t_i)[u_i - u_{(i-1)}] - \alpha| < \varepsilon.$$

where the number α is the Henstock integral of f on $[a,b]$. The family of all Henstock integrals function on $[a,b]$ is denoted by $H[a,b]$ with $\alpha = (H)\int_{[a,b]} f(x)dx$ and $f \in H[a,b]$.

Definition 2.4 [12] A function $f : [a,b] \to \mathbb{R}$ is Sequential Henstock integrable to $\alpha \in \mathbb{R}$ on $[a,b]$ if for any $\varepsilon > 0$ there exists a sequence of gauge functions $\delta_\mu(x) = \{\delta_n(x)\}_{n=1}^{\infty}$ such that for any $\delta_n(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$, we have

$$|\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha| < \varepsilon,$$

where the sum $\sum$ is over P_n, we write $\alpha = (SH)\int_{[a,b]} f(x)dx$ and $f \in SH[a,b]$.

Lemma 2.5[5] Let f, k be Sequential Henstock (SH)integrable functions on $[a,b]$, if $f \leq k$ is almost everywhere on $[a,b]$, then

$$\int_a^b f \leq \int_a^b k.$$

Definition 2.6 [11 and 15]
Let $I_{\mathbb{R}} = \{I = [I^-, I^+]$: I is a closed bounded interval on the real line $\mathbb{R}\}$.

For $X, Y \in I_{\mathbb{R}}$, we define
i. $X \leq Y$ if and only if $Y^- \leq X^-$ and $X^+ \leq Y^+$,
ii. $X + Y = Z$ if and only if $Z^- = X^- + Y^-$ and $Z^+ = X^+ + Y^+$,
iii. $X.Y = \{x.y : x \in X, y \in Y\}$, where

$$(X.Y)^- = \min\{X^-.Y^-, X^-.Y^+, X^+.Y^-, X^+.Y^+\}$$

and

$$(X.Y)^+ = \max\{X^-.Y^-, X^-.Y^+, X^+.Y^-, X^+.Y^+\}.$$

Define $d(X, Y) = \max(|X^- - Y^-|, |X^+ - Y^+|)$ as the distance between intervals X and Y.

Definition 2.7 [5]
An interval valued function $F : [a, b] \to L^p$ is Henstock integrable(l_p-$IH[a, b]$) to $I_0 \in L^p[0, 1]$ on $[a, b]$ if for every $\varepsilon > 0$ there exists a positive gauge function $\delta(x) > 0$ on $[a, b]$ such that for every $\delta(x) - fine$ tagged partitions $P = \{(u_{i-1}, u_i), t_i\}_{i=1}^n$, we have

$$\|\sum_{i=1}^{n\in\mathbb{N}} F(t_i)(u_i - u_{i-1}) - I_o\|_{L^p} < \varepsilon$$

We say that I_0 is the Henstock integral of F on $[a, b]$ with $(L^p[0, 1]\text{-}IH) \int_{[a,b]} F = I_0$ and $F \in L^p[0, 1]$-$IH[a, b]$.

Now, we will define the Sequential Henstock integral of $L^p[0, 1]$-interval valued function and then discuss some of the properties of the integral.

Definition 2.8
An interval valued function $F : [a, b] \to L^p$ is Sequential Henstock integrable($L^p[0, 1]$-$ISH[a, b]$) to $I_0 \in L^p[0, 1]$ on $[a, b]$ if for any $\varepsilon > 0$ there exists a sequence of positive gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that for every $\delta_n(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$, we have

$$\|\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - l_o\|_{L^p} < \varepsilon.$$

We say that $L^p[0, 1]$ is the Sequential Henstock integral of F on $[a, b]$ with $(L^p[0, 1]\text{-}ISH) \int_{[a,b]} F = \alpha$ and $F \in L^p[0, 1]$-$ISH[a, b]$.

In this section, we discuss some of the basic properties of the $L^p[0, 1]$-interval valued Sequential Henstock integrals.

Theorem 2.9
If $F \in L^p[0, 1]$-$ISH[a, b]$, then there exists a unique integral value.

Proof. Suppose the integral value are not unique. Let $\alpha_1 = (L^p[0, 1]\text{-}ISH) \int_{[a,b]} F$ and $\alpha_2 = (L^p[0, 1]\text{-}ISH) \int_{[a,b]} F$ with $\alpha_i \neq \alpha_2$. Let $\varepsilon > 0$ then there exists a $\{\delta_n^1(x)\}_{n=1}^{\infty}$ and $\{\delta_n^2(x)\}_{n=1}^{\infty}$ such that for each $\delta_n^1(x)$-fine tagged partitions P_n^1 of $[a, b]$ and for each $\delta_n^2(x)$-fine tagged partitions P_n^2 of $[a, b]$, we have

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha_1 \|_{L^p} < \frac{\varepsilon}{2},$$

and

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha_2 \|_{L^p} < \frac{\varepsilon}{2}.$$

respectively.
Define a positive gauge function $\delta_n(x)$ on $[a,b]$ by $\delta_n(x) = \min\{\delta_n^1(x), \delta_n^2(x)\}$. Let P_n be any $\delta_n(x)$-fine tagged partition of $[a,b]$ and let $\varepsilon = \frac{\|\alpha_1 - \alpha_2\|_p}{2^{\frac{1}{p}}}$. Then we have

$$\|\alpha_1 - \alpha_2\|_{L^p} = \| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha_1 + \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha_2 \|_{L^p}$$

$$\leq \| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha_1 \|_{L^p} + \| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha_2 \|_{L^p}$$

$$< 2^{\frac{1}{p}}(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}) = 2^{\frac{1}{p}}\varepsilon = \|\alpha_1 - \alpha_2\|_{L^p},$$

This is a contradiction. Thus $\alpha_1 = \alpha_2$. This completes the proof. ■

Theorem 2.10
An interval valued function $F \in L^p[0,1]$-$ISH[a,b]$ if and only if $F^-, F^+ \in L^p[0,1]$-$SH[a,b]$ and

$$(L^p[0,1]\text{-}ISH) \int_{[a,b]} F = [(l_p\text{-}SH) \int_{[a,b]} F^-, (L^p[0,1]\text{-}SH) \int_{[a,b]} F^+] \tag{2.1}$$

Proof. Let $F \in L^p[0,1]$-$ISH[a,b]$, from Definition 2.8 there is a unique interval number $I_o = [I_0^-, I_0^+]$ in the property, then for any $\varepsilon > 0$, there exists a $\{\delta_n(x)\}_{n=1}^{\infty}$, $n \geq \mu$ on $[a,b] \in \mathbb{R}$ such that for any $\delta_n(x)$-fine tagged partition P_n, we have

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - I_0 \|_{L^p} < \varepsilon.$$

Observe that

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - I_0 \|_{L^p} = \max(\| \sum_{i=1}^{m_n \in \mathbb{N}} F^-(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - I_0^- \|_{L^p},$$
$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F^+(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - I_0^+ \|_{L^p}.$$

Since $u_{i_n} - u_{(i-1)_n} \geq 0$ for $1 \leq i_n \leq m_n$, hen it follows that

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F^-(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - I_0^- \|_{L^p} < \varepsilon, \| \sum_{i=1}^{m_n \in \mathbb{N}} F^-(t_{i_n})(u_{i_n} - u_{(i-1)_n} - I_0^+)\|_p < \varepsilon.$$

for every $\delta_n(x)$-tagged partition $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$. Thus, by Definition 2.8, we obtain $F^+, F^- \in L^p[0,1]$-$SH[a,b]$ and

$$I_o^- = (L^p[0,1]\text{-}SH)\int_{[a,b]} F^-(x)dx$$

and

$$I_o^+ = (L^p[0,1]\text{-}SH)\int_{[a,b]} F^+(x)dx.$$

Conversely, Let $F^- \in L^p[0,1]$-$SH_{[a,b]}$. Then there exist a unique $\beta_1 \in \mathbb{R}$ with the property, let $\varepsilon > 0$ be given, then there exists a $\{\delta_n^1(x)\}_{n=1}^{\infty}$, such that for any $\delta_n^1(x)$-fine tagged partitions P_n^1 we have

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F^-(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \beta_1 \|_{L^p} < \varepsilon.$$

Similarly,
Let $F^+ \in L^p[0,1]$-$SH[a,b]$. Then there exist a unique $\beta_2 \in \mathbb{R}$ with the property, let $\varepsilon > 0$ be given, then there exists a $\{\delta_n^2(x)\}_{n=1}^{\infty}$, such that for any $\delta_n^2(x)$-fine tagged partitions P_n^2 we have

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F^+(t_{i_n})(u_{i_n} - u_{(i-1)_n} - \beta_2) \|_{L^p} < \varepsilon.$$

Let $\beta = [\beta_1, \beta_2]$. If $F^- \leq F^+$, then $\beta_1 \leq \beta_2$. We define $\delta_n(x) = \min(\delta_n^1(x), \delta_n^2(x))$, then for any $\delta_n(x) - fine$ tagged partitions P_n we have

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \beta \|_{L^p} < \varepsilon.$$

Hence, $F : [a,b] \to L^p$ is Sequential Henstock integrable on $[a,b]$.
This completes the proof.

■

Theorem 2.11
Let $F, K \in L^p[0,1]$-$ISH[a,b]$ with $F = [F^-, F^+]$ and $H = [K^-, K^+]$ and $\gamma, \xi \in \mathbb{R}$. Then $\gamma F, \xi K \in L^p[0,1]$-$ISH[a,b]$ and

$$(L^p[0,1]\text{-}ISH)\int_{[a,b]} (\gamma F + \xi K)dx = \gamma(L^p[0,1]\text{-}ISH)\int_{[a,b]} F dx + \xi(L^p[0,1]\text{-}ISH)\int_{[a,b]} K dx$$

Proof. (i) If $F, K \in L^p[0,1]$-$ISH[a,b]$, then $[F^-, F^+], K = [K^-, K^+] \in L^p[0,1]$-$SH[a,b]$ by Theorem 2.10. Hence, $\gamma F^- + \xi K^-, \gamma F^- + \xi K^+, \gamma F^+ + \xi K^-, \gamma F^+ + \xi K^+ \in L^p[0,1]$-$SH[a,b]$.
1) If $\gamma > 0$ and $\xi > 0$, then

$$\begin{aligned}(L^p[0,1]\text{-}SH)\int_{[a,b]} (\gamma F + \xi K)^- dx &= (L^p[0,1]\text{-}SH)\int_{[a,b]} (\gamma F^- + \xi K^-)dx \\ &= \gamma(L^p[0,1]\text{-}SH)\int_{[a,b]} F^- dx + \xi(L^p[0,1]\text{-}SH)\int_{[a,b]} K^- dx\end{aligned}$$

$$= \gamma((L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx)^- + \xi((L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx)^-$$
$$= (\gamma(L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx + \xi(L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx)^-.$$

2) If $\gamma < 0$ and $\xi > 0$, then

$$\begin{aligned}(L^p[0,1]\text{-}SH)\int_{[a,b]}(\gamma F+\xi K)^- dx &= (L^p[0,1]\text{-}SH)\int_{[a,b]}(\gamma F^+ + \xi K^+)dx\\ &= \gamma(L^p[0,1]\text{-}SH)\int_{[a,b]} F^+dx + \xi(L^p[0,1]\text{-}SH)\int_{[a,b]} K^+dx\\ &= \gamma((L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx)^+ + \xi((L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx)^+\\ &= (\gamma(L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx + \xi(L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx)^-.\end{aligned}$$

3) If $\gamma > 0$ and $\xi < 0$ (or $\gamma < 0$ and $\xi > 0$), then

$$\begin{aligned}(L^p[0,1]\text{-}ISH)\int_{[a,b]}(\gamma F+\xi K)^- dx &= (L^p[0,1]\text{-}SH)\int_{[a,b]}(\gamma F^- + \xi K^+)dx\\ &= \gamma(L^p[0,1]\text{-}SH)\int_{[a,b]} F^-dx + \xi(L^p[0,1]\text{-}SH)\int_{[a,b]} K^+dx\\ &= \gamma((L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx)^- + \xi((L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx)^+\\ &= (\gamma(L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx + \xi(L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx)^-.\end{aligned}$$

Similarly, for four cases above, we have

$$(L^p[0,1]\text{-}ISH)\int_{[a,b]}(\gamma F+\xi K)^+ dx = (\gamma(L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx + \xi(L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx)^+$$

Hence, by Theorem 2.10, $\gamma F, \xi K \in L^p[0,1]\text{-}ISH[a,b]$ and

$$(L^p[0,1]\text{-}ISH)\int_{[a,b]}(\gamma F+\xi K)dx = \gamma(L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx + \xi(L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx.$$

This completes the proof. ■

Theorem 2.12
Let $F, K \in L^p[0,1]\text{-}ISH[a,b]$ and $F(x) \le K(x)$ nearly everywhere on $[a,b]$, then

$$(L^p[0,1]\text{-}ISH)\int_{[a,b]} F(x)dx \le (L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx$$

Proof. If $F(x) \le K(x)$ nearly everywhere on $[a,b]$ and $F, K \in L^p[0,1]\text{-}ISH[a,b]$, then $F^-, F^+, K^-, K^+ \in L^p[0,1]\text{-}SH[a,b]$ and $F^- \le F^+, K^- \le K^+$ nearly everywhere on $[a,b]$. By Lemma 2.5

$$(L^p[0,1]\text{-}SH)\int_{[a,b]} F^-(x)dx \le (L^p[0,1]\text{-}SH)\int_{[a,b]} K^-dx$$

and

$$(L^p[0,1]\text{-}ISH)\int_{[a,b]} F^+(x)dx \leq (L^p[0,1]\text{-}ISH)\int_{[a,b]} K^+dx.$$

Hence by Theorem 2.10, we have

$$(L^p[0,1]\text{-}ISH)\int_{[a,b]} F(x)dx \leq (L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx.$$

This completes the proof. ■

Theorem 2.13 Let $k \in \mathbb{R}$.
1. If $F \in L^p[0,1]\text{-}ISH[a,b]$, then $kF \in L^p[0,1]\text{-}ISH[a,b]$. Moreover,

$$\int_a^b kF = k\int_a^b F.$$

2. If $F \in L^p[0,1]\text{-}ISH[a,b]$ and $G \in L^p[0,1]\text{-}ISH[c,b]$, then $(F+G) \in L^p[0,1]\text{-}ISH[a,b]$. Moreover

$$\int_a^b (F+G) = \int_a^b F + \int_a^b G.$$

Proof. (1) Suppose $F \in L^p[0,1]\text{-}ISH[a,b]$. The case $k = 0$ is obvious. Suppose $k \neq 0$ and $F \in L^p[0,1]\text{-}ISH[a,b]$, there exists a sequence of positive functions$\{\delta_n(x)\}_{n=1}^{\infty}$ on $[a,b]$ such that

$$\| \sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n} - \int_a^b F\|_{L^p} < \frac{\varepsilon}{|k|_{L^p}}$$

whenever P_n is $\delta_n(x)-fine$ tagged partitions of $[a,b]$. Then, exists a sequence of positive functions $\{\delta_n^2(x)\}_{n=1}^{\infty}$ on $[a,c]$ such that

$$\begin{aligned}\| \sum_{i=1}^{m_n\in\mathbb{N}} kF(t_{i_n})(u_{i_n} - u_{(i-1)_n} - k\int_a^b F\|_{L^p} &= \|k\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n} - k\int_a^b F\|_{L^p} \\ &< |k|_{L^p}\frac{\varepsilon}{|k|_{L^p}} \\ &= \varepsilon.\end{aligned}$$

(2) Let $\varepsilon > 0$ Suppose $\int_a^b F = \alpha_1$ and $\int_a^b G = \alpha_2$. Then there exists a sequence of positive functions$\{\delta_n^1(x)\}_{n=1}^{\infty}$ on $[a,b]$ such that

$$\| \sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n} - \alpha_1\|_{L^p} < \frac{\varepsilon}{2(2^{\frac{1}{p}})}$$

whenever P_n^1 is $\delta_n^1(x) - fine$ tagged partitions of $[a,b]$. Also, there exists a sequence of positive functions $\{\delta_n^2(x)\}_{n=1}^{\infty}$ on $[a,b]$ such that

$$\| \sum_{i=1}^{m_n\in\mathbb{N}} G(t_{i_n})(u_{i_n} - u_{(i-1)_n} - \alpha_2\|_{L^p} < \frac{\varepsilon}{2(2^{\frac{1}{p}})}$$

whenever P_n^2 is $\delta_n^2(x) - fine$ tagged partitions of $[a,b]$.
Define a positive gauge function $\delta_n(x)$ on $[a,b]$ by $\delta_n(x) = \min\{\delta_n^1(x), \delta_n^2(x)\}$. Let P_n be any $\delta_n(x)$-fine tagged partition of $[a,b]$. Then

$$\| \sum_{i=1}^{m_n\in\mathbb{N}} (F+G)(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - (\alpha_1+\alpha_2)\|_{L^p}$$

$$
\begin{aligned}
&= (\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) + \sum_{i=1}^{m_n \in \mathbb{N}} G(t_{i_n})(u_{i_n} - u_{(i-1)_n} - (\alpha_1 + \alpha_2) \|_{L^p}) \\
&\leq 2^{\frac{1}{p}} (\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n} - \alpha_1 \|_{L^p}) + 2^{\frac{1}{p}} (\| \sum_{i=1}^{m_n \in \mathbb{N}} G(t_{i_n})(u_{i_n} - u_{(i-1)_n} - \alpha_2 \|_{L^p}) \\
&< 2^{\frac{1}{p}} (\frac{\varepsilon}{2(2\frac{1}{p})} + \frac{\varepsilon}{2(2\frac{1}{p})}) \\
&= \varepsilon.
\end{aligned}
$$

This completes the proof. ■

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