ISSN 2319-3786
VOLUME 12, ISSUE 4, OCTOBER 2024

M
J
M

Malaya Journal of Matematik

an international journal of mathematical sciences

MKD PUBLISHING HOUSE
Dawn To Researchers
5, Venus Garden, Sappanimadai Road, Karunya Nagar (Post),
Coimbatore- 641114, Tamil Nadu, India.
www.mkdpress.com | www.malayajournal.org

The Malaya Journal of Matematik is published quarterly in single volume annually and four issues constitute one volume appearing in the months of January, April, July and October.

Subscription

The subscription fee is as follows:

USD 350.00 For USA and Canada

Euro 190.00 For rest of the world

Rs. 4000.00 In India. (For Indian Institutions in India only)

Prices are inclusive of handling and postage; and issues will be delivered by Registered Air-Mail for subscribers outside India.

Subscription Order

Subscription orders should be sent along with payment by Cheque/ D.D. favoring "Malaya Journal of Matematik" payable at COIMBATORE at the following address:

MKD Publishing House

5, Venus Garden, Sappanimadai Road, Karunya Nagar (Post),

Coimbatore- 641114, Tamil Nadu, India.

Contact No. : +91-9585408402

E-mail : info@mkdpress.com; editorinchief@malayajournal.org; publishingeditor@malayajournal.org

Website : https://mkdpress.com/index.php/index/index

Vol. 12 No. 04(2024): Malaya Journal of Matematik (MJM)

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **12(04)**(2024), 330–338.
http://doi.org/10.26637/mjm1204/001

Nonlinear partial completely continuous operators in a partially ordered Banach space and nonlinear hyperbolic partial differential equations

Bapurao C. Dhage[*1]
[1] *Kasubai, Gurukul Colony, Thodga Road, Ahmedpur - 413515, Dist. Latur, Maharashtra, India.*

Received 11 April 2024; Accepted 17 August 2024

Abstract. We prove a hybrid fixed point theorem for partial completely continuous operators in a partially ordered metric space and derive an applicable hybrid fixed point result in an ordered Banach space as a special case. As an application, we discuss a nonlinear hyperbolic partial differential equation for approximation result of local solutions by constructing the algorithms. Finally, an example is indicated to elaborate the hypotheses and abstract result of this paper.
AMS Subject Classifications: 47H10, 35A35

Keywords: Partially ordered metric space; Hybrid fixed point principle; Hyperbolic partial differential equation; Dhage iteration method; Local approximation result.

Contents

1. Introduction

Relaxing the convexity condition of the well-known Schauder fixed point theorem in a Banach space, the present author in Dhage [6] proved the following hybrid fixed point theorem in a partially ordered Banach space.

Theorem 1.1. *Let S be a non-empty, partially compact subset of a regular partially ordered Banach space $\big(X, \|\cdot\|, \preceq\big)$ and let every chain C in S be a Janhavi set. Suppose that $\mathcal{T} : S \to S$ is a partially continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has fixed point ξ^* and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 1.1 yields an applicable hybrid fixed point theorem in an ordered Banach space having numerous applications to nonlinear analysis. See Dhage [4], Dhage *et al.* [9–12], Dhage and Dhage [7], Dhage *et al.* [8] and references therein. Note that Theorem 1.1 removes the convexity condition from Schauder fixed pint theorem and replaced it by monotonicity condition of the operator in question. However, as a result we obtain an additional feature that it gives the algorithms which can be used to obtain the approximation of solution to the nonlinear problems. Now, the problem with the above hybrid fixed point theorem is that it is difficult to find the partially compact subset of an ordered Banach space always. To overcome this difficulty, here we relax the condition of existence of a partially compact subset and replace it by partial complete continuity of the operator $\mathcal{T}$ on S which is the main motivation of the present paper.

[*] **Corresponding author.** Email address: bcdhage@gmail.com (Bapurao C. Dhage)

2. A Hybrid Fixed Point Principle

Before going to the main hybrid fixed point result, we give some preliminary definitions which we need in what follows. The details appear in Dhage [4, 5] and references therein.

Let $(E, d, \preceq)$ be a partially ordered metric space and let $S \subset E$. E is called **regular** if a monotone nondecreasing (resp. monotone nonincreasing) sequence $\{x_n\}$ in E converges to x_*, then $x_n \preceq x_*$ (resp. $x_* \preceq x_n$) for all $n \in \mathbb{N}$. The metric d and the order relation $\preceq$ are said to be **compatible** in S if a monotone sequence $\{x_n\}$ in S has a convergent subsequence, then the original sequence $\{x_n\}$ is convergent and converges to the same limit point. S is called a **Janhavi set** if d and $\preceq$ are compatible in it. S is called **partial bounded** (resp. partially closed, partially compact) if every chain C in S is bounded (resp. closed, compact).

A mapping $\mathcal{T} : S \to S$ is called **monotone nondecreasing** (resp. monotone nonincreasing) if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ (resp. $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$). $\mathcal{T}$ is **monotone** if it is either monotone nondereasing or monotone nonincreasing. $\mathcal{T}$ is called **partial bounded** (resp. partially totally bounded or partially precompact) if $\mathcal{T}(S)$ is partially bounded (resp. partially totally bounded or partially precompact for partially bounded S). T is **partially continuous** if $\{x_n\} \subset S$ converges to x_* with $x_n \preceq x_*$, then $\mathcal{T}x_n \to \mathcal{T}x$. T is called **partial completely continuous** if it is partially continuous and partially totally bounded.

Now we are equipped with all the necessary details to state our main result if this section.

Theorem 2.1. *Let S be a non-empty, partial closed and partial bounded subset of a regular partially ordered complete metric space $(E, d, \preceq)$ and let every chain C in S be Janhavi set. Suppose that $\mathcal{T} : S \to S$ is a partial completely continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point ξ^* and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Proof. Assume first that we have an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ and define a sequence $\{x_n\}_{n=0}^{\infty}$ of points in S by

$$x_{n+1} = \mathcal{T}x_n, \ n = 0, 1, 2, \dots. \tag{2.1}$$

From the monotonic nndecreasing nature of $\mathcal{T}$, it follows that $\{x_n\}_{n=0}^{\infty}$ is a nondecreasing sequence of point in S, i.e., we have

$$x_0 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots. \tag{2.2}$$

Consequently, $\{x_n\}_{n=0}^{\infty}$ is a chain in S. Denote $C = \{x_n\}_{n=0}^{\infty}$. Then, C is bounded and by the construction of $\{x_n\}_{n=0}^{\infty}$, we have

$$\begin{aligned} C &= \{x_0, x_1, x_2, \dots\} \\ &= \{x_0\} \cup \{x_1, x_2, \dots\} \\ &= \{x_0\} \cup \mathcal{T}(C). \end{aligned} \tag{2.3}$$

As $\mathcal{T}$ is partially completely continuous, we have that $\overline{\mathcal{T}(C)}$ is compact. From (2.3), $\overline{C}$ is also a compact set in S. As a result, $\{x_n\}_{n=0}^{\infty}$ has a convergent subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ converging to a point, say, ξ^*. By hypothesis, $C = \{x_n\}_{n=0}^{\infty}$ is a Janhavi set in S, so the original sequence $\{x_n\}_{n=0}^{\infty}$ converges monotone nondecreasingly to ξ^*. Since $(E, \preceq, d)$ is a regular, we have that $x_n \to \xi^*$ and that $x_n \preceq \xi^*$ for all $n \in \mathbb{N}$. Finally, from partial continuity of $\mathcal{T}$, it follows that

$$\xi^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \mathcal{T}x_n = \mathcal{T}\left(\lim_{n\to\infty} x_n\right) = \mathcal{T}\xi^*.$$

Similarly, if $x_0 \succeq \mathcal{T}x_0$, it can be shown using analogous arguments that $\mathcal{T}$ has a fixed point ξ^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations converges monotone nonincreasingly to ξ^* Thus, in both the cases $\mathcal{T}$ has a fixed point ξ^* and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*. This completes the proof. □

Corollary 2.2. *Let S be a non-empty, partial closed and partial bounded subset of a regular partially ordered Banach space $\big(X, \|\cdot\|, \preceq\big)$ and let every chain C in S be Janhavi set. Suppose that $\mathcal{T}: S \to S$ is a partial completely continuous and monotone nondecreasing mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point ξ^* and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

If the Banach X is partially ordered by an order cone K in X, then in this case, we simply say that X is an ordered Banach space and we denote it by (X, K). The details of order cones and related fixed point theorems appear in the monographs Guo and Lakshmikantham [13] and Granas and Dugundji [14]. Then, we have the following useful results proved in Dhage [4, 5].

Lemma 2.3 (Dhage [4, 5])**.** *Every ordered Banach space (X, K) is regular.*

Lemma 2.4 (Dhage [4, 5])**.** *Every partially compact subset S of an ordered Banach space (X, K) is a Janhavi set in X.*

As a consequence of Lemmas 2.3 and 2.4 we obtain an applicable hybrid fixed point theorem in the area of nonlinear analysis and applications.

Theorem 2.5. *Let S be a non-empty, partially closed and partially bounded subset of an ordered Banach space (X, K) and let $\mathcal{T}: S \to S$ be a partially completely continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then $\mathcal{T}$ has a fixed point $\xi^* \in S$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 2.5 is an improvement of the following hybrid fixed point theorem of Dhage *et al.* [9] which is comparatively more convenient for applications to nonlinear equations.

Theorem 2.6 (Dhage *et al.* [9])**.** *Let S be a non-empty and partially compact subset of an ordered Banach space (X, K) and let $\mathcal{T}: S \to S$ be a partially continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then $\mathcal{T}$ has a fixed point $\xi^* \in S$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Remark 2.7. *We mention that Theorem 2.5 is an ordered Banach space version of the Schauder fixed point theorem wherein the convexity argument is altogether omitted and replaced by the monotonicity of the operator in question. The advantage of this approach over Schauder is that we obtain an algorithm which goes to the fixed point when applied repeatedly.*

3. Hyperbolic Partial Differential Equations

Given the closed and bunded intervals $J_a = [0, a]$ and $J_b = [0, b]$ in the real line $\mathbb{R}$, for some real numbers $a > 0$ and $b > 0$, consider the nonlinear IVP of hyperbolic partial differential equation (in short HPDE)

$$\frac{\partial^2 u}{\partial x \partial y} = f(x, y, u(x, y)), \ (x, y) \in J_a \times J_b, \tag{3.1}$$

satisfying the initial conditions

$$u(x, 0) = \phi(x) \quad \text{and} \quad u(0, y) = \psi(y), \tag{3.2}$$

where $f : J_a \times J_b \times \mathbb{R} \to \mathbb{R}$, $\phi : J_a \to \mathbb{R}$ and $\psi : J_b \to \mathbb{R}$ are continuous functions.

Definition 3.1. *By a solution of the HPDE* (3.1)-(3.2) *we mean a function $u \in C(J_a \times J_b, \mathbb{R})$ that satisfies the equations in* (3.1)-(3.2)*, where $C(J_a \times J_b, \mathbb{R})$ is the space of continuous real-valued functions defined on $J_a \times J_b$. If a solution u exists in a neighbourhood of a point $z \in C(J_a \times J_b, \mathbb{R})$, then we say that it is a local or neighbourhood solution of the HPDE* (3.1)-(3.2) *defined on $J_a \times J_b$.*

The HPDE (3.1)-(3.2) is fundamental in the theory of nonlinear hyperbolic partial differential equations and widely discussed in the literature for existence of solution. See Lakshmikantham and Pandit [15] and references therein. But to the knowledge of present author no approximation result is proved for local solution without the assumption of Lipschitz condition on the function f or without the assumption of existence of both lower as well as upper solution for the HPDE (3.1)-(3.2) on $J_a \times J_b$. Therefore, the approximation result of this section seem to be new to the theory of hyperbolic partial differential equations.

We put the HPDE (3.1)-(3.2) in the Banach space $C(J_a \times J_b, \mathbb{R})$. We introduce a supremum norm $\| \cdot \|$ in $C(J_a \times J_b, \mathbb{R})$ defined by

$$\|u\| = \sup_{(x,y)\in J_a \times J_b} |u(x,y)|. \tag{3.3}$$

and an order relation $\preceq$ in $C(J_a \times J_b, \mathbb{R})$ by the cone K given by

$$K = \{u \in C(J_a \times J_b, \mathbb{R}) \mid u(x,y) \geq 0 \; \forall \; (x,y) \in J_a \times J_b\}. \tag{3.4}$$

Thus,

$$u \preceq v \iff v - u \in K, \tag{3.5}$$

or equivalently,

$$u \preceq v \iff u(x,y) \leq v(x,y) \; \forall \; (x,y) \in J_a \times J_b.$$

It is known that the Banach space $C(J_a \times J_b, \mathbb{R})$ together with the order relations $\preceq$ becomes an ordered Banach space which we denote for convenience, by $\big(C(J_a \times J_b, \mathbb{R}), K\big)$. We denote the open and closed spheres centred at $z_0 \in C(J_a \times J_b, \mathbb{R})$ of radius r by

$$B_r(z_0) = \{u \in C(J_a \times J_b, \mathbb{R}) \mid \|u - z_0\| < r\} = B(z_0, r),$$

and

$$B_r[z_0] = \{u \in C(J, \mathbb{R}) \mid \|u - z_0\| \leq r\} = \overline{B(z_0, r)},$$

respectively.

Remark 3.2. *It is clear that an open ball $B(z_0, r)$ in $C(J_a \times J_b, \mathbb{R})$ centred at a point $z_0 \in C(J_a \times J_b, \mathbb{R})$ of radius $r > 0$ is a neighbourhood of the point z_0, so if a solution u^* of the HPDE (3.1)-(3.2) lies in a closed ball $\overline{B(z_0, r)}$ in $C(J_a \times J_b, \mathbb{R})$, then it is a* local solution *in view of the fact that $B(z_0, r) \subset \overline{B(z_0, r)} \subset B(z_0, r + \epsilon)$ for every $\epsilon > 0$. Note that the idea of local or nbhd-solution is different from the usual notion of a local solution as mentioned in Coddington [1].*

4. Local Approximation Results

We consider the following definition in the sequel.

Definition 4.1. *A function $f : J_a \times J_b \times \mathbb{R} \to \mathbb{R}$ is said to be $L^1_{\mathbb{R}}$-Carathéodory if*

(i) the map $(x, y) \mapsto f(x, y, u)$ is jointly measurable for each $u \in \mathbb{R}$,

(ii) the map $u \mapsto f(x, y, u)$ is continuous for each $(x, y) \in J_a \times J_b$, and

(iii) there exists a function $h \in L^1(J_a \times J_b, \mathbb{R})$ such that

$$|f(x, y, u)| \leq h(x, y) \text{ a.e. } (x, y) \in J_a \times J_b,$$

for all $u \in \mathbb{R}$.

Lemma 4.2 (Granas and Dugundji [14]). *If $f(x,y,u)$ is $L^1_{\mathbb{R}}$-Carathéodory, then the function $(x,y) \mapsto f(x,y,u(x,y))$ is jointly measurable for each $u \in C(J_a \times J_b, \mathbb{R})$.*

We need the following hypotheses in what follows.

(H_1) The function f is $L^1_{\mathbb{R}}$-Carathéodory.

(H_2) $f(x,y,u)$ is nondecreasing in u for each $(x,y) \in J_a \times J_b$.

(H_3) $f(x,y,z_0(x,y)) \geq 0$ for all $(x,y) \in J_a \times J_b$, where $z_0(x,y) = \psi(y) + \phi(x) - \phi(0)$.

Now, by using the theory of partial differentiation and integration, we obtain the following useful result.

Lemma 4.3. *If $h \in L^1(J_a \times J_b, \mathbb{R})$, then the IVP of ordinary second order linear hyperbolic partial differential equation*

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial y} &= h(x,y), \;\; (x,y) \in J_a \times J_b, \\ u(x,0) = \phi(x) \quad &\textit{and} \quad u(0,y) = \psi(y), \end{aligned} \tag{4.1}$$

is equivalent to the integral equation

$$u(x,y) = z_0(x,y) + \int_0^x \int_0^y h(s,t)\, ds\, dt, \;\; , (x,y) \in J_a \times J_b, \tag{4.2}$$

where $z_0(x,y) = \psi(y) + \phi(x) - \phi(0)$ is a continuous function on $J_a \times J_b$.

Theorem 4.4. *Suppose that the hypotheses (H_1), (H_2) and (H_3) hold. Furthermore, if $\|h\|_{L^1} \leq r$, then the HPDE (3.1)-(3.2) has a local solution u^* in $B_r[z_0]$ and the sequence $\{u_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$\left.\begin{aligned} u_0(x,y) &= z_0(x,y), \;\; (x,y) \in J_a \times J_b, \\ u_{n+1}(x,y) &= z_0(x,y) + \int_0^x \int_0^y f(s,t,u_n(s,t))\, ds\, dt, \;\; (x,y) \in J_a \times J_b, \end{aligned}\right\} \tag{4.3}$$

where $n = 0, 1, \ldots$; is monotone nondecreasing and converges to u^*.

Proof. Set $X = C(J_a \times J_b, \mathbb{R})$. Clearly, X is an ordered Banach space ordered by the cone K defined by (2.2). Let u_0 be a function on $J_a \times J_b$ such that $u_0 \equiv z_0$ on $J_a \times J_b$. Define a closed ball $B_r[z_0]$ in X, where $r \geq \|h\|_{L^1}$. By Lemma 4.2, the HPDE (3.1)-(3.2) is equivalent to the nonlinear hybrid integral equation (HIE)

$$u(x,y) = z_0(x,y) + \int_0^x \int_0^y f(s,t,u(s,t))\, ds\, dt, \;\; , (x,y) \in J_a \times J_b. \tag{4.4}$$

Now, define an operator $\mathcal{T}$ on $B_r[u_0]$ into X by

$$\mathcal{T}u(x,y) = z_0(x,y) + \int_0^x \int_0^y f(s,t,u(s,t))\, ds\, dt, \;\; , (x,y) \in J_a \times J_b. \tag{4.5}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.5 on $B_r[u_0]$ in the following series of steps.

Step I: *The operator $\mathcal{T}$ maps $B_r[z_0]$ into itself.*

Firstly, we show that $\mathcal{T}$ maps $B_r[z_0]$ into itself. Let $u \in B_r[z_0]$ be arbitrary element. Then, by hypothesis (H_1),

$$\begin{aligned}|\mathcal{T}u(x,y) - z_0(x,y)| &= \left|\int_0^x \int_0^y f(s,t,u(s,t))\,ds\,dt\right| \\ &\leq \int_0^x \int_0^y \big|f(s,t,u(s,t))\big|\,ds\,dt \\ &\leq \int_0^x \int_0^y h(s,t)\,ds\,dt \\ &\leq \|h\|_{L^1}.\end{aligned}$$

Taking the supremum over x and y in the above inequality yields

$$\|\mathcal{T}u - z_0\| \leq \|h\|_{L^1} = r$$

which implies that $\mathcal{T}u \in B_r[z_0]$ for all $u \in B_r[z_0]$.

Step II: *$\mathcal{T}$ is a monotone nondecreasing operator.*

Let $u, v \in B_r[z_0]$ be any two elements such that $u \succeq v$ on $J_a \times J_b$. Then,

$$\begin{aligned}\mathcal{T}u(x,y) &= z_0(x,y) + \int_0^x \int_0^y f(s,t,u(s,t))\,ds\,dt \\ &\geq z_0(x,y) + \int_0^x \int_0^y f(s,t,v(s,t))\,ds\,dt \\ &= \mathcal{T}v(x,y)\end{aligned}$$

for all $(x,y) \in J_a \times J_b$. So, $\mathcal{T}u \succeq \mathcal{T}v$, that is, $\mathcal{T}$ is monotone nondecreasing on $B_r[x_0]$.

Step III: *$\mathcal{T}$ is partially continuous operator.*

Let C be a chain in $B_r[z_0]$ and let $\{u_n\}$ be a sequence of points in C converging to a point $u \in C$. Then, by dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{T}u_n(x,y) &= \lim_{n\to\infty}\left[z_0(x,y) + \int_0^x \int_0^y f(s,t,u_n(s,t))\,ds\,dt\right] \\ &= z_0(x,y) + \lim_{n\to\infty}\int_0^x \int_0^y f(s,t,u_n(s,t))\,ds\,dt \\ &= z_0(x,y) + \int_0^x \int_0^y \Big[\lim_{n\to\infty} f(s,t,u_n(s,t))\Big]\,ds\,dt \\ &= z_0(x,y) + \int_0^x \int_0^y f(s,t,u(s,t))\,ds\,dt \\ &= \mathcal{T}u(x,y)\end{aligned}$$

for all $(x,y) \in J_a \times J_b$. Therefore, $\mathcal{T}u_n \to \mathcal{T}u$ pointwise on $J_a \times J_b$.

Next, we shows that $\mathcal{T}u_n$ is an equicontinuous sequence of functions on on the compact $J_a \times J_b$. Let $(x_1,y_1), (x_2,y_2) \in J_a \times J_b$ be arbitrary. Without loss of generality, we assume that $x_1 \leq x_2$ and $y_1 \leq y_2$. Then, by definition of $\mathcal{T}$, we have that

$$\begin{aligned}
&|\mathcal{T}u_n(x_1, y_1) - \mathcal{T}u_n(x_2, y_2)| \\
&\quad \leq |z_0(x_1, y_1)| - z_0(x_2, y_2)| + \int_{x_1}^{x_2} \int_{y_1}^{y_2} |f(s, t, u_n(s, t))| \, ds \, dt \\
&\quad \leq |z_0(x_1, y_1)| - z_0(x_2, y_2)| + \int_{x_1}^{x_2} \int_{y_1}^{y_2} h(s, t) \, ds \, dt \\
&\quad \to 0 \quad \text{as} \quad (x_1, y_1) \to (x_2, y_2), \qquad (4.6)
\end{aligned}$$

uniformly for all n, $n = 1, 2, \ldots$. This shows that $\mathcal{T}u_n$ is an equicontinuous sequence of functions on $J_a \times J_b$. As a result, we have that $\mathcal{T}u_n \to \mathcal{T}u$ uniformly on $J_a \times J_b$. Hence $\mathcal{T}$ is partially continuous operator on $B_r[z_0]$.

Step IV: *$\mathcal{T}$ is partially totally bounded.*

Firstly, we show that $\mathcal{T}$ is partially uniformly bounded. Let C be a chain in $B_r[z_0]$. Then, by monotonicity of $\mathcal{T}$, the set $\mathcal{T}(C)$ is again a chain in $\mathcal{T}\big(B_r[z_0]\big)$. Let $v \in \mathcal{T}(C)$ be arbitrary. Then, there is a point $u \in C$ such that $v(x, y) = \mathcal{T}u(x, y)$. Now, by hypothesis ($H_1$),

$$\begin{aligned}
|v(x, y)| &= \big|\mathcal{T}u(x, y)\big| \\
&\leq |z_0(x, y)| + \int_0^x \int_0^y |f(s, t, u(s, t))| \, ds \, dt \\
&\leq \|z_0\| + \int_0^x \int_0^y h(s, t) \, ds \, dt \\
&\leq \|z_0\| + \|h\|_{L^1} \qquad (4.7)
\end{aligned}$$

for all $(x, y) \in J_a \times J_b$. Taking the supremum over (x, y), we obtain $\|v\| \leq \|z_0\| + \|h\|_{L^1}$ for all $v \in \mathcal{T}(C)$. This shows that $\mathcal{T}$ is a partially uniformly bounded on $B_r[z]$. Next, proceeding as in the step III, it can be proved that $\mathcal{T}(C)$ is an equicontinuous chain of points in $\mathcal{T}\big(B_r[z_0]\big)$. As $\mathcal{T}(C)$ is uniformly bounded and equicontinuous set, it is precompact. Consequently $\mathcal{T}$ is partially precompact or partially totally bounded operator on $B_r[z_0]$. Now $\mathcal{T}$ is partially continuous and partially totally bounded, so it is partially completely continuous on $B_r[z_0]$.

Step V: *The element $u_0 = z_0 \in B_r[z_0]$ satisfies the order relation $u_0 \preceq \mathcal{T}u_0$.*

Since (H_3) holds, one has

$$\begin{aligned}
u_0(x, y) &= z_0(x, y) + \int_0^x \int_0^y f(s, t, u_0(s, t)) \, ds \, dt \\
&\leq u_0(x, y) + \int_0^x \int_0^y f(s, t, z_0(s, t) \, ds \, dt \\
&= z_0(x, y) + \int_0^x \int_0^y f(s, u_0(s, t)) \, ds \, dt \\
&= \mathcal{T}u_0(x, y)
\end{aligned}$$

for all $(x, y) \in J_a \times J_b$. As a result, we have $u_0 \preceq \mathcal{T}u_0$ on $J_a \times J_b$.

Thus, the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.5, and so $\mathcal{T}$ has a fixed point u^* in $B_r[z_0]$ and the sequence $\{\mathcal{T}^n u_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to u^*. This further implies that the HIE (3.4) and consequently the HPDE (3.1)-(3.2) has a local solution u^* and the sequence $\{u_n\}_{n=0}^{\infty}$ of successive approximations defined by (4.3) converges monotone nondecreasingly to u^*. This completes the proof. $\square$

Remark 4.5. *The conclusion of Theorems 4.4 also remains true if we replace the hypothesis (H$_3$) with the following one.*

(H$_4$) The function f satisfies $f(x, y, z_0(x, y)) \leq 0$ for all $(x, y) \in J_a \times J_b$.

In this case, the HPDE (3.1)-(3.2) has a local solution x^* defined on $J_a \times J_b$ and the sequence $\{u_n\}_{n=0}^{\infty}$ of successive approximations defined by (4.3) is monotone nonincreasing and converges to the solution u^*.

Remark 4.6. *If the initial condition (3.2) is such that $z_0(x, y) > 0$ for all $(x, y) \in J_a \times J_b$, then under the conditions of Theorem 4.4, the HPDE (3.1)-(3.2) has a local positive solution u^* defined on $J_a \times J_b$ and the sequence $\{u_n\}_{n=0}^{\infty}$ of successive approximations defined by (4.3) converges monotone nondecreasingly to u^*.*

Finally, we give an example to illustrate the abstract ideas involved in our main approximation result, Theorems 4.4.

Example 4.7. *Given a closed and bounded interval $J_1 = [0, 1]$ in $\mathbb{R}$, consider the IVP of nonlinear second order HPDE,*

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x \partial y} &= (x + y) \tanh u(x, y), \\ u(x, 0) &= \frac{x}{2} \ \text{ and } \ u(0, y) = \frac{y}{2}, \end{aligned}\right\} \tag{4.8}$$

for all $(x, y) \in [0, 1] \times [0, 1]$.

Here, $f(x, y, u) = (x + y) \tanh u$, $\phi(x) == \frac{x}{2}$ and $\psi(y) == \frac{y}{2}$ for $(x, y) \in [0, 1] \times [0, 1]$ and $u \in \mathbb{R}$. We show that f satisfies all the conditions of Theorem 4.4. Clearly, f is $L^!_r$-Carathéodory on $[0, 1] \times [0, 1] \times \mathbb{R}$ with $h(x, y) = x + y$, and so the hypothesis (H$_1$) is satisfied. Also the function $f(x, y, u)$ is nondecreasing in u for each $(x, y) \in [0, 1] \times [0, 1]$. Therefore, hypothesis (H$_2$) is satisfied. Next, we have $z_0(x, y) = \frac{x}{2} + \frac{y}{2}$. Therefore, $f(x, y, z_0(x, y)) = (x + y) \tanh \left(\frac{x+y}{2}\right) \geq 0$ for each $(x, y) \in [0, 1] \times [0, 1]$, and so the hypothesis (H$_3$) holds. Now, by an application of Theorem 4.4, the HPDE (4.8) has a local solution u^* in the closed ball $B_1[z_0]$ of $C([0, 1] \times [0, 1], \mathbb{R})$ which is positive in view of Remark 4.6. Furthermore, the sequence $\{u_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$\begin{aligned} u_0(t) &= \frac{x + y}{2}, \quad (x, y) \in [0, 1] \times [0, 1], \\ u_{n+1}(t) &= \frac{x + y}{2} + \int_0^x \int_0^y (t + s) \tanh u_n(s, t) \, ds \, dt, \quad (s, t) \in [0, 1] \times [0, 1], \end{aligned}$$

converges monotone nondecreasingly to u^*.

5. The Comparison

We observe that the existence of solutions of the HPDE (3.1) can also be obtained by an application of topological Schauder fixed point principle under the hypothesis (H$_1$) and restricted domain of intervals of the problem, but in that case we do not get any sequence of successive approximations that converges to the solution. Again, we can not apply analytical or geometric Banach contraction mapping principle to the problem (3.1) under the considered hypotheses (H$_1$)-(H$_3$) in order to get the desired conclusion, because here the nonlinear function f does not satisfy the usual Lipschitz condition on the domain $J_a \times J_b \times \mathbb{R}$. Similarly, we can not apply algebraic Tarski's fixed point theorem [16] or its extension obtained in Dhage [3] to HPDE (3.1) for proving the existence of solution, because the ordered Banach space $\big(C(J_a \times J_b, \mathbb{R}), \preceq\big)$ is not a complete lattice (see Davis [2]). Therefore, all these arguments show that our hybrid fixed point principle, Theorem 2.1 is very much advantageous to get more information about the solution of nonlinear equations in the subject of nonlinear analysis.

References

[1] E.A. Coddington, *An Introduction to Ordinary Differential Equations*, Dover Publications Inc. New York, 1989.

[2] A.C. Davis, A characterization of complete lattices, *Pacific J. Math.*, **5** (1965), 311-319.

[3] B.C. Dhage, An extension of lattice fixd point theorem and its applications, *Pure Appl. Math. Sci*, **25** (1987), 37-42.

[4] B.C. Dhage, A coupled hybrid fixed point theorem for sum of two mixed monotone coupled operators in a partially ordered Banach space with applications, *Tamkang J. Math.*, **50**(1) (2019), 1-36.

[5] B.C. Dhage, Coupled and mixed coupled hybrid fixed point principles in a partially ordered Banach algebra and PBVPs of nonlinear coupled quadratic differential equations, *Differ. Equ. Appl.*, **11**(1) (2019), 1-85.

[6] B.C. Dhage, A Schauder type hybrid fixed point theorem in a partially ordered metric space with applications to nonlinear functional integral equations, *Jñānābha*, **52** (2) (2022), 168-181.

[7] B.C. Dhage, S.B. Dhage, Approximating solutions of nonlinear first order ordinary differential equations, *GJMS Special Issue for Recent Advances in Mathematical Sciences and Applications-13, GJMS*, **2** (2) (2013), 25-35.

[8] B.C. Dhage, S.B. Dhage, S.K. Ntouyas, Approximating solutions of nonlinear hybrid differential equations, *Appl. Math. Lett.* **34** (2014), 76-80.

[9] B.C. Dhage, J.B. Dhage, S.B. Dhage, Approximating existence and uniqueness of solution to a nonlinear IVP of first order ordinary iterative differential equations, *Nonlinear Studies*, **29** (1) (2022), 303-314.

[10] J.B.Dhage, B.C. Dhage, Approximating local solution of an IVP of nonlinear first order ordinary hybrid differential equations,, *Nonlinear Studies*, **30** (3), (2023), 721-732.

[11] J.B. Dhage, B.C. Dhage, Approximating local solution of IVPs of nonlinear first order ordinary hybrid integrodifferential equations, *Malaya J. Mat.*, **11** (04) (2023), 344-355.

[12] J.B. Dhage, S.B. Dhage, B. C. Dhage, Dhage iteration method for an algorithmic approach to local solution of the nonlinear second order ordinary hybrid differential equations, *Electronic J. Math. Anal. Appl.*, **11** (2) No. 3, (2023), 1-10.

[13] D. Gua, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, London 1988.

[14] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer Verlag, New York, 2003.

[15] V. Lakshmikantham, S. G. Pandit, The Method of upper, lower solutions and hyperbolic partial differential equations, *J. Math. Anal. Appl.*, 105 (1985), 466-477.

[16] A. Tarski, A lattice theoretical fixed point theorem and its applications, *Pacific J. Math.*, **5** (1965), 285-309.

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **12(04)**(2024), 339–366.
http://doi.org/10.26637/mjm1204/002

Mathematical modeling and optimal control of the dynamics of terrorist ideologies

WENDPANGA ALAIN TAPSOBA*[1], YACOUBA SIMPORE[3] AND OUMAR TRAORE[1,2]

[1] *Laboratoire de Science et Technologie, Université Thomas SANKARA, 12 BP 417, Ouagadougou 12, Burkina Faso.*
[2] *Laboratoire d'Analyse Mathématique et d'Informatique, Université Joseph KI-ZERBO, 03 BP 7021 Ouagadougou 03, Burkina Faso E-mail : otraoret@gmail.com.*
[3] *Laboratoire d'Analyse Mathématique et d'Informatique, Université Joseph KI-ZERBO, 03 BP 7021 Ouagadougou 03, Burkina Faso DeustoTech Fundación, Deusto Avda Universidades, 24, 48007, Bilbao, Basque Country, Spain, E-mail: simplesaint@gmail.com*

***R*eceived 20 July 2024; *A*ccepted 11 September 2024**

Abstract. We describe the dynamics of the spread of terrorist ideologies within a population, described as an epidemic. The equations of the model are obtained using a contact process which gives us first-order autonomous non-linear differential equations. Next, the stability of the equilibrium point is established using the basic reproduction number technique; numerical simulations allow us to verify the mathematical results. Finally, optimal control analysis highlight the importance of synergy of action (numbers, equipment, strategy and training) within the defense and security forces, and the importance of patriotism in a nation. In addition, ongoing awareness-raising campaigns are helping to speed up the eradication process.

AMS Subject Classifications: 49K15, 93B05, 93C15, 93D23.

Keywords: Terrorism, modeling, basic reproduction number, optimal control.

Contents

*Corresponding author. Email address: **alansmiley7@gmail.com** (Wendpanga Alain TAPSOBA)

1. Introduction and Background

There is no an international accepted definition of terrorism. According to [1] terrorism is defined by Title 22 of the U.S. Code as politically motivated violence perpetrated in a clandestine manner against noncombatants. Experts on terrorism also include another aspect in the definition: the act is committed in order to create a fearful state of mind in an audience different from the victims. In [2] we have more than 260 other definitions of terrorism compiled by Joseph J. Easson and Alex P. Schmid. This means that terrorism is not easy concept to define because of its many manifestations: kidnappings of diplomats, sequestration of individuals not concerned by the defended cause, acts of sabotage, assassinations, hijackings of planes etc. [3]. Whether or not an act is considered as terrorism also depends on whether a legal, moral, or behavioral perspective is used to interpret the act, see [1] and [4]. Given definition by the Economists T. Sandler and W. Enders in [5] and [6] is very close: terrorism is "the premeditated use, or threat of use, of extra-normal violence to achieve a political objective, through intimidation or the fear of a large audience." The authors point out that an act without specific political motivation must be considered as a criminal offence rather than terrorist. They also consider violence to be targeted at vulnerable target populations not directly involved in political decision-making processes such as terrorists seek to influence. For [7], If a regime constrains the executive branch, then terrorism may be more prevalent. If, however, a regime allows all viewpoints to be represented, then grievances may be held in check, resulting in less terrorism. Regimes that value constituents' lives and property will also act to limit attacks.

Several models have been written in order to provide a good understanding of the problem, see [8], [9] and [10]. In [11] terrorism is described as a new challenge to Nigeria stability. In [12] C.G. Ngari purpose a mathematical model of Kenya domestic radicalization like a desease. Ngari incorporated rehabilitation centers in his model like A. Gambo and M.O. Ibrahim in [13]. M.R. Pooda and al in [14] study the dynamics of narcoterrorism int the Sahel and in [15] they state a multi-objective optimal control of counter-terrorism in the Sahel Region in Africa. All of theses models ignore that defense and security forces can evolve into terrorist. Our model has three major differences from existing models. Firstly, the death rates resulting from fighting are not constant coefficients. They depend on the balance of power between the defense and security forces and the terrorists. Secondly, terrorists are classified according to the roles they play on the chessboard, not in any hierarchical order. Finally, we incorporate into our model the fact that defense and security forces can also become terrorists. We propose in this paper a mathematical model of dynamics behavior of terrorism ideologies using contacts process. Without loss of generality, this model can be applied to the G5 Sahel countries and to any others similarity countries.

2. Model formulation

We divide the population in eight (08) compartments.

$$
\begin{aligned}
\mathbf{S}(t) &: \text{Susceptible ,}\\
\mathbf{D}(t) &: \text{Defense and Security Forces (DSF),}\\
\mathbf{H}(t) &: \text{Homeland Defense Volunteers (HDV),}\\
\mathbf{I}(t) &: \text{Internally Displaced Persons (IDP)}\\
\mathbf{P}(t) &: \text{Prisoners or People in Detention Centers}\\
\mathbf{T}(t) &: \text{Terrorist,}\\
\mathbf{T_S}(t) &: \text{Terrorist soldiers,}\\
\mathbf{T_L}(t) &: \text{Terrorist leaders.}
\end{aligned}
$$

We set

$$A = D + H + T + T_S + T_L \tag{2.1}$$
$$N = S + D + H + I + P + T + T_S + T_L \tag{2.2}$$

and Taking for initial conditions

$$S(0) > 0,\ D(0) > 0,\ H(0) > 0,\ I(0) \geq 0,\ P(0) \geq 0,\ T(0) \geq 0,\ T_S(0) \geq 0,\ T_L(0) \geq 0,\ N(0) \leqslant \frac{\Lambda}{\mu}. \tag{2.3}$$

We understand by susceptible any person capable of adhering to the terrorist ideology. This definition assumes that the person may or may not be aware of this ideology but has not adopted or accepted it. A susceptible is not a supporter of terrorist ideology and therefore she cannot propagate it.

A terrorist is a person who is a supporter of terrorist ideology. He can only propagate it by means which exclude the taking up of arms. As soon as a weapon is taken or violence is used, we have to deal with a terrorist soldier. We include in the class of terrorists all unarmed persons who provide assistance for the success of the terrorist activity. These include intelligence officers and civilians who supply them. Terrorists and terrorist soldiers are not only ideologically convinced people; some act out of coercion, or within certain limits to defend themselves. The terrorist leaders are the masters of the terrorist chessboard: they set the course. They are the ones who organize, decide on the areas to attack and instruct the actions to be carried out.

Internally Displaced Persons are, according to the United Nations High Commissioner for Refugees (U.N.H.C.R.) in [16], people forced to flee within their own country because of the attacks perpetuated by armed terrorist groups. In the practical dictionary of humanitarian law of Doctors Without Borders [17], we can read that they do not constitute a particular legal category and therefore do not benefit from specific protection under international law.

The regular army is designated by the term DSF. The Volunteers for the Defense of the Homeland (HDV) is a groups of armed combatants created by the government in order to better respond to the demands imposed on it by the terrorist hydra. We include in this class any self-defense groups and any other organization whose objective is to fight alongside the DSF for the defense of the homeland.

The term prison or detention center includes areas regularly set up to accommodate persons deprived of their freedom in connection with terrorism as well as probable detention areas which have been set up by the army for its needs and which meet the criteria of prison. The following assumptions complete the model formulation.

First of all, we assume that the compartments are homogeneous and contained within the same territory. Thus, the spatial distribution of terrorist ideology can be omit and everybody in the population has same average natural death rate μ.

As Castillo Chavez and Bao Song in [18], for $i = \overline{1,6}$ ε_i, q and e measure the strengh of the recruitement force and assumed to be proportionnal to the number of contacts per unit time as well as to the likelihood of success. We also denote Λ_5 and Λ_8 as the per-capita recovery rate. Hence, $1/\Lambda_5$ and $1/\Lambda_8$ are the average residence time respectively for terrorists and terrorist soldiers. This assumed that the residence times are exponentially distributed.

The model equations follow a contact process. In other words, the transition from a class A to a class B is obtained after contact with an individual of class B or an individual of another class who shares the convictions that emanate from class B. For example, an individual can only become a terrorist following contact with a terrorist, a terrorist soldier or a terrorist leader. Contact notion is any means by which individuals can stay in touch such as family ties, telephone calls, radio and television broadcasts, sending letters, coded or explicit messages, internet, etc.

Parameters definitions

Parameters	**Definitions**
η	Death rate due to detention conditions
μ	Natural mortality rate
Λ	Susceptible recruitment rate
Λ_1	DSF recruitment rate from S
Λ_2	DSF out-going rate
Λ_3	HDV recruitment rate from S
Λ_4	HDV drop-out rate
Λ_5	Terrorist soldiers repentance rate
Λ_6	Prisoners out-going rate
Λ_7	Force of radicalization
Λ_8	Terrorist repentance rate
Λ_9	Force of the determination in defense of the homeland
β_1	Terrorist-to-terrorist-soldiers conversion rate
β_2	Terrorist-to-terrorist-leaders conversion rate
β_3	Terrorist-soldiers-to-terrorist-leaders conversion rate
δ_1	DSF death rate due to violent extrmism
δ_2	HDV death rate due to violent extrmism
δ_3	Terrorists death rate due to counter-terrorist activities
δ_4	Terrorist soldiers death rate due to counter-terrorist activities
δ_5	Terrorist leaders death rate due to counter-terrorist activities
ε_1	Strength of the recruitment force from D into T
ε_2	Strength of the recruitment force from D into T_S
ε_3	Strength of the recruitment force from D into T_L
ε_4	Strength of the recruitment force from H into T
ε_4	Strength of the recruitment force from H into T_S
ε_6	Strength of the recruitment force from H into T_L
a	Undergoing judicial process rate from T_S
b	Strength of the recruitment force from P into T_S
h	Undergoing judicial process rate from T
k	Strength of the recruitment force from P into T
l_1	Undergoing judicial process rate from T_L
l_2	Strength of the recruitment force from P into T_L
n	HDV recruitment rate from IDP
m	DSF recruitment rate from IDP
π	DSF recruitment rate from HDV
e	Strength of the recruitment force from I into T_S
q	Strength of the recruitment force from I into T

the model equations are given by:

$$\frac{dS}{dt} = \Lambda + \Lambda_2 D + \Lambda_4 H + \Lambda_5 T_S + \Lambda_6 P + \Lambda_8 T - \left[\mu + \Lambda_9 \frac{T_S}{A+S} + \Lambda_1 \frac{D+H}{A+S} + \Lambda_3 \frac{T_S}{A+S} + \Lambda_7 \frac{T+T_S+T_L}{A+S}\right] S \tag{2.4}$$

$$\frac{dD}{dt} = \left(\frac{\Lambda_1 S}{A+S} + \frac{mI}{A+I}\right)(D+H) + \pi H - \left[\Lambda_2 + \mu + \delta_1 \frac{T_S}{A} + \varepsilon_1 \frac{T+T_S+T_L}{A} + \varepsilon_2 \frac{T_S+T_L}{A} + \varepsilon_3 \frac{T_L}{A}\right] D \tag{2.5}$$

$$\frac{dH}{dt} = \Lambda_3 \frac{T_S}{A+S} S + n \frac{D+H}{A+I} I - \left[\pi + \mu + \Lambda_4 + \delta_2 \frac{T_S}{A} + \varepsilon_4 \frac{T+T_S+T_L}{A} + \varepsilon_5 \frac{T_S+T_L}{A} + \varepsilon_6 \frac{T_L}{A}\right] H \tag{2.6}$$

$$\frac{dI}{dt} = \Lambda_9 \frac{T_S}{A+S} S - \left[\mu + (n+m)\frac{D+H}{A+I} + e\frac{T_S+T_L}{A+I} + q\frac{T+T_S+T_L}{A+I}\right] I \tag{2.7}$$

$$\frac{dP}{dt} = \left[hT + aT_S + l_1 T_L\right] \frac{D+H}{A} - \left[\mu + \eta + \Lambda_6 + l_2 \frac{T_L}{A+P} + b\frac{T_S+T_L}{A+P} + k\frac{T+T_S+T_L}{A+P}\right] P \tag{2.8}$$

$$\frac{dT}{dt} = \left[\Lambda_7 \frac{S}{A+S} + q\frac{I}{A+I} + k\frac{P}{A+P} + \frac{\varepsilon_1 D + \varepsilon_4 H}{A}\right](T+T_S+T_L) - \left[\Lambda_8 + \mu + (D+H)\left(\frac{h}{A} + \frac{\delta_3}{A}\right) + \beta_1 \frac{T_S+T_L}{A} + \beta_2 \frac{T_L}{A}\right] T \tag{2.9}$$

$$\frac{dT_S}{dt} = \left[\beta_1 \frac{T}{A} + \frac{\varepsilon_2 D + \varepsilon_5 H}{A} + e\frac{I}{A+I} + b\frac{P}{A+P}\right](T_S+T_L) - \left[\mu + \Lambda_5 + (D+H)\left(\frac{a}{A} + \frac{\delta_4}{A}\right) + \beta_3 \frac{T_L}{A}\right] T_S \tag{2.10}$$

$$\frac{dT_L}{dt} = \left[\beta_2 \frac{T}{A} + \beta_3 \frac{T_S}{A} + \frac{\varepsilon_3 D + \varepsilon_6 H}{A} + l_2 \frac{P}{A+P}\right] T_L - \left[\mu + (D+H)\left(\frac{l_1}{A} + \frac{\delta_5}{A}\right)\right] T_L \tag{2.11}$$

We get the terrorism network diagram.

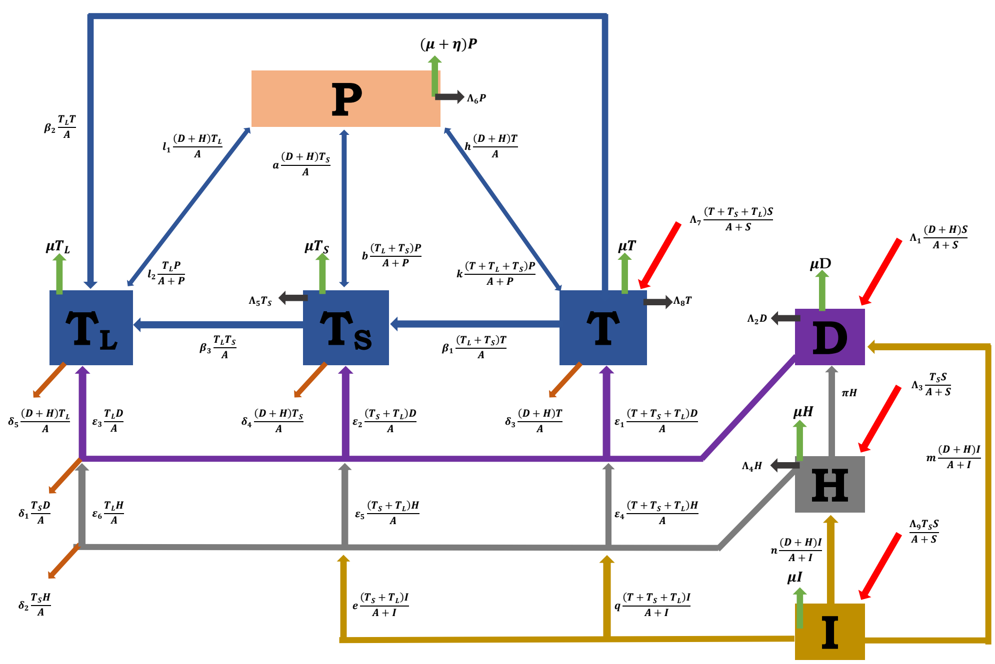

Figure 1: Flow diagram

3. Model Analysis

It's worth mentioning that the parameters of the formulated model are non-negative since the model describes the dynamics of an ideology in an human population. Consequently, it suffices to state that the solutions of the model are non-negative. We denote by $\mathbb{R}_+^8$ the set $[0;+\infty[$ and by Ω the set

$$\Omega = \left\{(S(t), D(t), H(t), I(t), P(t), T(t), T_S(t), T_L(t)) \in \mathbb{R}_+^8;\ and\ N \leq \frac{\Lambda}{\mu}\right\}. \tag{3.1}$$

Lemma 3.1. *The system* (2.4) - (2.11) *with initial contitions* (2.3) *has a unique solution in* Ω.

Proof. We follow [19] and apply Cauchy-Lypschitz theorem about the existence and the uniqueness of solutions for first-order autonomous systems with initial conditions (2.3). ∎

Theorem 3.1. *The feasible region* Ω *is positively invariant and attracting with respect the system* (2.3) - (2.11).

Proof. The vector field associated to the system (2.4) - (2.11) is denoted by

$$\vec{V} = \begin{pmatrix} \frac{dS}{dt} \\ \frac{dD}{dt} \\ \frac{dH}{dt} \\ \frac{dI}{dt} \\ \frac{dP}{dt} \\ \frac{dT}{dt} \\ \frac{dT_S}{dt} \\ \frac{dT_L}{dt} \end{pmatrix} \tag{3.2}$$

For this demonstration we follow [20], [21] and [22] using the barrier theorem by checking that the vector field is always tangent or pointing inside the boundary $\partial\mathbb{R}_+^8$ of $\mathbb{R}_+^8$.
$\partial\mathbb{R}_+^8 = \{S = 0\} \cup \{D = 0\} \cup \{H = 0\} \cup \{I = 0\} \cup \{P = 0\} \cup \{T = 0\} \cup \{T_S = 0\} \cup \{T_L = 0\}$.

On $\{S = 0\}$, the associated vector field is

$$\vec{V_1} = \begin{pmatrix} \Lambda + \Lambda_2 D + \Lambda_4 H + \Lambda_5 T_S + \Lambda_6 P + \Lambda_8 T \\ \left(\frac{dD}{dt}\right)_{S=0} \\ \left(\frac{dH}{dt}\right)_{S=0} \\ \left(\frac{dI}{dt}\right)_{S=0} \\ \left(\frac{dP}{dt}\right)_{S=0} \\ \left(\frac{dT}{dt}\right)_{S=0} \\ \left(\frac{dT_S}{dt}\right)_{S=0} \\ \left(\frac{dT_L}{dt}\right)_{S=0} \end{pmatrix}$$

We have $\vec{e_1} = (1,0,0,0,0,0,0,0)$ and

$$\vec{V_1} \cdot \vec{e_1} = \Lambda + \Lambda_2 D + \Lambda_4 H + \Lambda_6 P + \Lambda_8 T \geq 0$$

So, the vector field $\vec{V_1}$ is pointing inside the positive orthan.
The same reasoning can be done for $\{D = 0\}$, $\{H = 0\}$, $\{I = 0\}$, $\{P = 0\}$, $\{T = 0\}$, $\{T_S = 0\}$ and $\{T_L = 0\}$.
We deduce that the set Ω is positively invariant with respect the model.

Moreover,

$$\frac{dN}{dt} = \frac{d(S + D + H + I + P + T + T_S + T_L)}{dt}$$

$$\frac{dN}{dt} + \mu N \leqslant \Lambda \tag{3.3}$$

According to [23], the solution of (3.3) is given by

$$N(t) \leqslant N(0)\exp(-\mu t) + \frac{\Lambda}{\mu}[1 - \exp(-\mu t)] \tag{3.4}$$

$$N(0) \leqslant \frac{\Lambda}{\mu} \tag{3.5}$$

Furthermore, for $t \to +\infty$ in (3.4) the total population N approaches the caring capacity constant $\frac{\Lambda}{\mu}$. It means that $\limsup N(t)_{t\to+\infty} \leq \frac{\Lambda}{\mu}$, demonstrating that Ω is attractive within $\mathbb{R}^8_+$; see [13], [21] and [24]. ■

3.1. Basic reproduction number

The basic reproductive number, $\mathcal{R}_0$, is the average number of secondary infections produced by one infected individual during the entire course of infection in a completely susceptible population. I serves as a threshold parameter that predicts whether an infection dies out or keeps persistence in a population. We determine the basic reproduction number by using Watmough and Van den Driessche method in [25]. The population is divided in eight compartments in this order S, D, H, I, P, T, T_S and T_L. For our model, infected compartments are P, T, T_S and T_L and we can discard the compartment P because it doesn't change the basic reproduction number. The next generation matrice is obtained by calculated $G = FV^{-1}$.

Firstly we determine the Terrorist-free equilibrium (TFE) by solving the model equations for $T^* = T_S^* = T_L^* = 0$. It yields

$$E_0 = \left(\frac{\Lambda(\mu+\Lambda_2)}{\mu\Lambda_1}, \frac{\Lambda(\Lambda_1-\Lambda_2-\mu)}{\mu\Lambda_1}, 0, 0, 0, 0, 0\right) \tag{3.6}$$

Now, we state the basic reproduction number.
Considering $\mathcal{F}_T$, $\mathcal{F}_{T_S}$ and $\mathcal{F}_{T_L}$ as the rates of appearance of newly radicalized individuals respectively in compartments T, T_S and T_L and for $i \in \{T, T_S, T_L\}$ $\nu_i = \nu_i^- - \nu_i^+$ with ν_i^- the rate of transfers of individuals out the class i and ν_i^+ the rate of transfers of individuals into class i we get:

$$F = J_{\mathcal{F}}(E_0) = \begin{bmatrix} \frac{\partial \mathcal{F}_T}{\partial T} & \frac{\partial \mathcal{F}_T}{\partial T_S} & \frac{\partial \mathcal{F}_T}{\partial T_L} \\ \frac{\partial \mathcal{F}_{T_S}}{\partial T} & \frac{\partial \mathcal{F}_{T_S}}{\partial T_S} & \frac{\partial \mathcal{F}_{T_S}}{\partial T_L} \\ \frac{\partial \mathcal{F}_{T_L}}{\partial T} & \frac{\partial \mathcal{F}_{T_L}}{\partial T_S} & \frac{\partial \mathcal{F}_{T_L}}{\partial T_L} \end{bmatrix}(E_0) \quad and \quad V = J_{\nu}(E_0) = \begin{bmatrix} \frac{\partial \nu_T}{\partial T} & \frac{\partial \nu_T}{\partial T_S} & \frac{\partial \nu_T}{\partial T_L} \\ \frac{\partial \nu_{T_S}}{\partial T} & \frac{\partial \nu_{T_S}}{\partial T_S} & \frac{\partial \nu_{T_S}}{\partial T_L} \\ \frac{\partial \nu_{T_L}}{\partial T} & \frac{\partial \nu_{T_L}}{\partial T_S} & \frac{\partial \nu_{T_L}}{\partial T_L} \end{bmatrix}(E_0)$$

$$\text{where} \quad \mathcal{F} = \begin{bmatrix} \mathcal{F}_T \\ \mathcal{F}_{T_S} \\ \mathcal{F}_{T_L} \end{bmatrix} = \begin{bmatrix} \left[\Lambda_7 \frac{S}{A+S} + \frac{\varepsilon_1 D + \varepsilon_4 H}{A} + q\frac{I}{A+I}\right](T+T_S+T_L) \\ \left[\frac{\varepsilon_2 D + \varepsilon_5 H}{A} + e\frac{I}{A+I}\right](T_S+T_L) \\ \left[\frac{\varepsilon_3 D + \varepsilon_6 H}{A}\right] T_L \end{bmatrix}$$

This give us

$$F = \begin{bmatrix} \frac{\Lambda_7 S^*}{D^*+S^*} + \varepsilon_1 & \frac{\Lambda_7 S^*}{D^*+S^*} + \varepsilon_1 & \frac{\Lambda_7 S^*}{D^*+S^*} + \varepsilon_1 \\ 0 & \varepsilon_2 & \varepsilon_2 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}. \tag{3.7}$$

with

$$\frac{S^*}{D^*+S^*} = \frac{\mu+\Lambda_2}{\Lambda_1} \tag{3.8}$$

$$\frac{D^*}{D^*+S^*} = \frac{\Lambda_1-\Lambda_2-\mu}{\Lambda_1} \tag{3.9}$$

Now, we are locking for V and V^{-1}.

$$\nu = \begin{bmatrix} \nu_T \\ \\ \nu_{T_S} \\ \\ \nu_{T_L} \end{bmatrix}$$

$$= \begin{bmatrix} \left[\Lambda_8+\mu+(D+H)\left(\frac{h}{A}+\frac{\delta_3}{A}\right)+\beta_1\frac{T_S+T_L}{A}+\beta_2\frac{T_L}{A}\right]T \\ \\ \left[\mu+\Lambda_5+(D+H)\left(\frac{a}{A}+\frac{\delta_4}{A}\right)+\beta_3\frac{T_L}{A}\right]T_S-\left[\beta_1\frac{T}{A}\right](T_S+T_L) \\ \\ \left[\mu+(D+H)\left(\frac{l_1}{A}+\frac{\delta_5}{A}\right)\right]T_L-\left[\beta_2\frac{T}{A}+\beta_3\frac{T_S}{A}\right]T_L \end{bmatrix}$$

It comes that

$$V = \begin{bmatrix} \Lambda_8+\mu+h+\delta_3 & 0 & 0 \\ \\ 0 & \Lambda_5+\mu+a+\delta_4 & 0 \\ \\ 0 & 0 & \mu+l_1+\delta_5 \end{bmatrix} \tag{3.10}$$

and

$$V^{-1} = \begin{bmatrix} \frac{1}{\Lambda_8+\mu+h+\delta_3} & 0 & 0 \\ \\ 0 & \frac{1}{\Lambda_5+\mu+a+\delta_4} & 0 \\ \\ 0 & 0 & \frac{1}{\mu+l_1+\delta_5} \end{bmatrix} \tag{3.11}$$

The next generation matrix denotes $G = FV^{-1}$.
From (3.7), (3.11) and (3.6) we obtain the next generation matrix that is

$$G=\begin{bmatrix} \dfrac{\Lambda_7(\mu+\Lambda_2)+\Lambda_1\varepsilon_1}{\Lambda_1(\Lambda_8+\mu+h+\delta_3)} & \dfrac{\Lambda_7(\mu+\Lambda_1)+\Lambda_1\varepsilon_1}{\Lambda_1(\Lambda_5+\mu+a+\delta_4)} & \dfrac{\Lambda_7(\mu+\Lambda_1)+\Lambda_1\varepsilon_1}{\Lambda_1(\mu+l_1+\delta_5)} \\ 0 & \dfrac{\varepsilon_2}{\Lambda_5+\mu+a+\delta_4} & \dfrac{\varepsilon_2}{\mu+l_1+\delta_5} \\ 0 & 0 & \dfrac{\varepsilon_3}{\mu+l_1+\delta_5} \end{bmatrix} \tag{3.12}$$

The basic reproduction number is the spectral radius of the next generation matrix. Then

$$\mathcal{R}_0=\max\left\{\frac{\Lambda_7(\mu+\Lambda_2)+\Lambda_1\varepsilon_1}{\Lambda_1(\Lambda_8+\mu+h+\delta_3)}\,;\,\frac{\varepsilon_2}{\Lambda_5+\mu+a+\delta_4}\,;\,\frac{\varepsilon_3}{\mu+l_1+\delta_5}\right\}. \tag{3.13}$$

Applying theorem of Varga in [26], theorem 2 in [25] or theorem 6 in [27] and as [21] we claim the following local stability result.

Theorem 3.2. *The terrorist free equilibrium E_0 is locally asymptotically stable if $\mathcal{R}_0<1$.*

According to the theorem 3.2, as long as the value of $\mathcal{R}_0$ is less than one, terrorism can never take on the scale of an epidemic. Note that this interpretation depends on the initial conditions, in particular the number of terrorists, terrorist soldiers and leaders in the initial population. To get rid of this dependency, a global stability result is needed.

Theorem 3.3. *The terrorist free equilibrium E_0 is globally asymptotically stable if $\mathcal{R}_0<1$.*

Proof. According to theorem 3.2 the TFE is locally asymptotically stable and according to theorem 3.1 the domain Ω of the feasible solution is attractive. Thus, we follow [20], [21] and [24] to get the global asymptotic stability. ■

3.2. Endemic equilibrium

As soon as the basic reproduction number is greater than one, a single terrorist has a large recruitment capacity and can put the whole nation at risk. We're going to see an explosion in the number of terrorists, terrorist soldiers and terrorist leaders. With soldiers as the armed wing, the result will be more violence and an increase in the number of internally displaced people. There will be more deaths on the DSF and HDV sides. In the long term, the whole nation will be at risk, and in the worst case scenario, we'll have an occupation of the entire territory by armed terrorist groups.

Theorem 3.4. *if $\mathcal{R}_0>1$, the terrorist free equilibrium E_0 is unstable.*

Proof. We apply theorem 2 in [25]. ■

4. Numerical Analysis

In this section, we use numerical simulations to verified mathematical analysis results. This mean that for $\mathcal{R}_0<1$ we have to see that the populations of terrorists, terrorist soldiers and terrorist leaders are coming to disappear. In the verse, for $\mathcal{R}_0>1$ these populations are growing and terrorism ideology is spreading.

The Table 1 gives parameters settings for extinction and the Table 2 gives parameters settings for persistence. The populations of susceptible, DSF, HDV, IDP, prisoners, terrorists, terrorist soldiers and terrorist leaders at initials conditions ($t=0$) are given by :

$S=15\,089\,674$ $\quad D=23\,000$ $\quad H=50\,000$ $\quad I=1\,882\,391$

$P = 7\,041$ $\quad T = 470\,500$ $\quad T_S = 15\,089\,674$ $\quad T_L = 25$

Table 1 : Parameters settings for extinction	
Parameters and values	**Parameters and values**
$\eta = 0.0000025$	$\varepsilon_1 = 0.0001$
$\mu = 0.0034247$	$\varepsilon_2 = 0.0001$
$\Lambda = 600$	$\varepsilon_3 = 0.0001$
$\Lambda_1 = 0.05$	$\varepsilon_4 = 0.0000001$
$\Lambda_2 = 0.001$	$\varepsilon_4 = 0.0000001$
$\Lambda_3 = 0.005$	$\varepsilon_6 = 0.0000001$
$\Lambda_4 = 0.0001$	$a = 0.0008$
$\Lambda_5 = 0.0001$	$b = 0.001$
$\Lambda_6 = 0.001$	$h = 0.20635$
$\Lambda_7 = 0.056$	$k = 0.0016$
$\Lambda_8 = 0.0001$	$l_1 = 0.0001$
$\Lambda_9 = 0.01$	$l_2 = 0.0001$
$\beta_1 = 0.9 * 0.12$	$n = 0.01$
$\beta_2 = 0.0792$	$m = 0.001$
$\beta_3 = 0.00005$	$\pi = 0.005$
$\delta_1 = 0.00125$	$\delta_2 = 0.00125$
$\delta_3 = 0.0001$	$\delta_4 = 0.0005$
$\delta_5 = 0.0002$	$q = 0.005$
$e = 0.0025$	

Table 2 : Parameters settings for persistence	
Parameters and values	**Parameters and values**
$\eta = 0.00000025$	$\varepsilon_1 = 0.01$
$\mu = 0.00034247$	$\varepsilon_2 = 0.01$
$\Lambda = 600$	$\varepsilon_3 = 0.01$
$\Lambda_1 = 0.05$	$\varepsilon_4 = 0.001$
$\Lambda_2 = 0.001$	$\varepsilon_4 = 0.001$
$\Lambda_3 = 0.005$	$\varepsilon_6 = 0.001$
$\Lambda_4 = 0.0001$	$a = 0.00008$
$\Lambda_5 = 0.00001$	$b = 0.001$
$\Lambda_6 = 0.001$	$h = 0.0000010635$
$\Lambda_7 = 0.156$	$k = 0.0016$
$\Lambda_8 = 0.0000001$	$l_1 = 0.00001$
$\Lambda_9 = 0.01$	$l_2 = 0.01$
$\beta_1 = 0.1 * 0.12$	$n = 0.01$
$\beta_2 = 0.00792$	$m = 0.001$
$\beta_3 = 0.00005$	$\pi = 0.005$
$\delta_1 = 0.00125$	$\delta_2 = 0.00125$
$\delta_3 = 0.00001$	$\delta_4 = 0.00005$
$\delta_5 = 0.02$	$q = 0.05$
$e = 0.065$	

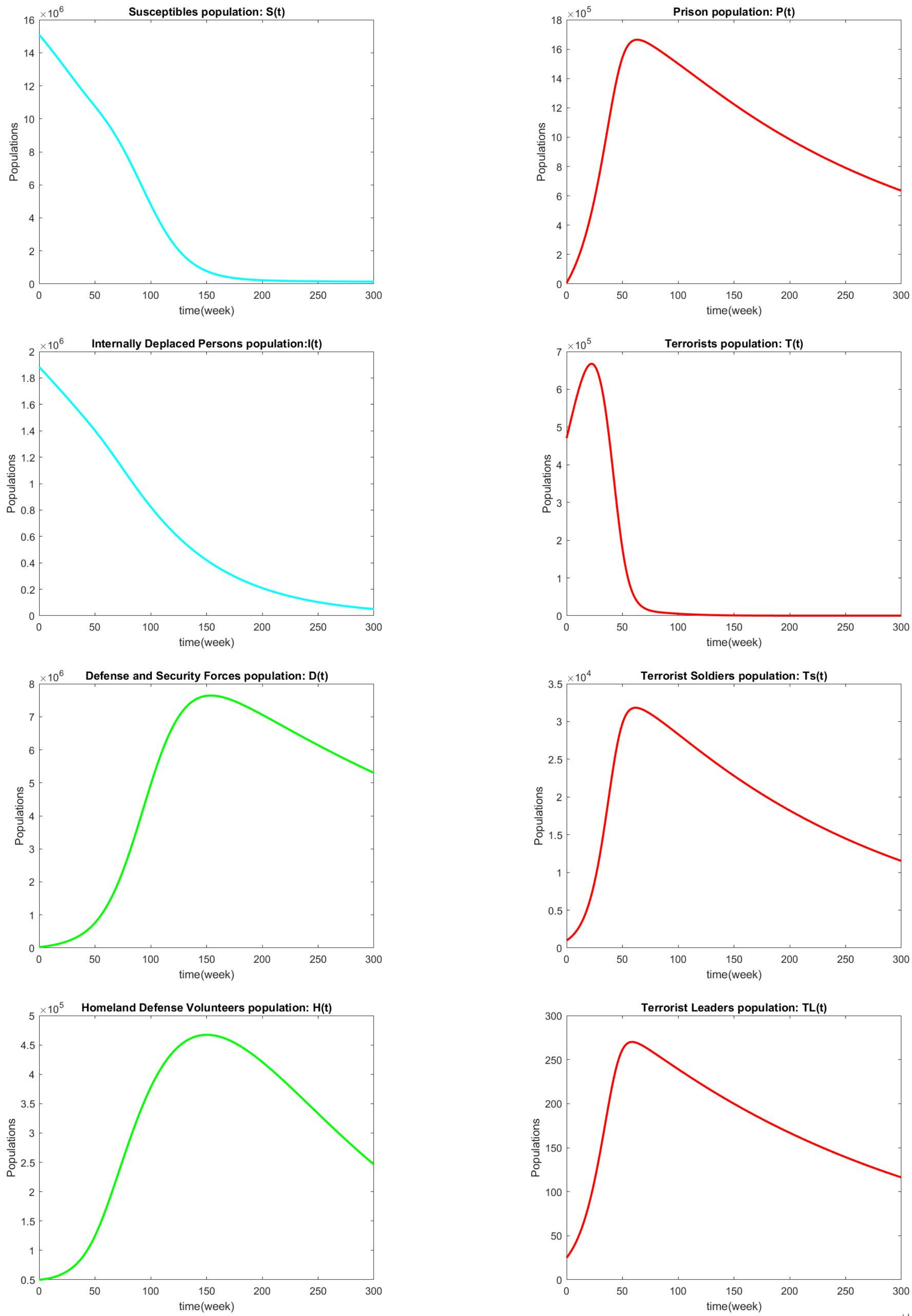

Figure 2: **Evolution of the different populations with extinction values : $\mathcal{R}_0 = 0,0268 < 1$.**

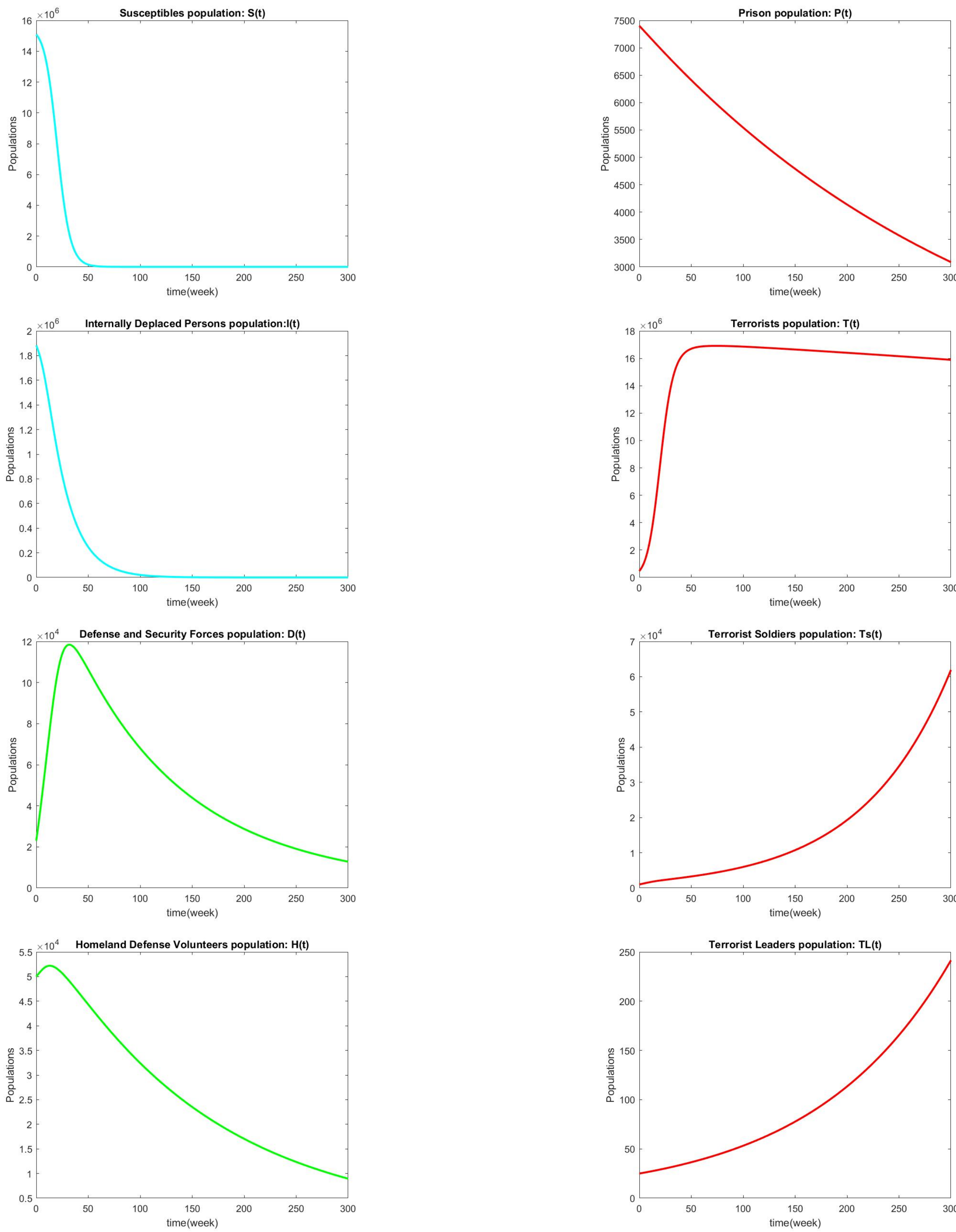

Figure 3: **Evolution of the different populations with persistence values :** $\mathcal{R}_0 = 40.1221 > 1$

Figures Comments:

Figure 2: It shows that terrorist, terrorist soldiers and leaders populations decrease until they stabilize at zero, meaning the extinction of the radicalization and the spread of terrorist ideologies. As HDV were created to help DSF, the decreasing of HDV compartment population is explained by the extinction of the spreading of terrorist ideologies. As a result, the influence that terrorists had within the general population, justifying the existence of the susceptible, no longer exists, hence the number of susceptible naturally stabilizes at zero. IDP who had fled their areas will be able to return and lead a peaceful life again. It therefore goes without saying that the IDV compartment is switched off. However, DSF population is growing. Indeed, the extinction of classes T, T_S and T_L induces the cancellation of the transfer coefficients from DSF class to classes T, T_S and T_L. The only coefficient which ensures the reduction in the number of individuals in the DSF class is the natural mortality rate which is relatively small. Finally, there is no one left to imprison because the terrorists, soldiers and leaders have all disappeared.

Figure 3 : The populations of terrorist, terrorist soldiers and leaders are continuously growing; showing that the terrorist ideology is spreading. There would be more violent and more deaths on the DSF and HDV sides explaining the decreasing of these populations. The slight growth that we are seeing in the first few weeks in the DSF and HDV compartments can be explained by the fact that the government, in response, will increase the recruitment of DSF and HDV to try to contain the growing hydra of terrorism. Since terrorist ideology will be predominant, susceptible and people in the IDP and prisoner compartments will spend less time in their respective compartments. They will be absorbed very quickly in compartments T, Ts and TL; thus contributing to the growth of the number of individuals in these compartments. Thus, in the long term, the whole nation will be in danger. This situation may result in the stabilization of the number of individuals in compartments T, T_s and T_L in the sense that the need for recruitment will decrease to a certain threshold which will be maintained in order to keep the territory under control. At this stage in the evolution of terrorism, the susceptible will no longer be susceptible but terrorists, which explains the stabilization of the number of susceptible at zero.

5. Optimal control model and analysis

5.1. Optimal control model formulation

We introduce three (03) time-dependent control $u_1(t)$, $u_2(t)$ and $u_3(t)$ which are described as follows.

(i) $u_1(t)$ covers all the actions undertaken by government, civil organizations, traditional authorities and political parties to raise awareness through public conferences, preaching and socio-religious seminars. This include television, radio and interactive broadcasts, as well as newspapers articles and pages used in the fight against terrorism.

(ii) $u_2(t)$ represents the ability of DSF and HDV to respond to attacks and carry out preventive operations. This capacity is expressed through military equipment, the quality of that equipment, military training, knowledge and control of the territory, the commitment of the players and their numbers.

(iii) $u_3(t)$ is any action that allows to identify and to neutralise terrorist leaders.

Adding the tree aforementioned time-dependent control we get the control system.

$$\frac{dS}{dt} = \Lambda+\Lambda_2 D+\Lambda_4 H+\Lambda_5 T_S+\Lambda_6 P+\Lambda_8 T-\left[\mu+\Lambda_9\frac{T_S}{A+S}+\Lambda_1\frac{D+H}{A+S}+\Lambda_3\frac{T_S}{A+S}+(1-u_1)\Lambda_7\frac{T+T_S+T_L}{A+S}\right]S \quad (5.1)$$

$$\frac{dD}{dt} = \left(\frac{\Lambda_1 S}{A+S}+\frac{mI}{A+I}\right)(D+H)+\pi H-\left[\Lambda_2+\mu+\delta_1\frac{T_S}{A}+(1-u_1)\varepsilon_1\frac{T+T_S+T_L}{A}+(1-u_2)\varepsilon_2\frac{T_S+T_L}{A}+(1-u_3)\varepsilon_3\frac{T_L}{A}\right]D \quad (5.2)$$

$$\frac{dH}{dt} = \Lambda_3\frac{T_S}{A+S}S+n\frac{D+H}{A+I}I-\left[\pi+\mu+\Lambda_4+\delta_2\frac{T_S}{A}+(1-u_1)\varepsilon_4\frac{T+T_S+T_L}{A}+(1-u_2)\varepsilon_5\frac{T_S+T_L}{A}+(1-u_3)\varepsilon_6\frac{T_L}{A}\right]H \quad (5.3)$$

$$\frac{dI}{dt} = \Lambda_9\frac{T_S}{A+S}S-\left[\mu+(n+m)\frac{D+H}{A+I}+(1-u_2)e\frac{T_S+T_L}{A+I}+(1-u_1)q\frac{T+T_S+T_L}{A+I}\right]I \quad (5.4)$$

$$\frac{dP}{dt} = [hT+aT_S+l_1T_L]\frac{D+H}{A}-\left[\mu+\eta+\Lambda_6+(1-u_3)l_2\frac{T_L}{A+P}+(1-u_2)b\frac{T_S+T_L}{A+P}+(1-u_1)k\frac{T+T_S+T_L}{A+P}\right]P \quad (5.5)$$

$$\frac{dT}{dt} = (1-u_1)\left[\Lambda_7\frac{S}{A+S}+q\frac{I}{A+I}+k\frac{P}{A+P}+\frac{\varepsilon_1 D+\varepsilon_4 H}{A}\right](T+T_S+T_L)-\left[\Lambda_8+\mu+(D+H)\left(\frac{h}{A}+\frac{\delta_3}{A}\right)+(1-u_2)\beta_1\frac{T_S+T_L}{A}+(1-u_3)\beta_2\frac{T_L}{A}\right]T \quad (5.6)$$

$$\frac{dT_S}{dt} = (1-u_2)\left[\beta_1\frac{T}{A}+\frac{\varepsilon_2 D+\varepsilon_5 H}{A}+e\frac{I}{A+I}+b\frac{P}{A+P}\right](T_S+T_L)-\left[\mu+\Lambda_5+(D+H)\left(\frac{a}{A}+\frac{\delta_4}{A}\right)+(1-u_3)\beta_3\frac{T_L}{A}\right]T_S \quad (5.7)$$

$$\frac{dT_L}{dt} = (1-u_3)\left[\beta_2\frac{T}{A}+\beta_3\frac{T_S}{A}+\frac{\varepsilon_3 D+\varepsilon_6 H}{A}+l_2\frac{P}{A+P}\right]T_L-\left[\mu+(D+H)\left(\frac{l_1}{A}+\frac{\delta_5}{A}\right)\right]T_L \quad (5.8)$$

5.2. Optimal control model analysis

Our goal is to seek the optimal solution required to minimize the number of terrorists, terrorist soldiers and terrorist leaders responsible for spreading the terrorist ideology int the population at minimum cost. Hence, the objective functional for this control problem is given by

$$\mathcal{J}(u_1,u_2,u_3)=\min_{0\le u_1,u_2,u_3\le 1}\int_0^T\left(\omega_1 T(t)+\omega_2 T_s(t)+\omega_3 T_L(t)+\omega_4 u_1^2+\omega_5 u_2^2+\omega_6 u_3^2\right)dt \quad (5.9)$$

where, constants $\omega_i, i=1,2,...,6$ are positive weights required to balance the corresponding terms in the objective functional. The optimal controls u_1^*, u_2^* and u_3^* we are looking for are the solutions of the problem

$$\mathcal{J}\left(u_1^*,u_2^*,u_3^*\right) = \min\{\mathcal{J}(u_1,u_2,u_3): u_1,u_2,u_3\in\mathcal{U}\}. \quad (5.10)$$

$$\mathcal{U} = \{(u_1,u_2,u_3):(u_1(t),\,u_2(t),\,u_3(t)) \text{ are measurable for } t\in[0;T]\} \quad (5.11)$$

Theorem 5.1. *The problem of optimal control (5.1)-(5.11) has a unique solution in $\mathcal{U}$.*

Proof. Luke's results [28] assure us of the existence of solutions for system (5.1)-(5.8). Since the state variables are bounded, the set containing the system's solutions is bounded. Consequently, we obtain the result by applying Flemming-Rishel's theorem; [29] and [30]. ∎

Pontryagin's maximum principle [31] gives the necessary conditions that the control u_1^*, u_2^* and u_3^* must satifsy. These conditions allow us to determine the optimal values of the control u_1^*, u_2^* and u_3^*, using the

Hamiltonian of the system. This Hamilton is given by

$$\begin{aligned}
\mathcal{H} &= \omega_1 T(t)+\omega_2 T_s(t)+\omega_3 T_L(t)+\omega_4 u_1^2+\omega_5 u_2^2+\omega_6 u_3^2\\
&+ \lambda_1\Big(\Lambda+\Lambda_2 D+\Lambda_4 H+\Lambda_5 T_S+\Lambda_6 P+\Lambda_8 T-\Big[\mu+\Lambda_9\frac{T_S}{A+S}+\Lambda_1\frac{D+H}{A+S}+\Lambda_3\frac{T_S}{A+S}+(1-u_1)\Lambda_7\frac{T+T_S+T_L}{A+S}\Big]S\Big)\\
&+ \lambda_2\Big(\Big(\frac{\Lambda_1 S}{A+S}+\frac{mI}{A+I}\Big)(D+H)+\pi H-\Big[\Lambda_2+\mu+\delta_1\frac{T_S}{A}+(1-u_1)\varepsilon_1\frac{T+T_S+T_L}{A}+(1-u_2)\varepsilon_2\frac{T_S+T_L}{A}+(1-u_3)\varepsilon_3\frac{T_L}{A}\Big]D\Big)\\
&+ \lambda_3\Big(\Lambda_3\frac{T_S}{A+S}S+n\frac{D+H}{A+I}I-\Big[\pi+\mu+\Lambda_4+\delta_2\frac{T_S}{A}+(1-u_1)\varepsilon_4\frac{T+T_S+T_L}{A}+(1-u_2)\varepsilon_5\frac{T_S+T_L}{A}+(1-u_3)\varepsilon_6\frac{T_L}{A}\Big]H\Big)\\
&+ \lambda_4\Big(\Lambda_9\frac{T_S}{A+S}S-\Big[\mu+(n+m)\frac{D+H}{A+I}+(1-u_2)e\frac{T_S+T_L}{A+I}+(1-u_1)q\frac{T+T_S+T_L}{A+I}\Big]I\Big)\\
&+ \lambda_5\Big([hT+aT_S+l_1T_L]\frac{D+H}{A}-\Big[\mu+\eta+\Lambda_6+(1-u_3)l_2\frac{T_L}{A+P}+(1-u_2)b\frac{T_S+T_L}{A+P}+(1-u_1)k\frac{T+T_S+T_L}{A+P}\Big]P\Big)\\
&+ \lambda_6\Big((1-u_1)\Big[\Lambda_7\frac{S}{A+S}+q\frac{I}{A+I}+k\frac{P}{A+P}+\frac{\varepsilon_1 D+\varepsilon_4 H}{A}\Big](T+T_S+T_L)-\Big[\Lambda_8+\mu+(D+H)\Big(\frac{h}{A}+\frac{\delta_3}{A}\Big)+(1-u_2)\beta_1\frac{T_S+T_L}{A}+(1-u_3)\beta_2\frac{T_L}{A}\Big]T\Big)\\
&+ \lambda_7\Big((1-u_2)\Big[\beta_1\frac{T}{A}+\frac{\varepsilon_2 D+\varepsilon_5 H}{A}+e\frac{I}{A+I}+b\frac{P}{A+P}\Big](T_S+T_L)-\Big[\mu+\Lambda_5+(D+H)\Big(\frac{a}{A}+\frac{\delta_4}{A}\Big)+(1-u_3)\beta_3\frac{T_L}{A}\Big]T_S\Big)\\
&+ \lambda_8\Big((1-u_3)\Big[\beta_2\frac{T}{A}+\beta_3\frac{T_S}{A}+\frac{\varepsilon_3 D+\varepsilon_6 H}{A}+l_2\frac{P}{A+P}\Big]T_L-\Big[\mu+(D+H)\Big(\frac{l_1}{A}+\frac{\delta_5}{A}\Big)\Big]T_L\Big)
\end{aligned}$$

where, λ_i for $i=1,2,3,...,8$, represent the adjoint variables associated with the state variables of the model (5.1)-(5.8).

Theorem 5.2. *Let (u_1^*,u_2^*,u_3^*) be a solution of the problem of minization (5.1)-(5.11). Then, the adjoint variables are given by*

$$\begin{aligned}
\dot\lambda_1 &= \lambda_1\mu+(\lambda_1-\lambda_6)\frac{\Lambda_7 A(T+T_S+T_L)(1-u_1)}{(A+S)^2}+(\lambda_1-\lambda_2)\frac{\Lambda_1 A(D+H)}{(A+S)^2}+(\lambda_1-\lambda_3)\frac{\Lambda_3 AT_S}{(A+S)^2}\\
&+ (\lambda_1-\lambda_4)\frac{\Lambda_9 AT_S}{(A+S)^2}
\end{aligned}$$

$$\begin{aligned}
\dot\lambda_2 &= \lambda_1\mu+(\lambda_2-\lambda_1)\Lambda_2+(\lambda_3-\lambda_1)\frac{\Lambda_3 T_S S}{(A+S)^2}+(\lambda_4-\lambda_1)\frac{\Lambda_9 T_S S}{(A+S)^2}+(\lambda_1-\lambda_2)\frac{\Lambda_1 S(S+T+T_S+T_L)}{(A+S)^2}\\
&+ (\lambda_6-\lambda_1)\frac{\Lambda_7 S(T+T_S+T_L)(1-u_1)}{(A+S)^2}+(\lambda_2-\lambda_6)\frac{\varepsilon_1(A-D)(T+T_S+T_L)(1-u_1)}{A^2}\\
&+ (\lambda_6-\lambda_3)\frac{\varepsilon_4 H(T+T_S+T_L)(1-u_1)}{A^2}+(\lambda_6-\lambda_4)\frac{qI(T+T_S+T_L)(1-u_1)}{(A+I)^2}\\
&+ (\lambda_6-\lambda_5)\frac{kP(T+T_S+T_L)(1-u_1)}{(A+P)^2}+(\lambda_6-\lambda_5)\frac{hT(T+T_S+T_L)}{A^2}\\
&+ (\lambda_4-\lambda_2)\frac{mI(I+T+T_S+T_L)}{(A+I)^2}+(\lambda_4-\lambda_3)\frac{nI(I+T+T_S+T_L)}{(A+I)^2}+(\lambda_7-\lambda_4)\frac{eI(T_S+T_L)(1-u_2)}{(A+I)^2}\\
&+ (\lambda_7-\lambda_6)\frac{\beta_1 T(T_S+T_L)(1-u_2)}{A^2}+(\lambda_2-\lambda_7)\frac{\varepsilon_2(A-D)(T_S+T_L)(1-u_2)}{A^2}\\
&+ (\lambda_7-\lambda_3)\frac{\varepsilon_5 H(T_S+T_L)(1-u_2)}{A^2}+(\lambda_7-\lambda_5)\frac{bP(T_S+T_L)(1-u_2)}{(A+P)^2}+(\lambda_7-\lambda_5)\frac{aT_S(T+T_S+T_L)}{A^2}\\
&+ (\lambda_8-\lambda_7)\frac{\beta_3 T_S T_L(1-u_3)}{A^2}+(\lambda_8-\lambda_6)\frac{\beta_2 TT_L(1-u_3)}{A^2}+(\lambda_8-\lambda_5)\frac{l_2 PT_L(1-u_3)}{(A+P)^2}\\
&+ (\lambda_8-\lambda_5)\frac{l_1 T_L(T+T_S+T_L)}{A^2}+(\lambda_8-\lambda_3)\frac{\varepsilon_6 HT_L(1-u_3)}{A^2}+(\lambda_2-\lambda_8)\frac{\varepsilon_3(A-D)T_L(1-u_3)}{A^2}\\
&+ \delta_1\lambda_2\frac{(A-D)T_S}{A^2}-\delta_2\lambda_3\frac{HT_S}{A^2}+\delta_3\lambda_6\frac{T(T+T_S+T_L)}{A^2}+\delta_4\lambda_7\frac{T_S(T+T_S+T_L)}{A^2}+\delta_5\lambda_8\frac{T_L(T+T_S+T_L)}{A^2}
\end{aligned}$$

$$
\begin{aligned}
\dot{\lambda}_3 &= \lambda_3\mu+(\lambda_3-\lambda_2)\pi+(\lambda_3-\lambda_1)\Lambda_4+(\lambda_1-\lambda_2)\frac{\Lambda_1 S(S+T+T_S+T_L)}{(A+S)^2}+(\lambda_3-\lambda_1)\frac{\Lambda_3 S T_S}{(A+S)^2}\\
&+ (\lambda_4-\lambda_1)\frac{\Lambda_9 S T_S}{(A+S)^2}+(\lambda_6-\lambda_1)\frac{\Lambda_7 S(T+T_S+T_L)(1-u_1)}{(A+S)^2}+(\lambda_4-\lambda_2)\frac{mI(I+T+T_S+T_L)}{(A+I)^2}\\
&+ (\lambda_4-\lambda_3)\frac{nI(I+T+T_S+T_L)}{(A+I)^2}+(\lambda_7-\lambda_4)\frac{eI(T_S+T_L)(1-u_2)}{(A+I)^2}+(\lambda_6-\lambda_4)\frac{qI(T+T_S+T_L)(1-u_1)}{(A+I)^2}\\
&+ (\lambda_8-\lambda_5)\frac{l_2 P T_L(1-u_3)}{(A+P)^2}+(\lambda_7-\lambda_5)\frac{bP(T_S+T_L)(1-u_2)}{(A+P)^2}+(\lambda_6-\lambda_5)\frac{kP(T+T_S+T_L)(1-u_1)}{(A+P)^2}\\
&+ (\lambda_6-\lambda_2)\frac{\varepsilon_1 D(T+T_S+T_L)(1-u_1)}{A^2}+(\lambda_7-\lambda_2)\frac{\varepsilon_2 D(T_S+T_L)(1-u_2)}{A^2}+(\lambda_8-\lambda_2)\frac{\varepsilon_3 D T_L(1-u_3)}{A^2}\\
&+ (\lambda_3-\lambda_6)\frac{\varepsilon_4(A-H)(T+T_S+T_L)(1-u_1)}{A^2}+(\lambda_3-\lambda_7)\frac{\varepsilon_5(A-H)(T_S+T_L)(1-u_2)}{A^2}\\
&+ (\lambda_3-\lambda_8)\frac{\varepsilon_6(A-H)T_L(1-u_3)}{A^2}+(\lambda_6-\lambda_5)\frac{hT(T+T_S+T_L)}{A^2}+(\lambda_6-\lambda_5)\frac{aT_S(T+T_S+T_L)}{A^2}\\
&+ (\lambda_8-\lambda_5)\frac{l_1 T_L(T+T_S+T_L)}{A^2}+(\lambda_7-\lambda_6)\frac{\beta_1 T(T_S+T_L)(1-u_2)}{A^2}+(\lambda_8-\lambda_6)\frac{\beta_2 T T_L(1-u_3)}{A^2}\\
&+ (\lambda_8-\lambda_7)\frac{\beta_3 T_S T_L(1-u_3)}{A^2}-\delta_1\lambda_2\frac{DT_S}{A^2}+\delta_2\lambda_3\frac{(A-H)T_S}{A^2}+\delta_3\lambda_6\frac{T(T+T_S+T_L)}{A^2}\\
&+ \delta_4\lambda_7\frac{T_S(T+T_S+T_L)}{A^2}+\delta_5\lambda_8\frac{T_L(T+T_S+T_L)}{A^2}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\lambda}_4 &= \lambda_4\mu+(\lambda_4-\lambda_2)\frac{mA(D+H)}{(A+I)^2}+(\lambda_4-\lambda_3)\frac{nA(D+H)}{(A+I)^2}+(\lambda_4-\lambda_6)\frac{qA(T+T_S+T_L)(1-u_1)}{(A+I)^2}\\
&+ (\lambda_4-\lambda_7)\frac{eA(T_S+T_L)(1-u_2)}{(A+I)^2}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\lambda}_5 &= \lambda_5(\mu+\eta)+(\lambda_5-\lambda_1)\Lambda_6+(\lambda_5-\lambda_8)\frac{l_2 A T_L(1-u_3)}{(A+P)^2}+(\lambda_5-\lambda_7)\frac{bA(T_S+T_L)(1-u_2)}{(A+P)^2}\\
&+ (\lambda_5-\lambda_6)\frac{kA(T+T_S+T_L)(1-u_1)}{(A+P)^2}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\lambda}_6 &= -\omega_1+\lambda_6\mu+(\lambda_6-\lambda_1)\Lambda_8+(\lambda_2-\lambda_1)\frac{\Lambda_1 S(D+H)}{(A+S)^2}+(\lambda_3-\lambda_1)\frac{\Lambda_3 S T_S}{(A+S)^2}+(\lambda_4-\lambda_1)\frac{\Lambda_9 S T_S}{(A+S)^2}\\
&+ (\lambda_1-\lambda_6)\frac{\Lambda_7 S(S+D+H)(1-u_1)}{(A+S)^2}+(\lambda_2-\lambda_4)\frac{mI(D+H)}{(A+I)^2}+(\lambda_3-\lambda_4)\frac{nI(D+H)}{(A+I)^2}\\
&+ (\lambda_7-\lambda_4)\frac{eI(T_S+T_L)(1-u_2)}{(A+I)^2}+(\lambda_4-\lambda_6)\frac{qI(I+D+H)(1-u_1)}{(A+I)^2}+(\lambda_8-\lambda_5)\frac{l_2 P T_L(1-u_3)}{(A+P)^2}\\
&+ (\lambda_7-\lambda_5)\frac{bP(T_S+T_L)(1-u_2)}{(A+P)^2}+(\lambda_5-\lambda_6)\frac{kP(P+D+H)(1-u_1)}{(A+P)^2}+(\lambda_2-\lambda_6)\frac{\varepsilon_1 D(D+H)(1-u_1)}{A^2}\\
&+ (\lambda_7-\lambda_2)\frac{\varepsilon_2 D(T_S+T_L)(1-u_2)}{A^2}+(\lambda_8-\lambda_2)\frac{\varepsilon_3 D T_L(1-u_3)}{A^2}+(\lambda_3-\lambda_6)\frac{\varepsilon_4 H(D+H)(1-u_1)}{A^2}\\
&+ (\lambda_7-\lambda_3)\frac{\varepsilon_5 H(T_S+T_L)(1-u_2)}{A^2}+(\lambda_8-\lambda_3)\frac{\varepsilon_6 H T_L(1-u_3)}{A^2}+(\lambda_6-\lambda_5)\frac{h(A-T)(D+H)}{A^2}\\
&+ (\lambda_5-\lambda_7)\frac{aT_S(D+H)}{A^2}+(\lambda_5-\lambda_8)\frac{l_1(D+H)T_L}{A^2}+(\lambda_6-\lambda_7)\frac{\beta_1(A-T)(1-u_2)}{A^2}\\
&+ (\lambda_6-\lambda_8)\frac{\beta_2(A-T)(1-u_3)}{A^2}+(\lambda_8-\lambda_7)\frac{\beta_3 T_S T_L(1-u_3)}{A^2}-\delta_1\lambda_2\frac{DT_S}{A^2}-\delta_2\lambda_3\frac{HT_S}{A^2}\\
&+ \delta_3\lambda_6\frac{(A-T)(D+H)}{A^2}-\delta_4\lambda_7\frac{T_S(D+H)}{A^2}-\delta_5\lambda_8\frac{T_L(D+H)}{A^2}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\lambda}_7 &= -\omega_2 + \lambda_7\mu + (\lambda_7-\lambda_1)\Lambda_5 + (\lambda_2-\lambda_1)\frac{\Lambda_1 S(D+H)}{(A+S)^2} + (\lambda_1-\lambda_3)\frac{\Lambda_3 S(A+S-T_S)}{(A+S)^2}\\
&+ (\lambda_1-\lambda_4)\frac{\Lambda_9 S(A+S-T_S)}{(A+S)^2} + (\lambda_1-\lambda_6)\frac{\Lambda_7(S+D+H)(1-u_1)}{(A+S)^2} + (\lambda_2-\lambda_4)\frac{mI(D+H)}{(A+I)^2}\\
&+ (\lambda_3-\lambda_4)\frac{nI(D+H)}{(A+I)^2} + (\lambda_4-\lambda_6)\frac{qI(I+D+H)(1-u_1)}{(A+I)^2} + (\lambda_4-\lambda_7)\frac{eI(I+T+D+H)(1-u_2)}{(A+I)^2}\\
&+ (\lambda_8-\lambda_5)\frac{l_2 PT_L(1-u_3)}{(A+P)^2} + (\lambda_5-\lambda_6)\frac{kP(P+D+H)(1-u_1)}{(A+P)^2} + (\lambda_5-\lambda_7)\frac{bP(P+T+D+H)(1-u_2)}{(A+P)^2}\\
&+ (\lambda_2-\lambda_6)\frac{\varepsilon_1 D(D+H)(1-u_1)}{A^2} + (\lambda_2-\lambda_7)\frac{\varepsilon_2 D(T+D+H)(1-u_2)}{A^2} + (\lambda_8-\lambda_2)\frac{\varepsilon_3 DT_L(1-u_3)}{A^2}\\
&+ (\lambda_3-\lambda_6)\frac{\varepsilon_4 H(D+H)(1-u_1)}{A^2} + (\lambda_3-\lambda_7)\frac{\varepsilon_5 H(T+D+H)(1-u_2)}{A^2} + (\lambda_8-\lambda_3)\frac{\varepsilon_6 HT_L(1-u_3)}{A^2}\\
&+ (\lambda_7-\lambda_5)\frac{a(A-T_S)(D+H)}{A^2} + (\lambda_5-\lambda_6)\frac{hT(D+H)}{A^2} + (\lambda_5-\lambda_8)\frac{l_1 T_L(D+H)}{A^2}\\
&+ (\lambda_6-\lambda_7)\frac{\beta_1 T(T+D+H)(1-u_2)}{A^2} + (\lambda_8-\lambda_6)\frac{\beta_2 TT_L(1-u_3)}{A^2} + (\lambda_7-\lambda_8)\frac{\beta_3 T_L(A-T_S)(1-u_3)}{A^2}\\
&+ \delta_1\lambda_2\frac{D(A-T_S)}{A^2} + \delta_2\lambda_3\frac{H(A-T_S)}{A^2} - \delta_3\lambda_6\frac{T(D+H)}{A^2} + \delta_4\lambda_7\frac{(A-T_S)(D+H)}{A^2} - \delta_5\lambda_8\frac{T_L(D+H)}{A^2}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\lambda}_8 &= -\omega_3 + \lambda_8\mu + (\lambda_2-\lambda_1)\frac{\Lambda_1 S(D+H)}{(A+S)^2} + (\lambda_3-\lambda_1)\frac{\Lambda_3 ST_S}{(A+S)^2} + (\lambda_4-\lambda_1)\frac{\Lambda_9 ST_S}{(A+S)^2}\\
&+ (\lambda_1-\lambda_6)\frac{\Lambda_7(S+D+H)(1-u_1)}{(A+S)^2} + (\lambda_2-\lambda_4)\frac{mI(D+H)}{(A+I)^2} + (\lambda_3-\lambda_4)\frac{nI(D+H)}{(A+I)^2}\\
&+ (\lambda_4-\lambda_6)\frac{qI(I+D+H)(1-u_1)}{(A+I)^2} + (\lambda_4-\lambda_7)\frac{eI(I+T+D+H)(1-u_2)}{(A+I)^2}\\
&+ (\lambda_5-\lambda_8)\frac{l_2 P(A+P-T_L)(1-u_3)}{(A+P)^2} + (\lambda_5-\lambda_7)\frac{bP(P+T+D+H)(1-u_2)}{(A+P)^2}\\
&+ (\lambda_5-\lambda_6)\frac{kP(P+D+H)(1-u_1)}{(A+P)^2} + (\lambda_2-\lambda_6)\frac{\varepsilon_1 D(D+H)(1-u_1)}{A^2}\\
&+ (\lambda_2-\lambda_7)\frac{\varepsilon_2 D(T+D+H)(1-u_2)}{A^2} + (\lambda_2-\lambda_8)\frac{\varepsilon_3 D(A-T_L)(1-u_3)}{A^2}\\
&+ (\lambda_3-\lambda_6)\frac{\varepsilon_4 H(D+H)(1-u_1)}{A^2} + (\lambda_3-\lambda_7)\frac{\varepsilon_5 H(T+D+H)(1-u_2)}{A^2}\\
&+ (\lambda_3-\lambda_8)\frac{\varepsilon_6 H(A-T_L)(1-u_3)}{A^2} + (\lambda_8-\lambda_5)\frac{l_1(A-T_L)(D+H)}{A^2} + (\lambda_5-\lambda_6)\frac{hT(D+H)}{A^2}\\
&+ (\lambda_5-\lambda_7)\frac{aT_S(D+H)}{A^2} + (\lambda_6-\lambda_7)\frac{\beta_1 T(T+D+H)(1-u_2)}{A^2} + (\lambda_6-\lambda_8)\frac{\beta_2 T(A-T_L)(1-u_3)}{A^2}\\
&+ (\lambda_7-\lambda_8)\frac{\beta_3 T_S(A-T_L)(1-u_3)}{A^2} - \delta_1\lambda_2\frac{DT_S}{A^2} - \delta_2\lambda_3\frac{HT_S}{A^2} - \delta_3\lambda_6\frac{T(D+H)}{A^2} - \delta_4\lambda_7\frac{T_S(D+H)}{A^2}\\
&+ \delta_5\lambda_8\frac{(A-T_L)(D+H)}{A^2}
\end{aligned}
$$

Further, the optimal control (u_1^*, u_2^*, u_3^*) is

$$
\begin{aligned}
u_1^* &= \max\{0, \min\{1; \tau_1^*\}\}\\
u_2^* &= \max\{0, \min\{1; \tau_2^*\}\}\\
u_3^* &= \max\{0, \min\{1; \tau_3^*\}\}
\end{aligned}
$$

where

$$
\tau_1^* = \frac{1}{2\omega_4}\left[\frac{(\lambda_6-\lambda_1)\Lambda_7 S}{A+S} + \frac{(\lambda_6-\lambda_2)\varepsilon_1 D}{A} + \frac{(\lambda_6-\lambda_3)\varepsilon_4 H}{A} + \frac{(\lambda_6-\lambda_4)qI}{A+I} + \frac{(\lambda_6-\lambda_5)kP}{A+P}\right](T+T_S+T_L)
$$

$$\tau_2^* = \frac{1}{2\omega_5}\left[\frac{(\lambda_7-\lambda_2)\varepsilon_2 D}{A} + \frac{(\lambda_7-\lambda_3)\varepsilon_5 H}{A} + \frac{(\lambda_7-\lambda_4)eI}{A+I} + \frac{(\lambda_7-\lambda_5)bP}{A+P} + \frac{(\lambda_7-\lambda_6)\beta_1 T}{A}\right](T_S+T_L)$$

$$\tau_3^* = \frac{1}{2\omega_6}\left[\frac{(\lambda_8-\lambda_2)\varepsilon_3 D}{A} + \frac{(\lambda_8-\lambda_3)\varepsilon_6 H}{A} + \frac{(\lambda_8-\lambda_5)l_2 P}{A+P} + \frac{(\lambda_8-\lambda_6)\beta_2 T}{A} + \frac{(\lambda_8-\lambda_7)\beta_3 T_S}{A}\right]T_L$$

Proof. Following [30] and [32], we determine the differential of the Hamiltonian with respect the system variables and deduce the adjoint system

$$\begin{array}{llll}
\dot{\lambda}_1 = -\dfrac{\partial\mathcal{H}}{\partial S} & \dot{\lambda}_2 = -\dfrac{\partial\mathcal{H}}{\partial D} & \dot{\lambda}_3 = -\dfrac{\partial\mathcal{H}}{\partial H} & \dot{\lambda}_4 = -\dfrac{\partial\mathcal{H}}{\partial I} \\
\dot{\lambda}_5 = -\dfrac{\partial\mathcal{H}}{\partial P} & \dot{\lambda}_6 = -\dfrac{\partial\mathcal{H}}{\partial T} & \dot{\lambda}_7 = -\dfrac{\partial\mathcal{H}}{\partial T_S} & \dot{\lambda}_8 = -\dfrac{\partial\mathcal{H}}{\partial T_L}
\end{array}$$

To obtain the optimal control formulation we solve the given equation by the Hamiltonian differential $\mathcal{H}$ with respect to (u_1, u_2, u_3). It follows that

$$u_1^* = \begin{cases} 0 & \text{if} \quad \tau_1^* \leqslant 0 \\ \tau_1^* & \text{if} \quad 0 < \tau_1^* < 1 \\ 1 & \text{if} \quad \tau_1^* \geqslant 1 \end{cases},$$

$$u_2^* = \begin{cases} 0 & \text{if} \quad \tau_2^* \leqslant 0 \\ \tau_2^* & \text{if} \quad 0 < \tau_2^* < 1 \\ 1 & \text{if} \quad \tau_2^* \geqslant 1 \end{cases}$$

and

$$u_3^* = \begin{cases} 0 & \text{if} \quad \tau_3^* \leqslant 0 \\ \tau_3^* & \text{if} \quad 0 < \tau_3^* < 1 \\ 1 & \text{if} \quad \tau_3^* \geqslant 1 \end{cases}.$$

with

$$\tau_1^* = \frac{1}{2\omega_4}\left[\frac{(\lambda_6-\lambda_1)\Lambda_7 S}{A+S} + \frac{(\lambda_6-\lambda_2)\varepsilon_1 D}{A} + \frac{(\lambda_6-\lambda_3)\varepsilon_4 H}{A} + \frac{(\lambda_6-\lambda_4)qI}{A+I} + \frac{(\lambda_6-\lambda_5)kP}{A+P}\right](T+T_S+T_L)$$

$$\tau_2^* = \frac{1}{2\omega_5}\left[\frac{(\lambda_7-\lambda_2)\varepsilon_2 D}{A} + \frac{(\lambda_7-\lambda_3)\varepsilon_5 H}{A} + \frac{(\lambda_7-\lambda_4)eI}{A+I} + \frac{(\lambda_7-\lambda_5)bP}{A+P} + \frac{(\lambda_7-\lambda_6)\beta_1 T}{A}\right](T_S+T_L)$$

$$\tau_3^* = \frac{1}{2\omega_6}\left[\frac{(\lambda_8-\lambda_2)\varepsilon_3 D}{A} + \frac{(\lambda_8-\lambda_3)\varepsilon_6 H}{A} + \frac{(\lambda_8-\lambda_5)l_2 P}{A+P} + \frac{(\lambda_8-\lambda_6)\beta_2 T}{A} + \frac{(\lambda_8-\lambda_7)\beta_3 T_S}{A}\right]T_L$$

$\blacksquare$

5.3. Numerical simulation of the optimal control problem

The aim of the control is to determine a cost-effective control strategy. To achieve this, we use numerical simulations to study the impact of each control. In addition, we will carry out a comparative study of the impact of each control in the fight against ideological terrorism. This study will enable us to highlight the impact of the strategies linked to each control function, and thus to identify an effective counter-terrorism strategy at a lower cost in terms of time, human and financial resources.

In order to observe the impact of the values taken by the control functions, we'll use the same parameter values as in the case $R_0 = 0.0268 < 1$. Indeed, for $R_0 < 1$, the terrorist ideology is already in extinction, and for control values tending towards 1, we should be able to observe curves showing faster decay in the compartments P, T, T_S and T_L. This will show that by acting on the factors represented by the three controls, the State can significantly increase the effectiveness of the fight.

Figures Comments:

Figure 4: We assume that the government and its partners and the entire population, in a spirit of patriotism, are working to defeat terrorism. This action is reflected in continuous awareness-raising actions, the reinforcement and acquisition of increasingly efficient military equipment accompanied by solid and adapted military training. As the figure shows, the implementation of a synergy of action based on the three control functions makes it possible to defeat terrorism in all its forms in a relatively short time.

Figure 5: Among the infected classes P, T, T_s and T_L, terrorists are the class that is most in contact with the other compartments. Thus, by putting all the available means on the control u_1, we see that the measures taken in this direction reach all the other compartments. For $u_1 = 1$, we assume that the entire population has fully integrated the fact that no one should adhere to terrorist ideology. So the terrorist compartment will gradually empty out, stabilizing at zero, and there will be no more opportunities for recruitment. Then, since terrorist soldiers recruit mainly from the T class and leaders from the T and Ts classes, the extinction of the terrorist class inexorably leads to the extinction of the Ts and T_L classes.

Figure 6: The control u_2 represents the ability of DSF and HDV to respond attack and carry out preventive operations. So, for $u_2 = 1$ DFS and HDV are the DFS and HDV are well trained, equipped and qualified for combat. However, military equipment and training are designed only for the army. This can be seen in the figure. In fact, the consequences of the control u_2 are effective on soldier terrorists, but have no effect on other compartments. The soldier terrorists will certainly be eradicated, but the terrorist ideology will remain through the persistence of the T and TL compartments. The latter will always work to create the compartment of terrorist soldiers. The struggle can go on forever, which means that terrorism cannot be defeated by military action alone.

Figure 7: Putting all resources into u_3 control is probably the least effective control strategy. Leading terrorists are eliminated, but all other classes remain intact, and the curves are confounded. Eliminating the leaders will disorganize the fight and spread panic among the terrorists. However, the transition from soldier to leader allows a renewal of the leader class. The change of leaders within terrorist groups can also, against all odds, instill a new dynamic and reinvigorate the terrorists. It's worth noting that new leaders, coming from the soldier class of terrorists, because they have fought in combat, are more familiar with the context of the struggle and may prove to be more competent. These new, potentially more effective leaders can reverse the trend of the struggle and succeed in leading the terrorists to an undesirable victory for a country.

Figure 8: Joint actions on controls $u_1(t)$, $u_2(t)$ and $u_3(t)$ have a significant impact on the compartments T, T_S and T_L that we aim to stabilize at 0. This is a further proof of the importance of synergy of action (numbers, equipment, strategy and training) within the defense and ongoing awareness-raising campaigns.

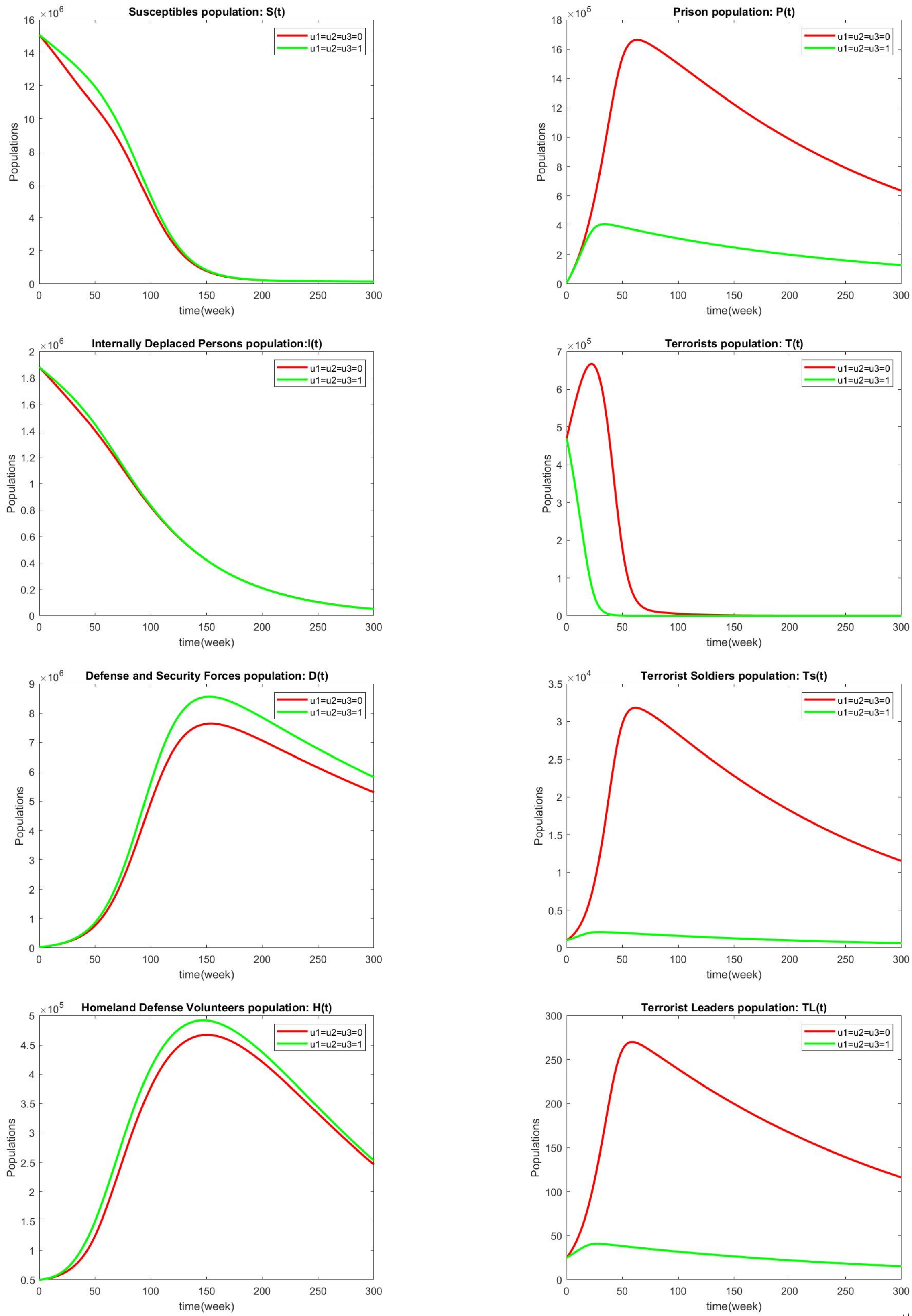

Figure 4: **Comparative evolution of the different populations, setting :** $u_1 = u_2 = u_3 = 0$ **and** $u_1 = u_2 = u_3 = 1$**.**

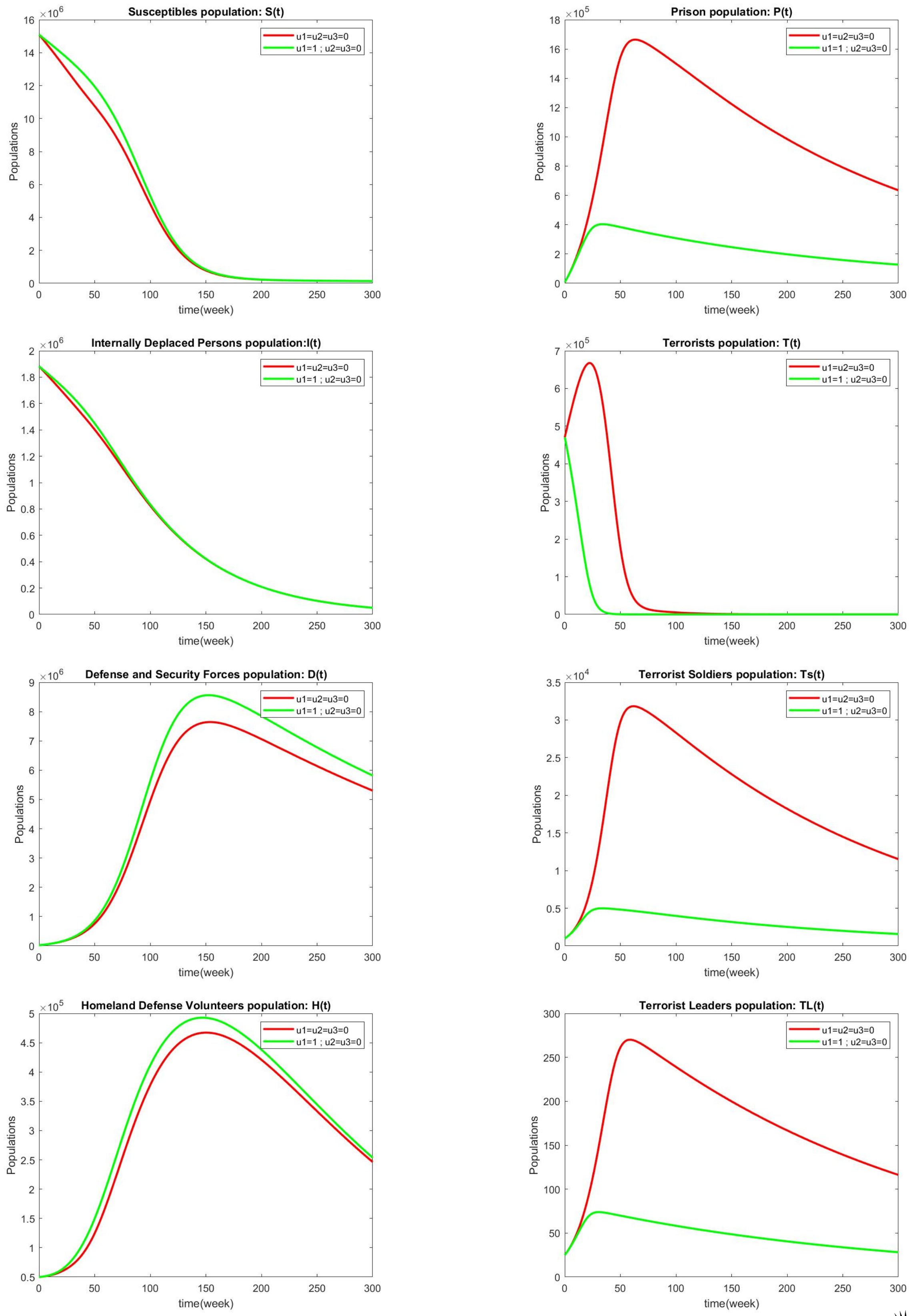

Figure 5: **Comparative evolution of the different populations, Setting :** $u_1 = u_2 = u_3 = 0$ **and** $u_1 = 1; u_2 = u_3 = 0$.

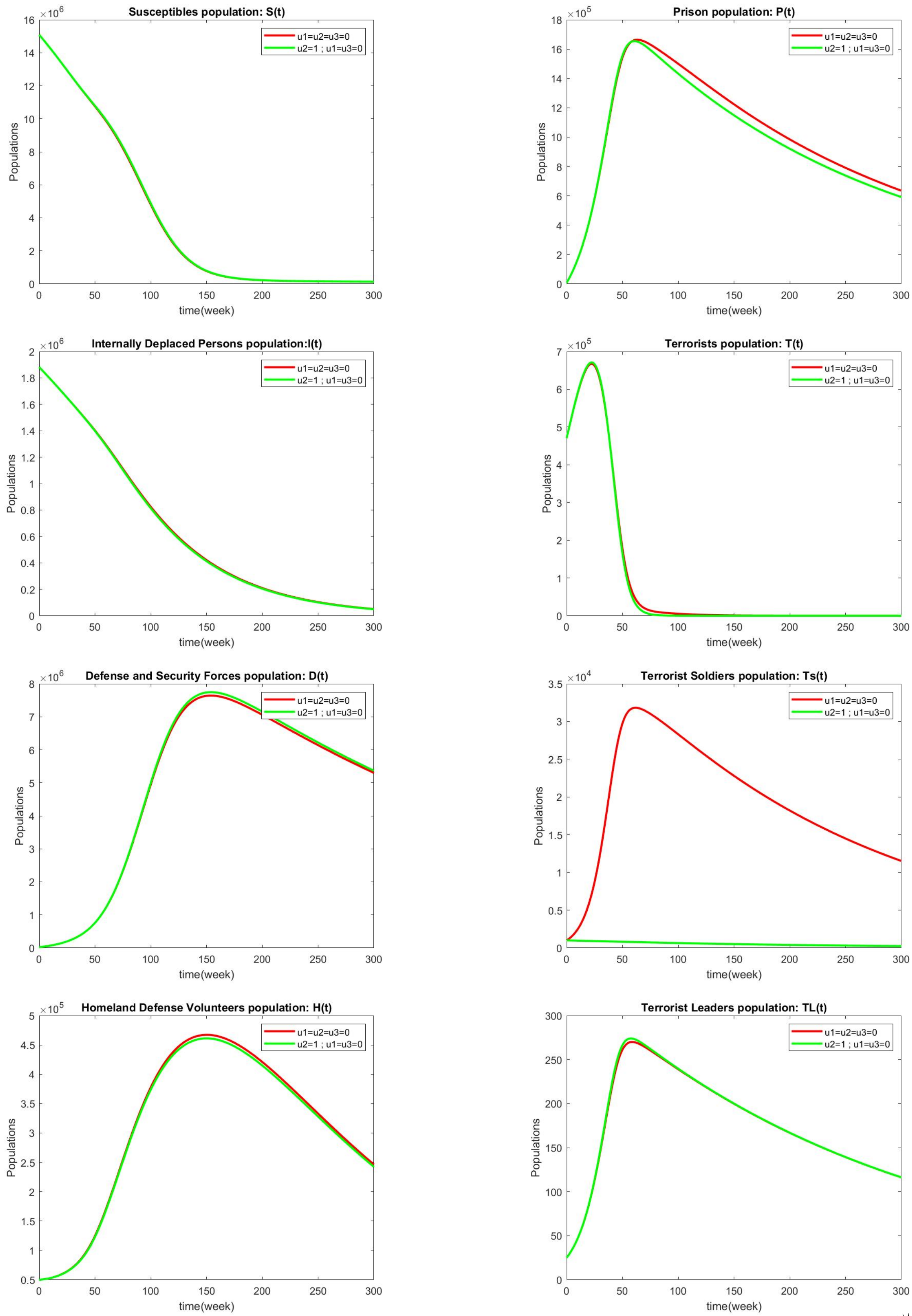

Figure 6: **Comparative evolution of the different populations, Setting :** $u_1 = u_2 = u_3 = 0$ **and** $u_2 = 1; u_1 = u_3 = 0$.

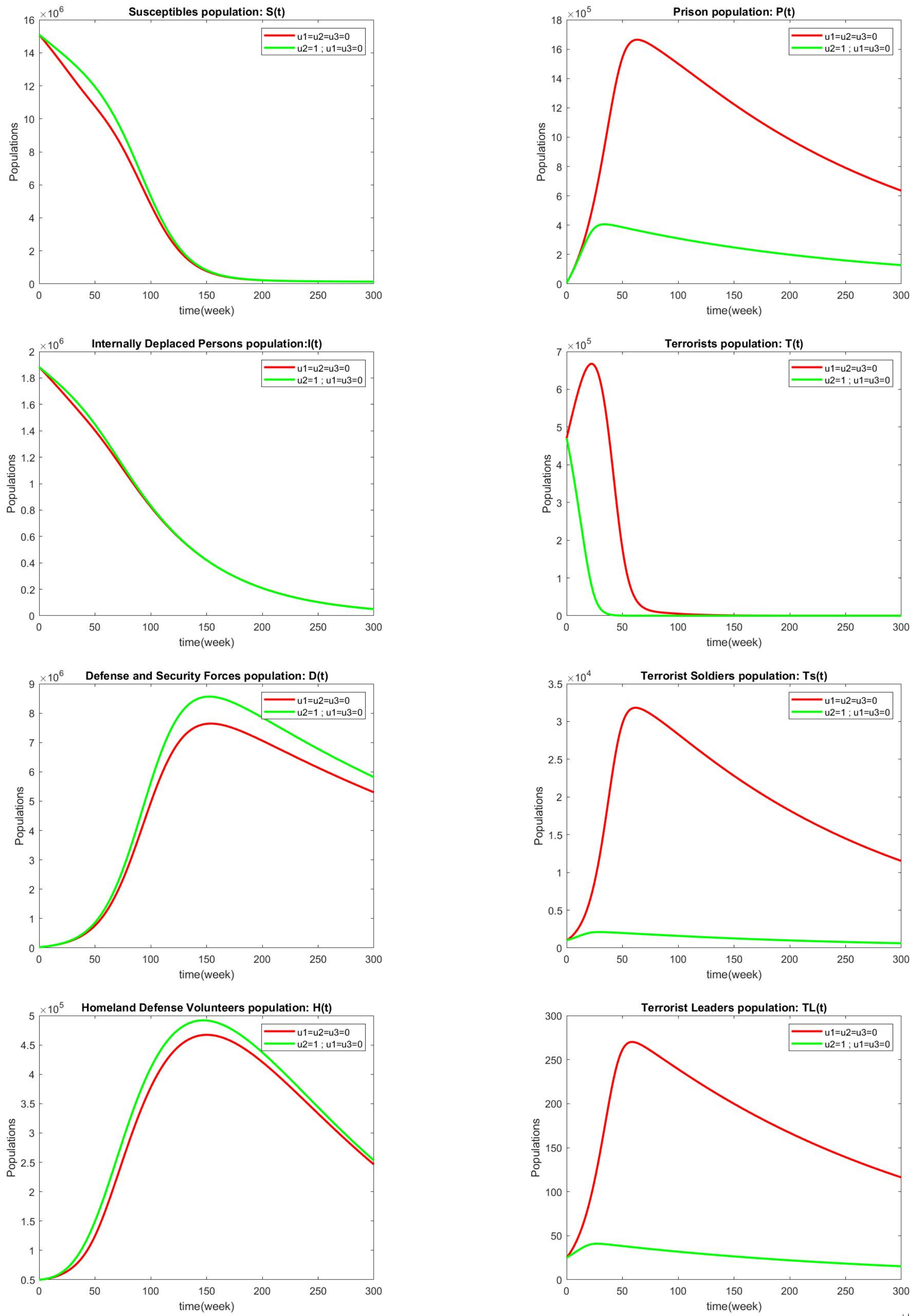

Figure 7: **Comparative evolution of the different populations, Setting :** $u_1 = u_2 = u_3 = 0$ **and** $u_3 = 1; u_1 = u_2 = 0$.

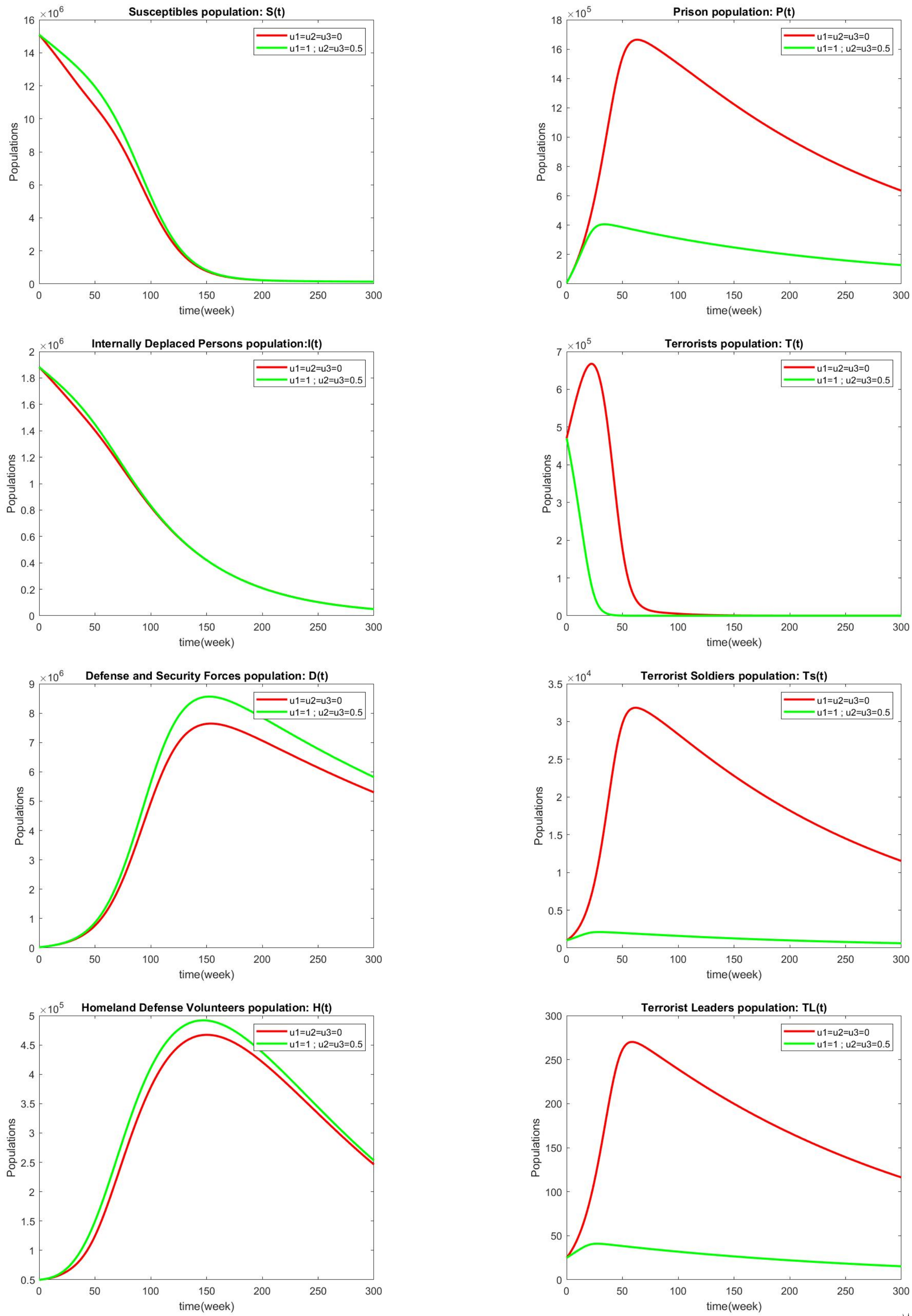

Figure 8: **Comparative evolution of the different populations, Setting :** $u_1 = u_2 = u_3 = 0$ **and** $u_1 = 1; u_2 = u_3 = 0.5$**.**

6. Discussing

The analysis of the optimal control problem shows that it's important that there be synergy of action in the fight against terrorism. Indeed, each of the three axes of struggle that we have identified and controlled must be considered and substantial resources must be injected into them. When all the three axes are stimulated simultaneously, terrorism can be eradicated in a time of about 300 weeks, see figure 4. On the other hand, when one of the axes is abandoned, after the same period of 300 weeks, there are still some individuals in compartments T, T_S and T_L, see figures 5, 6 and 7. This will result in a longer time of struggle, during which time uncontrolled events could change the course of the struggle. It should also be noted that the u_1 control is the most sensitive, see figure 5 where the evolution of the populations in compartments T, T_S and T_L is like in figure 4 where $u_1 = u_2 = u_3 = 1$; this means that the government, in coordination with civil organizations and religious institutions, is carrying out large-scale awareness-raising actions in order to rekindle the patriotic flame in the hearts of the people. According to our model, such actions will reduce the number of individuals in compartment T to zero. As a result, the compartments T_S and T_L will no longer be able to recruit and will be emptied; the figure 8 is supporting this idea.

7. Conclusion

This paper present a mathematical modeling and control of the dynamics of terrorist ideologies. In particular, the model takes into account the fact that military personnel, FDS and HDV, can be led to radicalize. Subdividing the population in eight compartments we have constructed a deterministic model using contacts process. The theoretical analysis of the model highlights the existence of a disease-free equilibrium that is globally asymptotically stable. Consequently the spread of the terrorist ideologies can be effectively controlled in the population, whatever the number of infectious people individuals initially introduced into the completely susceptible population. This is how we have introduced tree (03) time-dependent control $u_1(t)$, $u_2(t)$ and $u_3(t)$ with the aim of limiting or even eradicating the spreading of terrorist ideologies in the population.

Terrorism is a new challenge for our country. The fight has made progress, but the threat persists and has diversified, according to the United Nation. It is up to each State, according to its realities, to find endogenous and lasting solutions to effectively and definitively eradicate the terrorist hydra.

8. Acknowledgement

The author is thankful to the referee for his valuable suggestions which improved the presentation of the paper.

References

[1] C.L. Ruby, The definition of terrorism, *Analyses of social issues and public policy, Wiley Online Library*, **2**(2002), No.1, 9-14.

[2] A.P. Shmid, The revised academic consensus definition of terrorism, *Perspectives on Terrorism*, **6**(2012), No.2.

[3] J.F. Prévost, Les aspects nouveaux du terrorisme international, *Annuaire français de droit international, Persée-Portail des revues scientifiques en SHS*, **19**(1973), No.1, 579-600.

[4] B. Hoffman, D.K. Hoffman, The Rand-St Andrews chronology of international terrorist incidents, *Taylor & Francis*, (1996).

[5] W. Enders and T. Sandler, What do we know about the substitution effect in transnational terrorisme, in Research on terrorism , *Routledge*, (2003), 139-157.

[6] T. Deffarges, Sur la nature et les causes du terrorisme. Une revue de la littérature économique, *Revue Tiers Monde, Cairn/Isako*, **2**(2003), 369-392.

[7] K. Gaibulloev, J. Piazza and T. Sandler, Regime types and terrorism, *International organization, Cambridge University Press*, **71**(2017), No.3, 491–522.

[8] C.J. Lumsden, Color categorization: a possible concordance between genes and culture, *Proceedings of the National Academy of Sciences, National Acad Sciences*, **82**(1985), No.17, 5805-5808.

[9] M.W Feldman and L. Cavalli-Sforza, On the theory of evolution under genetic and cultural transmission with application to the lactose absorption problem, *Stanford Institute for Population and Resource Studies*, (1985).

[10] DC. Dennett, Darwin's Dangerous Idea Simon & Schuster,*New York*, (1995).

[11] A. Isyaku, Terrorism: A New Challenge to NigeriaS Stability in the 21 st Century, *Terrorism*, **12**(2013).

[12] C.G. Ngari, Modeling Kenya Domestic Radicalization like A Disease Incorporating Rehabilitation Centers, *Academic Research Publishing Group*, **2**(2016), 27-39.

[13] A. Gambo, Mathematical modeling of dynamics behavior of terrorism and control, *Caspian Journal of Mathematical Sciences (CJMS), University of Mazandaran*, **9**(2020), No 1, 68–85.

[14] M. Pooda, Y. Simpore and O. Traoré, Modeling and optimal control of the dynamics of narcoterrorism in the Sahel, *Malaya Journal of Matematik*, **12**(2024), No.2, 163-185.

[15] M. Pooda, Y. Simpore and O. Traoré, Modeling and Multi-objective Optimal Control of The dynamics of Counterterrorism in the Sahel Region in Africa, *European Journal of Pure and Applied Mathematics*, **17**(2024), No.2, 870-904.

[16] United Nations High Commissioner for Refugees, Internally Displaced People, *https://www.unhcr.org/internally-displaced-people.html* **2022-08-06**.

[17] Doctors without borders, The Practical Guide to Humanitarian Law, *https://guide-humanitarian-law.org/content/article/3/internally-displaced-persons/*, **2022-08-06**.

[18] C. Castillo-Chavez and B. Song, Models for the transmission dynamics of fanatic behaviors, In Bioterrorism: mathematical modeling applications in homeland security, *SIAM*, (2003), 155-172.

[19] K.L. Cooke and P. van den Driessche, Analysis of an SEIRS epidemic model with two delays, *Journal of Mathematical Biology*, *https://doi.org/10.1007/s002850050051*, **35**(1996), No.2, 240-260, ISSN:1432-1416.

[20] Y. Wang and F. Bu,Modeling radicalization of terrorism under the influence of multiple ideologies,*AIMS Mathematics*, **7-3**, (2022), 4833-4850.

[21] H. Ouedraogo, D. Ouedraogo, I. Ibrango and A. Guiro, A study of stability of SEIHR model of infectious disease transmission,*Non autonomous Dynamical Systems, De Gruyter*, **8**(2021), No.1, 307-327.

[22] P. Bonnard, N. Kalach, J.F. Cadranel, F. Remoue, G. Riveau and A. Capron, Manifestations digestives et hépatiques de la bilharziose, *Gastroenterologie Clinique Et Biologique; https://api.semanticscholar.org/CorpusID:204072827*, **24**(2000), 409-419.

[23] G. Birkhoff and G.C. Rota, Ordinary Differential Equations, *Search in, Ginn, Boston*, (1982).

[24] M. SANTOPRETE AND X. FEI, Global stability in a mathematical model of de-radicalization, *Physica A: Statistical Mechanics and its Applications, Elsevier*, **509**(2018), 151-161.

[25] P. VAN DEN DRIESSCHE AND J. WATMOUGH, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Mathematical biosciences, Elsevier*, **180**(2002), 29-148.

[26] R.S. VARGA, Matrix iterative analysis: Englewood Cliffs, *XIV-NY: Prentice-Hall*, (1962).

[27] O. DIEKMANN AND J.A.P. HEESTERBEEK, Mathematical epidemiology of infectious diseases: model building, analysis and interpretation, *John Wiley & Sons*, **5**, (2000).

[28] D.L. LUKES, Differential equations: classical to controlled, (1982).

[29] W.H. FLEMMING AND W.R. RISHEL, Deterministic and stochastic optimal control, *Springer Science & Business Media*, **1**(2012).

[30] S. OLANIYI, K. OKOSUN, A. OLADIPO AND S. ADEWALE, Mathematical modelling and optimal cost-effective control of COVID-19 transmission dynamics, *The European Physical Journal Plus*, *Springer Berlin Heidelberg*, **135**(2020), 938.

[31] L. SEMENOVICH, Mathematical theory of optimal processes, *Routledge*, (2018).

[32] E. TRELAT, Contrôle optimal: théorie et applications, *Vuibert, Collection" Mathématiques Concrètes*, (2005).

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **12(04)**(2024), 367–387.
http://doi.org/10.26637/mjm1204/003

Analysis and optimal control for SEIR mathematical modeling of COVID-19

Lassina Ouattara*[1], Harouna Ouedraogo[2], Dramane Ouedraogo[3], Aboudramane Guiro[4]

[1,3,4] *Département de Mathématiques, Université Nazi BONI, Bobo-Dioulasso, Burkina Faso.*
[2] *Département de Mathématiques, Université Joseph Ki Zerbo, Ouagadougou, Burkina Faso.*

***R*eceived 23 March 2024; *A*ccepted 18 August 2024**

Abstract. In this paper a mathematical model of SEIR type is formulated. represented by modeling the coronavirus epidemic. In this present study, we consider a mathematical model that incorporates the whole population and variability in transmission between reported and unreported populations. The global stability of the disease free equilibrium (DFE) point is established. The basic reproduction number R_0 is calculated. We introduce into our model two controls which are vaccination of susceptible humans denoted by u and treatment of infected humans designed by v. In addition, this model takes into consideration the control of contact (γ) between infectious individuals and susceptible persons. A numerical simulation of the model is made.

AMS Subject Classifications: 34A34, 34H05, 34H15, 65D25

Keywords: COVID-19, mathematical modeling, SEIR, disease-free equilibrium, stability, basic reproduction number, optimal control, Hamiltonian.

Contents

*Corresponding author. Email address: **lassmaths2@gmail.com** (Lassina Ouattara)

1. Introduction

The pandemic of COVID-19 is an infectious disease caused by the virus Sars-CoV-2 and characterized by severe acute respiratory syndrome. In late December 2019, the disease COVID-19 was first identified in China precisely in city of Wuhan ([37]). This virus has caused several deaths in the world and deserves the attention of researchers. In Burkina Faso, the first case is detected on March 9, 2020 ([28, 35]). Today, there is no effective treatment that has been accepted. In response to this epidemic, the state has taken a number of measures to reduce the spread of the virus. The best way to fight COVID-19 is to find ways to limit the spread of the virus in public spaces. The whole world is now concerned with the transmission of the disease by trying out vaccines, treatments and barrier measures in order to control the disease. In the literature, several mathematical models have been studied in order to show the dynamics of the infectious disease (see the references [8, 22, 38, 39]). Wu et al. ([38]) developed a susceptible exposed infectious recovered model (SEIR) to clarify the transmission dynamics and global spread of disease. Tang et al. ([32]) proposed a compartmental deterministic model that would combine the clinical development of the disease, the patient's state of health and intervention measures. Researchers found that the amount of control reproduction number may be as high as $R_0 = 6.47$, and that the methods of intervention, including contact followed by quarantine and isolation would effectively minimize COVID-19 cases ([9, 33]). Several modeling studies have already been performed for the COVID-19 outbreak (see [20, 27, 31–33]). Recent mathematical models with optimal control have been developed to study the COVID-19 pandemic. Hongzhi Lin and Yongping Zhang are studying a COVID-19 model to determine the optimal deployment of cordon sanitaires in terms of minimum queueing delay time with available health testing resources (see [14]). Shou Chens and Chen Xiao are studying a COVID-19 model to determine the associated credit risk contagion among financial institutions (see [2]). According to the models and the epidemiological characteristics of COVID-19 ([5]), we propose a SEIR type model to study the dynamics of this current pandemic (see [11, 20, 25, 29]). Our model is described by differential equations system and gives a comprehensive mechanism for the dynamics of COVID-19 transmission. In this model, we take into consideration the control of contact (γ) between infectious individuals and susceptible persons. We introduce into our model two controls which are vaccination of susceptible humans denoted by u and treatment of infected humans designed by v.

The organization of this paper is as follow: In Section 2, we formulate the mathematical model for COVID-19. In Section 3, we give Mathematical properties of the model (estimation of R_0, parameters with biological interpretation of model, positivity and boundedness of the solution). In Section 4, we establish the global stability of disease free equilibrium (DFE). In Section 5, we give a numerical simulation in order to illustrate the theoretical results. In Section 6, we give the optimal control problem and we derive the necessary condition for existence optimal control and we present the resulting numerical simulation. Finally, in Section 7, we give the conclusion.

2. Mathematical model

In this section, we formulate the mathematical model. Considering the characteristics of the COVID-19 pandemic, we have the following compartments:

- $S(t)$ Susceptible persons at time t.
- $E(t)$ Exposed and infectious persons at time t.
- $I(t)$ Infected and infectious persons at time t.
- $I_r(t)$ Symptomatic infected and infectious persons at time t (the number of persons infected who are reported and isolated at time t).
- $I_u(t)$ Asymptomatic infected and infectious persons at time t (the number of persons who are infected but do not have symptoms at time t).

- $R_r(t)$ Recovery of infected reported persons at time t.
- $R_u(t)$ Recovery of infected unreported persons at time t.

The class of infected individuals I is subdivided into two classes infected reported persons (I_r) and infected unreported persons (I_u) for the following reasons. Firstly COVID-19 patients do not test for COVID-19 because the test is expensive in this country. As a result, infected people do not show signs of the disease. These infectious persons move freely in the susceptible population and continue to infect them. They are the most vulnerable in the infection of COVID-19 and spread the disease more. Each individual in this class called infected unreported persons (I_u), heals alone and enters the class R_u. Secondly, those infected with COVID-19 who are tested positive are detected, isolated then treated. They are less infectious. Each individual of this class infected reported persons (I_r), heals by treatment and enters in the class R_r.
Therefore, we have the following transfer diagram:

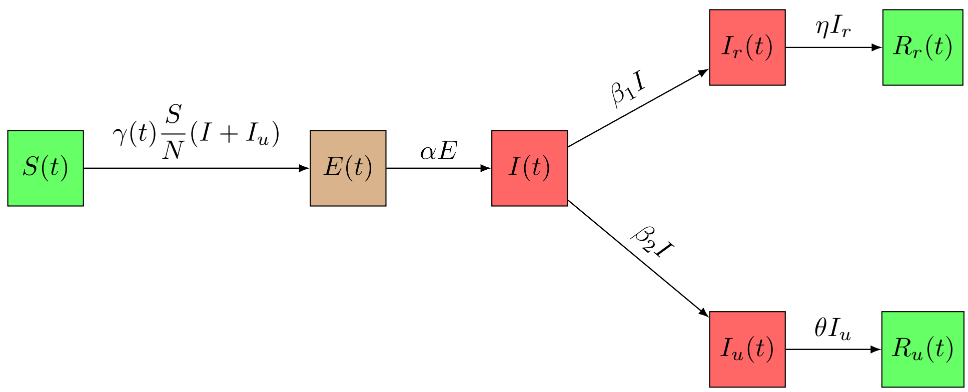

Figure 1: The transfer diagram.

According to the Figure 1 the corona virus mathematical model is

$$\begin{cases} \dfrac{dS}{dt} = \dfrac{-\gamma(t)S(I+I_u)}{N}, \\ \dfrac{dE}{dt} = \dfrac{\gamma(t)S(I+I_u)}{N} - \alpha E, \\ \dfrac{dI}{dt} = \alpha E - (\beta_1+\beta_2)I, \\ \dfrac{dI_r}{dt} = \beta_1 I - \eta I_r, \\ \dfrac{dI_u}{dt} = \beta_2 I - \theta I_u, \\ \dfrac{dR_r}{dt} = \eta I_r, \\ \dfrac{dR_u}{dt} = \theta I_u. \end{cases} \tag{2.1}$$

The initial conditions are:

$$S(0) = S_0 > 0, \quad E(0) = E_0 \geq 0, \quad I(0) = I_0 \geq 0, \quad I_r(0) = I_{r0} \geq 0, \quad I_u(0) = I_{u0} \geq 0,$$

$$R_u(0) = R_{u0} \geq 0, \quad R_r(0) = R_{r0} \geq 0.$$

The total population at time t is given by:

$$N(t) = S(t) + E(t) + I(t) + R_r(t) + I_r(t) + I_u(t) + R_u(t).$$

and the total population $N_0 = S_0 + E_0 + I_0 + I_{r0} + I_{u0} + R_{r0} + R_{u0}$ at the initial time $t_0 = 0$ is constant. Parameters with biological interpretation of model (2.1)

- $\gamma(t)$: the contact rate of a person in state S at time t.
- α : the transition rate of a person in state E.
- β_1 : the transition rate between E and I_r.
- β_2 : the transition rate between E and I_u.
- θ : the transition rate of a person in state I_u to the state R_u.
- η : the transition rate of a person in state I_r to the state R_r.

3. Mathematical properties of the model

3.1. Estimation of R_0

The disease free equilibrium (DFE) of the model (2.1) is $X_0 = (S^0, E^0, I^0, I_u^0, I_r^0, R_r^0, R_u^0) = (N_0, 0, 0, 0, 0, 0, 0)$. We determine the basic reproduction number R_0 by applying Van Den Driesche and Watmougth method ([36]).

Proposition 3.1. *The basic reproduction number of model (2.1) is defined by*

$$R_0 = \frac{\gamma_0(\theta + \beta_2)}{\theta(\beta_1 + \beta_2)}. \tag{3.1}$$

Proof.

$$\mathcal{F} = \begin{pmatrix} \dfrac{\gamma S(I + I_u)}{N} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathcal{V} = \begin{pmatrix} -\alpha E \\ \alpha E - \beta I \\ \beta_1 I - \eta I_R \\ \beta_2 I - \theta I_u \end{pmatrix} \quad \text{where} \quad \beta = \beta_1 + \beta_2. \tag{3.2}$$

$\mathcal{F}$ is the new infection or contact function and $\mathcal{V}$ is the transition function.
$F = \left(\dfrac{\partial \mathcal{F}_j}{\partial x_i}\right)$ with $1 \leq i, j \leq 4$ and similarly $V = \left(\dfrac{\partial \mathcal{V}_j}{\partial x_i}\right)$ with $1 \leq i, j \leq 4$ and

$$X = \begin{pmatrix} E \\ I \\ I_r \\ I_u \end{pmatrix}.$$

which gives

$$V = \begin{pmatrix} -\alpha & 0 & 0 & 0 \\ \alpha & -\beta & 0 & 0 \\ 0 & -\beta_1 & -\eta & 0 \\ 0 & \beta_2 & 0 & -\theta \end{pmatrix} \quad \Leftrightarrow \quad V^{-1} = \begin{pmatrix} \frac{-1}{\alpha} & 0 & 0 & 0 \\ \frac{-1}{\beta} & \frac{-1}{\beta} & 0 & 0 \\ \frac{-\beta_1}{\beta\eta} & \frac{-\beta_1}{\beta\eta} & \frac{-1}{\eta} & 0 \\ \frac{-\beta_2}{\beta\theta} & \frac{-\beta_2}{\beta\theta} & 0 & \frac{-1}{\theta} \end{pmatrix}$$

and

$$F = \begin{pmatrix} 0 & \frac{\gamma_0 S^0}{N_0} & 0 & \frac{\gamma_0 S^0}{N_0} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

V^{-1} is determined by $VX = Y$, then we express the coordinates of vector X as a function of Y.

$$-FV^{-1} = \begin{pmatrix} \frac{\gamma_0\theta S^0 + \beta_2\gamma_0 S^0}{\theta N_0(\beta_1+\beta_2)} & \frac{\gamma_0\theta S^0 + \beta_2\gamma_0 S^0}{\theta N_0(\beta_1+\beta_2)} & 0 & \frac{\gamma_0 S^0}{N_0\theta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

this gives

$$\rho(-FV^{-1}) = \frac{\gamma_0\theta S^0 + \beta_2\gamma_0 S^0}{\theta N_0(\beta_1+\beta_2)} = R_0.$$

The basic reproduction number with γ_0 constant is:

$$R_0 = \frac{\gamma_0\,(\theta+\beta_2)}{\theta(\beta_1+\beta_2)} \tag{3.3}$$

Therefore

$$R_e(t) = \frac{\gamma(t)S(t)\,(\theta+\beta_2)}{N\theta(\beta_1+\beta_2)}$$

$R_e(t)$ is called the effective reproduction number at time t, it is defined as the number of cases that one infected person generates during his infectious period at time t in the presence of barrier measures controlled by $\gamma(t)$. After taking the measures, the number of contacts decreases and $\gamma(t)$ decreases as a function of time t. The disease slows when $R_e(t) < 1$. The basic reproduction number R_0 is defined as the number of cases that one infected person generates on average during his infectious period, in an uninfected population and without any special control measures. This number does not change during the spread of the disease. Furthermore, $\gamma(0) = \gamma_0$ and $R_e(0) = R_0$.

3.2. Positivity and boundedness of the solution for the Model

In this subsection, we show the positivity of the solution of model (2.1), let pose

$$\Omega = \left\{ (S(t), E(t), I(t), I_r(t), I_u(t), R_r(t), R_u(t)) \in \mathbb{R}_+^7; \begin{pmatrix} S(0) \\ E(0) \\ I(0) \\ I_r(0) \\ I_u(0) \\ R_r(0) \\ R_u(0) \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

In general, the following lemma is used to show the positivity of the solutions of time-delay system where τ is the time-delay. In our model, the time-delay $\tau = 0$, therefore if the initial conditions are positive the lemma can be used. In the literature, the following lemma is used by T. Sarda et al. ([30]) and O. Harouna et al. (see [23]) to show the positivity of the solutions of ordinary differential equations.

Lemma 3.2. *([12]) Let $\Omega \subset \mathbb{R} \times \mathbb{C}^n$ an open and $f_i \in C(\Omega, \mathbb{R})$, $i = 1, ..., n$, if $f_i|_{x_i=0} \geq 0$ for $(x_1, ..., x_n) = X_t \in \mathbb{C}_{+0}^n$ then $\mathbb{C}_{+0}^n = \left\{ \phi = (\phi_1, ..., \phi_n) : \quad \phi \in C([-\tau; 0], \mathbb{R}_+^n) \right\}$
is the invarious domain of the following equations:*

$$\frac{dx_i(t)}{dt} = f_i(t, X_t), \quad t \geq \tau, \quad i = 1, ..., n. \tag{3.4}$$

Where $\mathbb{R}_+^n = \{(X_1, ..., X_n) \in \mathbb{R}^n : \quad X_i \geq 0; \quad i = 1, ..., n\}$.

Proof. We consider the following equation

$$\frac{dx_i(t)}{dt} = f_i(t, X(t)) + \frac{1}{m}, \quad t \geq \tau, \quad i = 1, ..., n, \quad n, m \in \mathbb{N}^*. \tag{3.5}$$

Let $x_i(t)$ be the solution of (3.5) and $x_i(t) \geq 0$, $t \in [l - t, l]$, with $x_i(l) > 0$, $i = 1, ..., n$. If there is a $\tau > l$, $X_\tau \notin \mathbb{C}_{+0}^n$, then there must be i and t_0 such that $x_i(t_0) = 0$, $X_{it_0} \geq 0$, $t \in [l, t_0]$. This implies $\dfrac{dx_i(t_0)}{dt} \leq 0$. It contradicts because $\dfrac{dx_i(t_0)}{dt} = f_i(t_0, X_{t_0}) + \dfrac{1}{m} > 0$. So we can say that $\mathbb{C}_{+0}^n$ is the invarious domain of (3.5). Letting $m \longrightarrow +\infty$ we get that $\mathbb{C}_{+0}^n$ is the invarious domain of (3.4)

Proposition 3.3. *The set Ω is positively invariant, moreover the system (2.1) has a unique solution in Ω.*

Proof. We use the same technique as Harouna et al. ([23]) and Sardar et al. ([30]) to show the positivity of the solutions of system (2.1). The system (2.1) can be rewrited as follow

$$\frac{dX_i(t)}{dt} = f_i(t, X(t)), \quad X(0) = X_0 \geq 0, \quad i = 1, ..., 7,$$

where $X(t) = (S, E, I, I_r, I_u, R_r, R_u)$.
We can note that $\dfrac{dS}{dt}\big|_{(S=0)} = 0 \geq 0$, $\dfrac{dE}{dt}|_{(E=0)} = \dfrac{\gamma(t)S(I + I_u)}{N} \geq 0$,
$\dfrac{dI}{dt}|_{(I=0)} = \alpha E \geq 0$, $\dfrac{dI_u}{dt}|_{(I_u=0)} = \beta_2 I \geq 0$,
$\dfrac{dI_r}{dt}|_{(I_r=0)} = \beta_1 I \geq 0$, $\dfrac{dR_r}{dt}|_{(R_r=0)} = \eta I_r \geq 0$,

$\frac{dR_u}{dt}|_{(R_u=0)} = \theta I_u \geq 0.$

Then it follows from the Lemma 3.2 that Ω is an invariant set for the system (2.1).

For the second part of the proof, we use the same techniques as [34] to show the uniqueness of the system solutions (2.1). Let's now consider the following function

$$\dot{Y}(t) = g(t, Y(t)), \text{ where } \quad Y \in \Omega \tag{3.6}$$

and

$$g : \mathbb{R}^+ \times \mathbb{R}^7 \longrightarrow \mathbb{R}^7, \tag{3.7}$$

such as

$$g(t, Y(t)) = \begin{pmatrix} -\frac{\gamma(t)S(t)(I(t)+I_u(t))}{N} \\ \frac{\gamma(t)S(t)(I(t)+I_u(t))}{N} - \alpha E(t) \\ \alpha E(t) - (\beta_1 + \beta_2)I(t) \\ \beta_1 I(t) - \eta I_r(t) \\ \beta_2 I(t) - \theta I_r(t) \\ \eta I_r(t) \\ \theta I_u(t) \end{pmatrix}.$$

The function $g(.,.)$ is continuous and $t \mapsto g(t,.)$ is lipschitzian. By application of theorem.2.2.1 and theorem.2.2.3 of Hale and Verduyn Lunel ([7]), the system (2.1) has a unique solution in Ω.

Proposition 3.4. *The solution of system* (2.1) *is bounded in*
$\Omega_1 = \{(S, E, I, I_r, I_u, R_r, R_u) \in \Omega : \quad S + E + I + I_r + I_u + R_r + R_u \leq N_0\}.$

Proof. $N(t) = S(t) + E(t) + I(t) + I_u(t) + I_r(t) + R_r(t) + R_u(t)$, by using the system (2.1) we get

$\frac{dN}{dt} = 0 \Longleftrightarrow N$ is constant i.e $\forall t \geq 0, \quad N(t) = N(0) = N_0$. Therefore, for any $t \geq 0$ we obtain

$$0 \leq S(t) \leq N_0; \quad 0 \leq E(t) \leq N_0; \quad 0 \leq I(t) \leq N_0; \quad 0 \leq I_r(t) \leq N_0;$$

$$0 \leq R_u(t) \leq N_0; \quad 0 \leq R_r(t) \leq N_0.$$

Hence the system (2.1) is bounded in Ω_1.

4. Global stability of disease-free equilibrium (DFE)

In this section, we prove the global stability of the disease free equilibrium (DFE) point.

Theorem 4.1. *The DFE of the model* (2.1) *is globally asymptotically stable in* Ω *whenever* $R_0 \leq 1$.

Proof. We use the Lyapunov function technique. Let consider the follow candidate Lyapunov function:

$$V = \theta(E + I) + \gamma_0 I_u.$$

By definition, V is positive because the parameters of model (2.1) are positive. V is zero at DFE (X_0). We take the function V derivated with respect to t.

$$
\begin{aligned}
\dot{V} &= \theta(\dot{E}+\dot{I})+\gamma_0\dot{I}_u \\
&= \theta\left[\frac{\gamma(t)S}{N}(I+I_u)-\alpha E+\alpha E-\beta I\right]+\gamma_0(\beta_2 I-\theta I_u) \\
&= \theta\frac{\gamma(t)S}{N}(I+I_u)-\theta\beta I+\gamma_0\beta_2 I-\gamma_0\theta I_u \\
&= \theta\frac{\gamma(t)S}{N}(I+I_u)-\theta\beta I+\gamma_0\beta_2 I-\theta\gamma_0 I_u+\beta_2\gamma(t)I_u-\beta_2\gamma(t)I_u \\
&= \theta\frac{\gamma(t)S}{N}(I+I_u)+\gamma_0\beta_2(I+I_u)-\beta\theta I-\theta\gamma_0 I_u-\gamma_0\beta_2 I_u \\
&\leq (\theta\gamma_0+\beta_2\gamma_0)(I+I_u)-\beta\theta(I+I_u)+[\beta\theta-\theta\gamma_0-\beta_2\gamma_0]I_u \\
&\leq [\theta\gamma_0+\beta_2\gamma_0](I+I_u)-\beta\theta(I+I_u)+\beta\theta\left(1-\frac{\theta\gamma_0+\beta_2\gamma_0}{\beta\theta}\right)I_u \\
&\leq (\theta\gamma_0+\beta_2\gamma_0-\beta\theta)\,(I+I_u)+\beta\theta\left(1-\frac{\theta\gamma_0+\beta_2\gamma_0}{\beta\theta}\right)I_u \\
&\leq \beta\theta\left(\frac{\theta\gamma_0+\beta_2\gamma_0}{\beta\theta}-1\right)(I+I_u)+\beta\theta\,(1-R_0)\,I_u \\
&\leq \beta\theta\,(R_0-1)\,(I+I_u)+\beta\theta\,[1-R_0]\,I_u \\
&\leq \beta\theta\,[(R_0-1)I+(R_0-1)I_u-(R_0-1)I_u] \\
&\leq \beta\theta(R_0-1)I.
\end{aligned}
$$

Since all the parameters of the model (2.1) are non negative, it follows that $\dot{V}\leq 0$ for $R_0\leq 1$. Hence V is Lyapunov function on Ω. Therefore, by using the Lasalle invariance principle ([12]), we have : $(E(t),I(t),I_u(t))\longrightarrow(0,0,0)$ as $t\longrightarrow+\infty$.
Since $\lim\limits_{t\to+\infty} supE(t)=0$, $\lim\limits_{t\to+\infty} supI(t)=0$, $\lim\limits_{t\longrightarrow+\infty} supI_u(t)=0$. It follows that for sufficiently small $\epsilon\geq 0$, there exist constant $t_1\geq 0$, $t_2\geq 0$ and $t_3\geq 0$ such that
$\lim\limits_{t\to+\infty} supE(t)\leq\epsilon$, for all $t\geq t_1$
$\lim\limits_{t\to+\infty} supI(t)\leq\epsilon$, for all $t\geq t_2$ and $\lim\limits_{t\to+\infty} supI_u(t)\leq\epsilon$, for all $t\geq t_3$
Hence, it follows from the fifth equations of the model (2.1)
$\frac{dI_r}{dt}\leq\beta\epsilon-\eta I_r$. Therefore using comparison theorem

$$
I_r^{\infty}=\lim_{t\to+\infty} supI_r(t)\leq\frac{\beta\epsilon}{\eta}\longrightarrow 0 \text{ as } \epsilon\longrightarrow 0. \tag{4.1}
$$

Similarity (by using $\lim\limits_{t\to+\infty} infI_r(t)=0$)

$$
I_{r\infty}=\lim_{t\to+\infty} infI_r(t)=0. \tag{4.2}
$$

It follows from the two relations (4.1) and (4.2) above
$\lim\limits_{t\to+\infty} I_r(t)=0.$
It can also be shown that
$\lim\limits_{t\to+\infty} R_u(t)=0$, $\lim\limits_{t\to+\infty} R_r(t)=0$, $\lim\limits_{t\to+\infty} S(t)=N_0$.
Therefore by combining all equations above, it follows that each solution of the model equation (2.1), with initial conditions in Ω, approaches X_0 as $t\to+\infty$ for $R_0\leq 1$.

5. Numerical simulation

In this section, we propose the numerical simulation of mathematical model (2.1). The following curves are obtained by using scilab. Estimated values of the model (2.1) parameters and unknown initial conditions $(S_0, E_0, I_0, I_{r0}, I_{u0}, R_{r0}, R_{u0}) = (8000, 198, 2, 2, 0, 0, 0)$ are provided by [5]. The parameters values are given by the table 1.

Symbol	Values of model (2.1)	Source	Values of model (6.1)	source
α	0.1818	[5]	0.1818	[5]
γ_0	0.19	[5]	0.19	[5]
θ	0.0714	fixed	1/14	[5]
η	0.823	[5]	$1/14$	[5]
β_1	0.418	[5]	0.28	fixed
β_2	0.415	[5]	0.31	fixed
μ	0.127	[5]	0.42	fixed
A_1			8	fixed
A_2			10	[1]

Table 1: The values of the parameters for the simulation of model (2.1) and (6.1)

After 14 days, strong government measures in the country, such as isolation, quarantine, and the wearing of face mask, allowed the reduction of the transmission of new cases. For that we use an exponential decrease for the transmission rate $\gamma(t)$ given by ([5, 19])

$$\gamma(t) = \begin{cases} \gamma_0, \quad 0 < t < 14, \\ \\ \gamma_0 \exp(-\mu(t-14)), \quad t \geq 14. \end{cases}$$

For the simulation of model (2.1), we use the ode method in scilab given by following algorithm. The system (2.1) can be rewrited $\dot{x} = f(t, x)$ where $f(t, x) = f_i(t, x)$, i=1,...,7 and $x = (S, E, I, I_r, I_u, R_r, R_u)$.

Algorithm
function Xdot=f(t,X)
X_1dot=$f_1(t, X)$
X_2dot=$f_2(t, X)$
...
X_7dot=$f_7(t, X)$
endfunction
X=ode(X_0, t_0, f)
X_0 is the initial conditions at $t_0 = 0$, $t = 0 : 0.1 : 900$

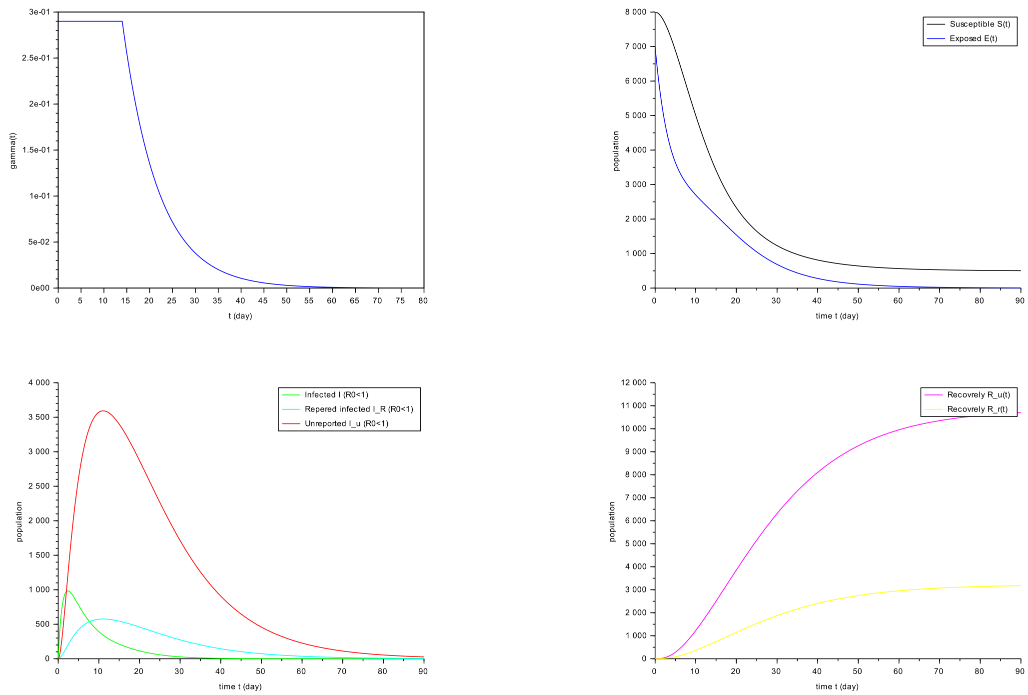

Figure 2: The variation of contact $\gamma(t)$ and population dynamic

The $\gamma(t)$ curve in the Figure 2 represents the variation of the contact : from 0 to 14 days, the infected remained in constant contact with the susceptible individuals. After the 14 days, the measures taken by the government permitted to reduce the contact between the infected and susceptible persons . In this case the contact function decreases and is canceled after 65 days when all measures taken by the government are respected.

The curves describing the dynamics of the susceptible (S) and the exposed (E) in Figure 2 decrease and stabilizes after 50 days. This decrease is due to the respect of the barrier measures taken by the government.

The curves describing the dynamics of infected individuals in Figure 2 show two phases. The increase of the curves in the first phase is due to the fact that there were no measures before the 14 days. After the 14 days, the measures that are taken allowed the reduction of the infected. If all the measures are respected then the disease disappears after 40 days.

6. The optimal control problem

The best way to control the COVID-19 epidemic is to respect the barrier measures which are represented here by $\gamma(t)$. The implementation of these measures is very complicated in practice because there are unreported infectious diseases. For this we need another alternative to control the disease. Furthermore, we first prove the existence of the two optimal controls u^*, v^* and we give their characterization.

6.1. Presentation of the problem

In this section we use the optimal control theory to analyze the behavior of the model (6.1). Our goal is to maximize the number of persons who have survived the disease (recovered) and to minimize the infected individuals during the course of an epidemic and the cost of this strategy. In the model (2.1), we introduce two controls $u; v$ which are defined as follow.

- The function $u(t) \in [0,1]$ is the control corresponding to the vaccination ([26]). The rate at which individuals gain immunity through vaccination is denoted by $u(t) \in [0,1]$ with $t \in [0,t_f]$. Because asymptomatic infected may not be aware of their infection, we assume that susceptible and asymptomatic infected are indistinguishable with respect to vaccination. Vaccinating asymptomatic infected individuals has no effect, but still implies a cost. The ideal is to vaccinate the entire population in this case $u = 1$. In reality this is not possible, so we try to vaccinate as many people. To find the maximum number of people we take $u = u_{max}$. u_{max} represents the proportion of susceptible persons receiving serum of vaccine.

- The second control $v(t) \in [0,1]$ represents the treatment of patients over the interval $[0;t_f]$. The control v that we consider here can therefore represent the treatment of symptomatic or the isolation of patients in hospitals to avoid possible new contamination.

By inserting the controls u and v in the model (2.1), we obtain the following controlled equations:

$$\begin{cases} \dfrac{dS}{dt} = \dfrac{-(1-u)\gamma(t)S(I+I_u)}{N}, \\ \dfrac{dE}{dt} = \dfrac{(1-u)\gamma(t)S(I+I_u)}{N} - \alpha E, \\ \dfrac{dI}{dt} = \alpha E - (\beta_1+\beta_2)I, \\ \dfrac{dI_r}{dt} = \beta_1 I - (\eta+v)I_r, \\ \dfrac{dI_u}{dt} = \beta_2 I - \theta I_u, \\ \dfrac{dR_r}{dt} = (\eta+v)I_r, \\ \dfrac{dR_u}{dt} = \theta I_u. \end{cases} \tag{6.1}$$

$$S(t_0) = S_0 > 0, \quad E(t_0) = E_0 > 0, \quad I(t_0) = I_0 > 0, \quad I_r(t_0) = I_{r0} > 0, \quad I_u(t_0) = I_{u0} > 0,$$
$$R_u(t_0) = R_{u0} > 0, \quad R(t_0) = R_{r0} > 0.$$

Mathematically, for a fixed terminal time t_f, we minimize the functional objective J on $[0,t_f]$.

$$J(u,v) = \int_0^{t_f} \left(I_r(t) - R_r(t) + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t) \right) dt. \tag{6.2}$$

$A_1 > 0$ is the weight which allows to regulate the control u and $A_2 > 0$ the weight which allows to regulate the control v.

6.2. Study of optimal control problem

In this section, we define the Hamiltonian associated with the control problem. Then, we characterize the solutions of control problem (6.1) after proving their existence. Our work is to determine the optimal controls (u^*, v^*) such as

$$J(u^*,v^*) = min\,\{J(u,v) : (u,v) \in U \times V\} \tag{6.3}$$

U and V are the set of admissible controls defined by:

$$U = \left\{u(t) \in \mathbb{R}/ \quad 0 \leq u(t) \leq u_{max} < 1, \quad t \in [0, t_f], \quad u \in L^2([0; t_f], \mathbb{R})\right\}$$

and

$$V = \left\{v(t) \in \mathbb{R}/ \quad 0 \leq v(t) \leq 1; v \in L^2([0; t_f], \mathbb{R})\right\}.$$

Definition 6.1. *(Hamiltonian of the minimization problem)*
The Pontryagin's maximum principle [21] converted (6.1) *,* (6.2) *and* (6.3) *into problem of minimizing an Hamiltonian,* H*, defined by:*

$$H = I_r - R_r + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t) + \sum_{i=1}^{7} \lambda_i f_i.$$

Where f_i *are the right side of the differential equations state variable and* λ_i, $i = 1, ..., 7$ *are the adjoints variables associated with their respective states.*

Theorem 6.2. *Consider the optimal control problem* (6.1) *subject to* (6.2)*. Then there exists an optimal pair of controls* (u^*, v^*) *and a corresponding optimal states* $(S^*, E^*, I^*, I_u^*, I_r^*, R_r^* R_u^*)$ *that minimizes the objective function* $J(u, v)$ *over set of admissible controls* $U \times V$*.*

Proof. The existence of optimal control can be proved by using the results from ([13] see Theorem 2.1) and Fleming's results (Theorem III.4.1, [4]), we must verify the following conditions:

- the set of admissible controls is nonempty,
- the admissible sets U, V are convex and closed,
- the vector field of the state system is bounded by a linear function of control,
- the objective function is convex,
- there exists constants $c_1, c_2 > 0$ such as the integrand of the objective function be bounded by $c_1(|u|^2 + |v|^2)^{\frac{p}{2}} - c_2$.

(1) We verify these conditions, thanks to a result of Lukes et al. [24] which assures the existence of solutions for the state system (6.1).

(2) The set U and V are convex and bounded by definition.

(3) The right-hand side of the state system (6.1) is bounded by a linear function in the state and control variables.

(4) The integrand of the objective functional is

$$f^0(x, u, v) = I_r - R_r + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t).$$

The hessian matrix of $f^0(X, u, v)$ is given by :

$$M_{f^0} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

$spec(M_{f^0}) = \{A_1, A_2\} \subset \mathbb{R}_+^*$.
So f^0 is strictly convex over $U \times V$.

(5) We have,

$$
\begin{aligned}
f^0(x,u,v) &= I_r - R_r + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t)\\
&\geq \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t) - R_r\\
&\geq \frac{1}{2}\min\{A_1, A_2\}\left(|u|^2(t) + |v|^2(t)\right)^k - R_r\\
&\geq c_1\left(|u|^2(t) + |v|^2(t)\right)^k - c_2
\end{aligned}
$$

where $c_1 = \frac{1}{2}\min\{A_1, A_2\} > 0$, $c_1 \leq R_r \leq c_2$ and $k \geq 1$. Therefore the last assertion is verified.

6.3. Characterization of optimal control

In this section, we characterize the solutions of system (6.1).

Theorem 6.3. *Given an optimal $w^* = (u^*, v^*) \in U \times V$ and corresponding states $X^* = (S^*, E^*, I^*, I_u^*, I_r^*, R_r^*\ R_u^*)$ of system (6.1), there exist adjoint functions satisfying the following system.*

$$
\begin{cases}
\dfrac{d\lambda_1(t)}{dt} = \dfrac{(\lambda_1(t)-\lambda_2(t))\gamma(t)}{N}(1-u(t))(I+I_u),\\
\dfrac{d\lambda_2(t)}{dt} = (\lambda_2-\lambda_3)\alpha,\\
\dfrac{d\lambda_3(t)}{dt} = \dfrac{(\lambda_1(t)-\lambda_2(t))\gamma(t)S(t)}{N}(1-u(t))(\lambda_3-\lambda_4)\beta_4 + (\lambda_3-\lambda_5)\beta_2,\\
\dfrac{d\lambda_4(t)}{dt} = -1 + \lambda_4(\beta_1+\beta_2) - \lambda_6(\eta+v),\\
\dfrac{d\lambda_5(t)}{dt} = \dfrac{(\lambda_1(t)-\lambda_2(t))\gamma(t)S(t)}{N}(1-u(t)) + \theta(\lambda_5-\lambda_7),\\
\dfrac{d\lambda_6(t)}{dt} = 1,\\
\dfrac{d\lambda_7(t)}{dt} = 0
\end{cases}
\tag{6.4}
$$

with the transversality conditions
$\lambda_1(t) = 0, \lambda_2(t) = 0, \lambda_3(t) = 0, \lambda_4(t) = 0, \lambda_5(t) = 0, \lambda_6(t) = 0, \lambda_7(t) = 0.$
Let's up $N^* = S^* + E^* + I^* + I_u^* + I_r^* + R_u^* + R_r^*$.
Furthermore, the optimal controls are characterized by:

$$
\begin{aligned}
u^* &= \max\left\{0, \min\left\{u_{max}, \left(\frac{\lambda_2(t)-\lambda_1(t)}{A_1}\right)\gamma(t)\frac{S^*}{N^*}(I^*+I_u^*)\right\}\right\},\\
v^* &= \max\left\{0, \min\left\{1, \frac{(\lambda_4(t)-\lambda_6(t))}{A_2}I_r^*\right\}\right\}.
\end{aligned}
\tag{6.5}
$$

Proof. The differential equations for the adjoints are standard results from Pontryagin's Maximum Principle. Let $w^* = (u^*, v^*)$ corresponding solution $X^* = (S^*, E^*, I^*, I_R^*, I_u^*, R_r^*\ R_u^*)$ that minimizes $J(u,v)$ over $U \times V$. By applying the Pontryagin's maximum principle (see [21]) there exists adjoint functions,

$p(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), \lambda_5(t), \lambda_6(t), \lambda_7(t))$ $(t \in [0, t_f])$ verifying the following conditions:

$$\frac{dp(t)}{dt} = -\frac{\partial H}{\partial X} \tag{6.6}$$

$$\frac{dX(t)}{dt} = \frac{\partial H}{\partial p} \tag{6.7}$$

$$\frac{\partial H}{\partial u} = 0, \quad \frac{\partial H}{\partial v} = 0. \tag{6.8}$$

$$\frac{dp(t)}{dt} = -\frac{\partial H}{\partial X} \Longleftrightarrow \begin{cases} \dfrac{d\lambda_1}{dt} = -\dfrac{\partial H}{\partial S} \\ \dfrac{d\lambda_2}{dt} = -\dfrac{\partial H}{\partial E} \\ \dfrac{d\lambda_3}{dt} = -\dfrac{\partial H}{\partial I} \\ \dfrac{d\lambda_4}{dt} = -\dfrac{\partial H}{\partial I_r} \\ \dfrac{d\lambda_5}{dt} = -\dfrac{\partial H}{\partial I_u} \\ \dfrac{d\lambda_6}{dt} = -\dfrac{\partial H}{\partial R_r} \\ \dfrac{d\lambda_7}{dt} = -\dfrac{\partial H}{\partial R_u}, \end{cases} \tag{6.9}$$

$$\lambda_i(t_f) = 0 \quad (i = 1, ..., 7).$$

Therefore, the system (6.9) yields (6.4).
By applying the optimality conditions to the (6.8), we obtain:

$$\frac{\partial H}{\partial u}|_{u^*} = 0, \tag{6.10}$$

$$\frac{\partial H}{\partial v}|_{v^*} = 0. \tag{6.11}$$

$\Rightarrow$ On the set $\{0 \leq u^*(t) \leq u_{max}\}$,$\{0 \leq v^*(t) \leq 1\}$, we have:

the conditions (6.10) and (6.11) give:

$$\begin{cases} \lambda_1(t)\gamma(t)\dfrac{S^*}{N}(I^* + I_u^*) - \lambda_2\gamma(t)\dfrac{S^*}{N}(I^* + I_u^*) + A_1 u^* = 0 \\ \\ -\lambda_4 I_r^* + \lambda_6 I_r^* + A_2 v^* = 0. \end{cases} \tag{6.12}$$

As $-A_1 < 0$ and $-A_2 < 0$, so (6.12) becomes:

$$\begin{cases} 0 \geq \dfrac{(-\lambda_2(t) + \lambda_1(t))\,\gamma(t) S^*(I^* + I_u^*)}{-N^* A_1}, \\ \\ 0 \geq \dfrac{-\lambda_4 I_r^* + \lambda_6 I_r^*}{-A_2}. \end{cases} \tag{6.13}$$

We obtain

$$\begin{aligned} u^* &= \max\left\{0, \left(\frac{\lambda_2(t) - \lambda_1(t)}{A_1}\right)\gamma(t)\frac{S^*}{N^*}(I^* + I_u^*)\right\}, \\ v^* &= \max\left\{0, \frac{(\lambda_4(t) - \lambda_6(t))}{A_2} I_r^*\right\}. \end{aligned} \tag{6.14}$$

$\Rightarrow$ $\{u^*(t) = u_{max}\}$ and $\{v^*(t) = 1\}$.
The equation (6.12) gives

$$\begin{cases} -NA_1 u_{max} \geq \left(-\lambda_2(t) + \lambda_1(t)\right)\gamma(t)S^*(I^* + I_u^*) \\ -A_2 \geq -\lambda_4 I_r^* + \lambda_6 I_r^*. \end{cases}$$

This gives,

$$\begin{cases} u_{max} \leq \dfrac{\left(-\lambda_2(t) + \lambda_1(t)\right)\gamma(t)S^*(I^* + I_u^*)}{-N^* A_1} \\ 1 \leq \dfrac{-\lambda_4 I_r^* + \lambda_6 I_r^*}{-A_2} \end{cases}$$

and thus

$$\begin{aligned} u^* &= \min\left\{u_{max}, \left(\frac{\lambda_2(t) - \lambda_1(t)}{A_1}\right)\gamma(t)\frac{S^*}{N^*}(I^* + I_u^*)\right\} \\ v^* &= \min\left\{1, \frac{(\lambda_4(t) - \lambda_6(t))}{A_2} I_r^*\right\}. \end{aligned} \tag{6.15}$$

The systems (6.15) and (6.14) give the result :

$$\begin{aligned} u^* &= \max\left\{0, \min\left\{u_{max}, \left(\frac{\lambda_2(t) - \lambda_1(t)}{A_1}\right)\gamma(t)\frac{S^*}{N^*}(I^* + I_u^*)\right\}\right\} \\ u^* &= \max\left\{0, \min\left\{1, \frac{(\lambda_4(t) - \lambda_6(t))}{A_2} I_r^*\right\}\right\}. \end{aligned}$$

6.4. Numerical simulation of the controlled model

Several modeling studies have already been performed for the simulation of optimal contol model like Liu et al. ([16–18]). Here, we present the numerical results of the system (6.1) by using python and the same method of [1]. The boundary conditions of optimality system at times $t_0 = 0$ and t_f are separated. We put $N_0 = 200000$ representing the number of the total population of a city in our country. We use the Euler method of step h=0.1 to solve the optimality system (6.1). We discretize the model in interval $[t_0, t_f]$ at time $t_i = t_0 + ih$ (i= 0,1,...,n), where $h = 0.1$ is the time step such that $t_n = t_f = 90$ days, $t_0 = 0$. The value n=900 is the number of points of the discretization. Our algorithm is inspired by [1, 3, 6, 10, 15] to approximate the solutions. A combination of forward and backward difference, we obtain the following approximation:

$$\begin{cases} \dfrac{S_{i+1}-S_i}{h} = -(1-u_i)\gamma_i \dfrac{S_{i+1}}{N_0}(I_i + I_u^i) \\ \dfrac{E_{i+1}-E_i}{h} = (1-u_i)\gamma_i \dfrac{S_{i+1}}{N_0}(I_i + I_u^i) - \alpha E_{i+1} \\ \dfrac{I_{i+1}-I_i}{h} = \alpha E_{i+1} - (\beta_1+\beta_2) I_{i+1} \\ \dfrac{I_r^{i+1}-I_r^i}{h} = \beta_1 I_{i+1} - (\eta + v_i) I_r^{i+1} \\ \dfrac{I_u^{i+1}-I_u^i}{h} = \beta_2 I_{i+1} - \theta I_u^{i+1} \\ \dfrac{R_r^{i+1}-R_r^i}{h} = (\eta + v_i) I_r^{i+1}. \\ \dfrac{R_u^{i+1}-R_u^i}{h} = \theta I_u^{i+1}. \end{cases}$$

By using a similar technique in [1], we approximate the time derivative of the adjoint variables by their first order backward difference and we use the appropriate scheme as follows:

$$\begin{cases} \dfrac{\lambda_1^{n-i}-\lambda_1^{n-i-1}}{h} = \dfrac{(\lambda_1^{n-i-1}-\lambda_2^{n-i})\gamma_i}{N_0}(1-u_i)(I_{i+1}+I_u^{i+1}) \\ \dfrac{\lambda_2^{n-i-1}-\lambda_2^{n-i}}{h} = \alpha(\lambda_2^{n-i-1}-\lambda_3^{n-i}) \\ \dfrac{\lambda_3^{n-i}-\lambda_3^{n-i-1}}{h} = \dfrac{(\lambda_1^{n-i-1}-\lambda_2^{n-i-1})\gamma_i}{N_0}(1-u_i)S_{i+1} + \beta_1(\lambda_3^{n-i-1}-\lambda_4^{n-i}) \\ \qquad + \beta_2(\lambda_3^{n-i-1}-\lambda_3^{n-i}) - \beta_1\lambda_4^{n-i} \\ \dfrac{\lambda_4^{n-i}-\lambda_4^{n-i-1}}{h} = \lambda_4^{n-i-1}(\eta+v_i) - \lambda_6^{n-i}(\eta+v_i) - 1 \\ \dfrac{\lambda_5^{n-i}-\lambda_5^{n-i-1}}{h} = \dfrac{(\lambda_1^{n-i-1}-\lambda_2^{n-i-1})\gamma_i S_{i+1}}{N_0}(1-u_i) + \theta(\lambda_5^{n-i-1}-\lambda_7^{n-i}) \\ \dfrac{\lambda_6^{n-i}-\lambda_6^{n-i-1}}{h} = 1 \\ \dfrac{\lambda_7^{n-i}-\lambda_7^{n-i-1}}{h} = 0. \end{cases}.$$

The algorithm describing the approximation method to give the optimal control is the following.
Algorithm2.
Step1.

$$S(0) = S_0, \quad E(0) = E_0, \quad I(0) = I_0, \quad I_r(0) = I_{r0}, \quad I_u(0) = I_{u0},$$
$$R_u(0) = R_{u0}, \quad R_r(0) = R_{r0}, \quad \lambda_i(t_f) = 0, \quad (i = 1, ..., 7), \quad u(0) = v(0) = 0.$$

Step2.
For $i = 1, ..., n+1$ do,

$$S_{i+1} = \frac{N_0 S_i}{N_0 + \gamma_i h(1-u_i)(I_i + I_u^i)}, \qquad E_{i+1} = \frac{N_0 E_i + h(1-u_i)\gamma_i S_{i+1}(I_i + I_u^i)}{N_0(1+h\alpha)}$$

$$I_{i+1} = \frac{I_i + h\alpha E_{i+1}}{1 + h\beta_1 + h\beta_2}, \qquad I_r^{i+1} = \frac{I_r^i + h\beta_1 I_{i+1}}{1 + h(\eta + v_i)},$$

$$I_u^{i+1} = \frac{I_u^i + \beta_2 h I_{i+1}}{1 + h\theta}, \qquad R_r^{i+1} = R_r^i + h(\eta + v_i) I_r^{i+1}.$$

$$\begin{aligned}
\lambda_1^{n-i-1} &= \frac{\lambda_1^{n-i} N_0 + h\lambda_2^{n-i}(1 - u_i)(I_{i+1} + I_u^{i+1})}{N_0 + h(1 - u_i)(I_{i+1} + I_u^{i+1})\gamma_i} \\
\lambda_2^{n-i-1} &= \frac{\lambda_1^{n-i} + h\alpha\lambda_3^{n-i}}{1 + h\alpha} \\
\lambda_3^{n-i-1} &= \frac{\lambda_3^{n-i} + h(\lambda_2^{n-i-1} - \lambda_1^{n-i-1})\gamma_i(1 - u_i)S_{i+1} + N_0 h\beta_2\lambda_5^{n-i} + N_0 h\beta_1\lambda_4^{n-i}}{N_0(1 + h\beta_1 + h\beta_2)} \\
\lambda_4^{n-i-1} &= \frac{\lambda_4^{n-i} + h(\eta + v_i)\lambda_6^{n-i} + h}{1 + h\beta_1 + h v_i} \\
\lambda_5^{n-i-1} &= \frac{N_0\lambda_5^{n-i} + h(\lambda_2^{n-i} - \lambda_1^{n-i-1})\gamma_i(1 - u_i)S_{i+1} + hN_0\theta\lambda_7^{n-i}}{N_0 + N_0 h\theta} \\
\lambda_6^{n-i-1} &= h + \lambda_6^{n-i} \\
\lambda_7^{n-i-1} &= \lambda_7^{n-i} \\
M_{i+1} &= \left(\frac{(\lambda_1^{n-i-1} - \lambda_2^{n-i-1})}{A_1}\right)\gamma_i \frac{S_{i+1}^*}{N_0}(I_{i+1}^* + I_u^{*(i+1)}) \\
Z_{i+1} &= \frac{\lambda_4^{n-i-1} - \lambda_6^{n-i-1}}{A_2} I_r^{*(i+1)} \\
u_{i+1} &= \max\left(0, \min\left(u_{max}, M_{i+1}\right)\right) \\
v_{i+1} &= \max\left(0, \min\left(1, Z_{i+1}\right)\right).
\end{aligned}$$

Step3.
For i =0,...,n, do
$S^*(t_i) = S_i$, $E^*(t_i) = E_i$, $I^*(t_i) = I_i$, $I_r^*(t_i) = I_r^i$, $I_u^*(t_i) = I_u^i$, $R_r^*(t_i) = R_r^i$,
$u^*(t_i) = u_i$, $v^*(t_i) = v_i$. The curves in this simulation are obtained by python. Certain values of the simulation are taken in [1, 5] and $(S_0, E_0, I_0, I_{r0}, I_{u0}, R_{r0}, R_{u0}) = (N_0, 198, 2, 2, 0, 0, 0)$.

The curves of infected reported persons in the Figure 3 are obtained by simulating the symptomatic infectious population. If left unchecked, the disease infection stabilizes within 120 days. But after application of control u (vaccination) and taking the control of the barrier measures $\gamma(t)$, the reported infected immediately decrease and stabilize in I_0. This is explained by the treatment of patients who are immediately isolated. The curves describing the dynamics of recovered persons in Figure 3 show the evolution of individuals cured of the disease by applying the reported individuals (I_r) the treatment (control v). The curves of unreported infectious persons in the Figure 3 show the evolution of individuals unreported by applying in the susceptible individuals (S) the vaccination (control u). After vaccination of susceptible, there is no effect of contact with unreported infected.

The curves of susceptible persons in the Figure 3 represent the dynamics of the susceptible population (S) for different aspects of control. After operation of the measures, the vaccination u and treatment v, the populations susceptible stabilizes.

The curves of exposed persons in the Figure 3 represent the dynamic of exposed population (E) for different

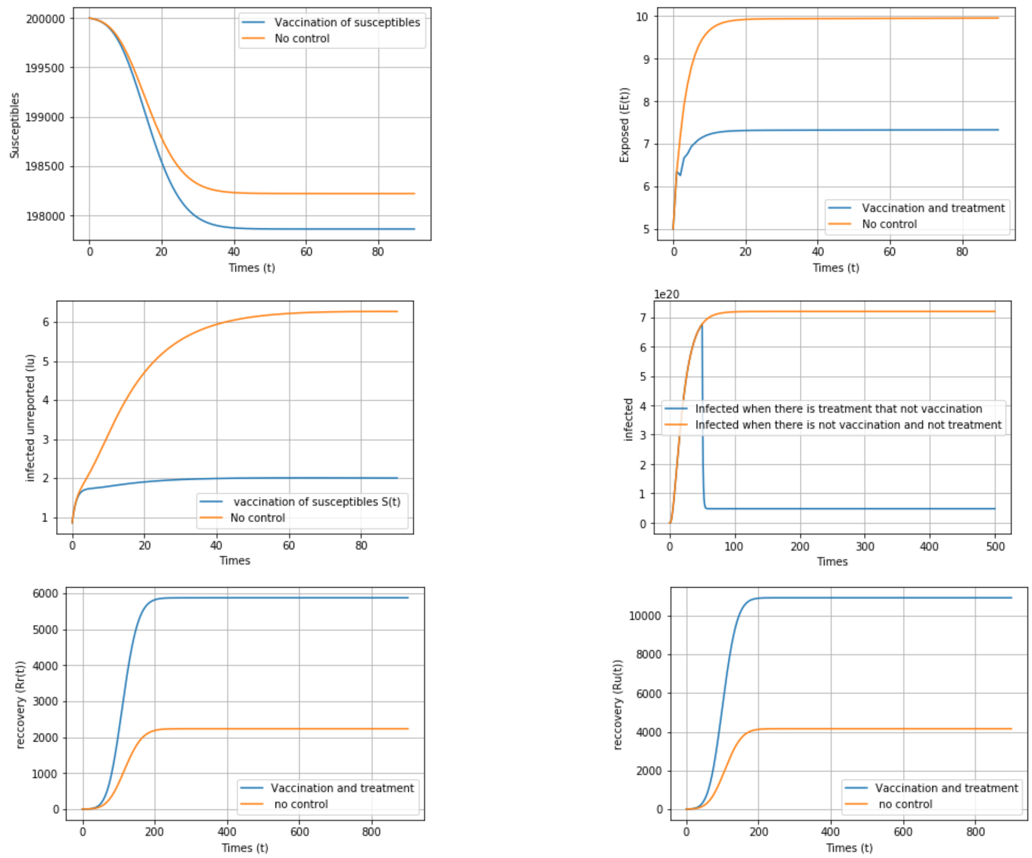

Figure 3: Population dynamic with and without control

aspects of control. The orange curve is the evolution of exposed population in application of u (vaccination) and v (treatment) controls. The blue curve is uncontrolled ($u = 0$ and $v = 0$).

The curves of unreported infectious persons in the Figure 3 represent the dynamics of the unreported infected and infectious population for different aspects of control u (vaccination) and v (treatment). The blue curve represents the evolution of the infected unreported population (I_u) with u and v control ($u \neq 0$ and $v \neq 0$). The orange curve represents the evolution of infected people who have not been brought back without control ($u = 0$ and $v = 0$).

The curves of recovered persons in the Figure 3 represent the dynamics of the reported cured (R_r) population for different aspects of controls. The blue curve is the evolution of cured reported in application of controls ($u \neq 0$ and $v \neq 0$). The orange curve is without control ($u = 0$ and $v = 0$).

The curves of unreported persons in the Figure 3 represent the dynamics of the unreported cured (R_u) population for different aspects of control u (vaccination) and v (treatment).

7. Conclusion

We have developed a model of the COVID-19 epidemic in China (see [20, 27, 31–33]). In this present study, we consider a mathematical model of COVID-19 transmission that incorporates the exposed populations. In our model, we also consider transmission variability between symptomatic and asymptomatic population with former being a fast spreader of the disease. The basic reproduction number is calculated by applying the Van den Driesch method [36]. We also construct the Lyapunov function to show the global stability of disease free equilibrium.

Next, we consider model (2.1) with the controls u (vaccination) and v (treatment of infected). In this model, the existence and uniqueness of the solution associated to the optimal controls are proven. The Hamiltonian function is constructed converting (6.1) into problem of minimizing an Hamiltonian. The $\gamma(t)$ function makes it possible to control the contact between infected individuals and those susceptible at time t. It takes into account all the measures taken by the government of a country. Finally a numerical simulation allows us to interpret the results on the curves. The study shows that the most infectious individuals are the unreported infected.

References

[1] M. Barro, A. Guiro, and D. Ouedraogo, Optimal control of a sir epidemic model with general incidence function and a time delays. *Cubo (Temuco)*, 20(2):53–66, 2018.

[2] S. Chen and C. Xiao, Financial risk contagion and optimal control. *Journal of Industrial and Management Optimization*, 19(4):2915–2935, 2023.

[3] M. Elhia, O. Balatif, J. Bouyaghroumni, E. Labriji, and M. Rachik, Optimal control applied to the spread of influenza a (h1n1). *Applied Mathematical Sciences*, 6(82):4057–4065, 2012.

[4] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control*, volume 1. Springer Science & Business Media, 2012.

[5] A. Guiro, B. Koné, and S. Ouaro, Mathematical model of the spread of the coronavirus disease 2019 (covid-19) in burkina faso. *Applied Mathematics*, 11(11):1204–1218, 2020.

[6] A. Gumel, P. Shivakumar, and B. Sahai, A mathematical model for the dynamics of hiv-1 during the typical course of infection. *Nonlinear Analysis, Theory, Methods and Applications*, 47(3):1773–1783, 2001.

[7] J. K. Hale, Verduyn lunel s m. introduction to functional differential equations. *Appl. Math. Sciences*, 99, 1993.

[8] B. Ivorra, M. Ferrández, M. Vela-Pérez, and A. Ramos, Mathematical modeling of the spread of the coronavirus disease 2019 (covid-19) considering its particular characteristics. the case of china. *Communications in Nonlinear Science and Numerical Simulation*, 88:105303, 2020.

[9] B. Ivorra, M. R. Ferrández, M. Vela-Pérez, and A. M. Ramos, Mathematical modeling of the spread of the coronavirus disease 2019 (covid-19) taking into account the undetected infections. the case of china. *Communications in nonlinear science and numerical simulation*, 88:105303, 2020.

[10] J. Karrakchou, M. Rachik, and S. Gourari, Optimal control and infectiology: application to an hiv/aids model. *Applied Mathematics and Computation*, 177(2):807–818, 2006.

[11] A. J. Kucharski, T. W. Russell, C. Diamond, Y. Liu, J. Edmunds, S. Funk, R. M. Eggo, F. Sun, M. Jit, J. D. Munday, Early dynamics of transmission and control of covid-19: a mathematical modelling study. *The Lancet Infectious Diseases*, 20(5):553–558, 2020.

[12] J. P. La Salle, *The Stability of Dynamical Systems*. SIAM, 1976.

[13] M. Lhous, M. Rachik, and A. Larrache, Free optimal time control problem for a seir-epidemic model with immigration of infective. *International Journal of Computer Applications*, 159, 02 2017.

[14] H. Lin and Y. Zhang, Health resource allocation method at cordon sanitaire for pandemic control and prevention. *Journal of Industrial & Management Optimization*, 19(7), 2023.

[15] C. Liu, Z. Gong, C. Yu, S. Wang, and K. L. Teo, Optimal control computation for nonlinear fractional time-delay systems with state inequality constraints. *Journal of Optimization Theory and Applications*, 191(1):83–117, 2021.

[16] C. Liu, R. Loxton, Q. Lin, and K. L. Teo, Dynamic optimization for switched time-delay systems with state-dependent switching conditions. *SIAM Journal on Control and Optimization*, 56(5):3499–3523, 2018.

[17] C. Liu, R. Loxton, and K. L. Teo, A computational method for solving time-delay optimal control problems with free terminal time. *Systems & Control Letters*, 72:53–60, 2014.

[18] C. Liu, R. Loxton, K. L. Teo, and S. Wang, Optimal state-delay control in nonlinear dynamic systems. *Automatica*, 135:109981, 2022.

[19] Z. Liu, P. Magal, O. Seydi, and G. Webb, Understanding unreported cases in the covid-19 epidemic outbreak in wuhan, china, and the importance of major public health interventions. *Biology*, 9(3):50, 2020.

[20] S. S. Nadim, I. Ghosh, and J. Chattopadhyay, Short-term predictions and prevention strategies for covid-2019: A model based study. *arXiv preprint arXiv:2003.08150*, 2020.

[21] N. Ostianu, L. Pontryagin, and R. Gamkrelidze, Geometry - 1. *Journal of Mathematical Sciences (New York)*, 01 1998.

[22] L. Ouattara, D. Ouedraogo, H. Ouedraogo, and A. Guiro, Stability analysis and optimal control of seirs mathematical model. *Discussiones Mathematicae: Differential Inclusions, Control & Optimization*, 43, 2023.

[23] H. Ouedraogo, D. Ouedraogo, I. Ibrango, and A. Guiro, A study of stability of SEIHR model of infectious disease transmission. *Nonautonomous Dynamical Systems*, 8(1):307–327, 2021.

[24] K. Palmer, Differential equations: classical to controlled (dahlard l. lukes), 1984.

[25] L. Peng, W. Yang, D. Zhang, C. Zhuge, and L. Hong, Epidemic analysis of covid-19 in china by dynamical modeling. *arXiv preprint arXiv:2002.06563*, 2020.

[26] C. Piccolo III and L. Billings, The effect of vaccinations in an immigrant model. *Mathematical and Computer Modelling*, 42(3-4):291–299, 2005.

[27] B. J. Quilty, S. Clifford, S. Flasche, R. M. Eggo, Effectiveness of airport screening at detecting travellers infected with novel coronavirus (2019-ncov). *Eurosurveillance*, 25(5):2000080, 2020.

[28] T. Rouamba, S. Samadoulougou, B. Bonnechère, B. Chiêm, and F. Kirakoya-Samadoulougou, What can we learn from burkina faso covid-19 data? using phenomenological models to characterize the initial growth dynamic of the outbreak and to generate short-term forecasts. *Multidisciplinary Digital Publishing Institute*, 2020.

[29] T. Sardar, I. Ghosh, X. Rodó, and J. Chattopadhyay, A realistic two-strain model for mers-cov infection uncovers the high risk for epidemic propagation. *PLoS Neglected Tropical Diseases*, 14(2):e0008065, 2020.

[30] T. Sardar, S. Nadim, and J. Chattopadhyay, Assessment of 21 days lockdown effect in some states and overall india: A predictive mathematical study on covid-19 outbreak. 04 2020.

[31] M. Shen, Z. Peng, Y. Xiao, and L. Zhang, Modeling the epidemic trend of the 2019 novel coronavirus outbreak in china. *The Innovation*, 1(3):100048, 2020.

[32] B. Tang, N. L. Bragazzi, Q. Li, S. Tang, Y. Xiao, and J. Wu, An updated estimation of the risk of transmission of the novel coronavirus (2019-ncov). *Infectious Disease Modelling*, 5:248–255, 2020.

[33] B. Tang, X. Wang, Q. Li, N. L. Bragazzi, S. Tang, Y. Xiao, J. Wu, Estimation of the transmission risk of the 2019-ncov and its implication for public health interventions. *Journal of Clinical Medicine*, 9(2):462, 2020.

[34] B. Traore, B. Sangare, and S. Traore, A mathematical model of malaria transmission in a periodic environment. *Journal of Biological Dynamics*, 2018.

[35] W. Van Damme, Evolution of the covid-19 pandemic over six weeks in four french-speaking countries in west africa. *Journal of Global Health*, 2021.

[36] P. Van den Driessche and J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Mathematical Biosciences*, 180(1-2):29–48, 2002.

[37] C. Wang, P. W. Horby, F. G. Hayden, and G. F. Gao, A novel coronavirus outbreak of global health concern. *The Lancet*, 395(10223):470–473, 2020.

[38] J. T. Wu, K. Leung, and G. M. Leung, Nowcasting and forecasting the potential domestic and international spread of the 2019-ncov outbreak originating in wuhan, china: a modelling study. *The Lancet*, 395(10225):689–697, 2020.

[39] Y. YODA, D. OUEDRAOGO, H. OUEDRAOGO, A. GUIRO, Optimal control of seihr mathematical model of covid-19. *Electronic Journal of Mathematical Analysis and Applications*, 11(1):134–161, 2023.

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **12(04)**(2024), 388–411.
http://doi.org/10.26637/mjm1204/004

Polynomial stability of a Rayleigh system with distributed delay

INNOCENT OUEDRAOGO*[1], DÉSIRÉ SABA[2], CHEIKH SECK[3], GILBERT BAYILI [4]

[1,2,4] *Laboratoire de Mathématiques et d'Informatique(LAMI), Ecole Doctorale Sciences et Technologies Université Joseph Ki-Zerbo, 03 BP 7021 Ouagadougou Burkina Faso*
[3] *Laboratoire d'Analyse Mathématiques, Statistiques et Applications de la Faculté des Sciences et Techniques (FST),Université Cheikh Anta Diop, BP 5036 Dakar Sénégal.*

Received 22 August 2024; Accepted 17 September 2024

Abstract. In this paper we study the polynomial stability of a Rayleigh system with distributed delay in dynamic control. After studying the existence and uniqueness of the solution, we showed polynomial stability and finally proved that this polynomial stability is the best that can be had by establishing that there is no exponential stability. Our contribution is the introduction of the distributed delay term in the control.

AMS Subject Classifications: 34D20, 35B40, 35L70.

Keywords: Rayleigh beam equation, dynamic boundary control, distributed delay, spectral analysis, Rational stabilization.

Contents

1. Introduction

In this paper we focus on the Rayleigh problem subject to a single dynamic control with a distributed delay as follows

*Corresponding author. Email address: **innocentouedraogo850@yahoo.fr** (Innocent OUEDRAOGO)

$$\begin{cases} u_{tt}(x,t) - \gamma u_{xxtt}(x,t) + u_{xxxx}(x,t) = 0 \text{ in }]0,1[\times(0,+\infty) \\ u(0,t) = u_x(0,t) = 0 \\ u_{xx}(1,t) + \eta(t) = 0, \\ u_{xxx}(1,t) - \gamma u_{xtt}(1,t) = 0,\ \forall\, t \in (0,+\infty) \\ \eta_t(t) - u_{xt}(1,t) + \beta_1\eta(t) + \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s)\eta(t-s)ds = 0,\ \ \forall\, t \in (0,+\infty) \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1 \text{ in }]0,1[, \quad \eta(0) = \eta_0 \in \mathbb{C} \\ \eta(-t) = f_0(.,-t),\ \forall\, t \in (0,\tau_2), \end{cases} \tag{1.1}$$

where η denotes the dynamical control, $\displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s)\eta(t-s)ds$ is the time delay, β_1 is a positive constants and the initial data (u_0, u_1, f_0) belong to a suitable space. The damping of the system is made via the indirect damping mechanism.
Throughout this paper, we assume that $\beta_2 : [\tau_1;\tau_2] \to \mathbb{R}$, β_2 is in $L^{+\infty}$ and is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} \beta_2(s)ds < \beta_1. \tag{1.2}$$

It should be that D. Mercier and al. studied in [9] the problem

$$\begin{cases} u_{tt}(x,t) - \gamma u_{xxtt}(x,t) + u_{xxxx}(x,t) = 0 \text{ in }]0,1[\times(0,+\infty) \\ u(0,t) = u_x(0,t) = 0 \\ u_{xx}(1,t) + \eta(t) = 0, \\ u_{xxx}(1,t) - \gamma u_{xtt}(1,t) = 0\ \ \forall\, t \in (0,+\infty) \\ \eta_t(t) - u_{xt}(1,t) + \beta\eta(t) = 0\ \ \forall\, t \in (0,+\infty) \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1 \text{ in }]0,1[, \quad \eta(0) = \eta_0 \in \mathbb{C} \end{cases} \tag{1.3}$$

where β is a positive constant and η the dynamical control.
A study in which they showed the polynomial decay of the solution of the system (1.3).

Then, the important and interesting case when the Rayleigh beam equation is damped by only one dynamical boundary with distributed delay remaine open. The aim of this paper is to fill this gap by considering a clamped Rayleigh beam equation subject to only one dynamical boundary feedback whith distributed delay (1.1).

The paper is organized as follows: In the second part we will establish the well posedness of problems (1.1) using semi-group theory. In the sections 3 and 4 respectively we will establish the strong and polynomial stability and finally in section 5 the absence of an exponential decay.

2. Existence and uniqueness of solution

Here we study the well posedness for the problem (1.1) using the semigroup theory.
As we did in [11, 12] and [13] let's

$$z(\rho,t,s)=\eta(t-s\rho),\quad \rho\in(0,1), s\in(\tau_1,\tau_2),\ t>0. \tag{2.1}$$

Now the problem (1.1) is equivalent to

$$\begin{cases} u_{tt}(x,t)-\gamma u_{xxtt}(x,t)+u_{xxxx}(x,t)=0 \text{ in }]0,1[\times(0,+\infty)\\ sz_t(\rho,t)+z_\rho(\rho,t)=0 \text{ in } (0,1)\times(0,+\infty)\\ u(0,t)=u_x(0,t)=0\\ u_{xx}(1,t)+\eta(t)=0,\\ u_{xxx}(1,t)-\gamma u_{xtt}(1,t)=0\ \forall\, t\in(0,+\infty)\\ \eta_t(t)-u_{xt}(1,t)+\beta_1\eta(t)+\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)z(1,t,s)ds=0\ \ \forall\, t\in(0,+\infty)\\ u(\cdot,0)=u_0,\quad u_t(\cdot,0)=u_1 \text{ in }]0,1[,\quad \eta(0)=\eta_0\in\mathbb{C}\\ z(\rho,0,s)=f_0(.,-\rho\tau)\ \ \forall\,\rho\in(0,1), s\in(\tau_1,\tau_2),\\ z(0,t,s)=\eta(t)\ \ \forall\, t\in(0,+\infty)\end{cases} \tag{2.2}$$

The well posedness of problem (1.1) follows from standard semigroup theory.

Now let

$$V=\left\{u\in H^1(0,1), u(0)=0\right\},\ \ \|u\|_V^2=\int_0^1(|u|^2+\gamma\,|u_x|^2)dx$$

$$W=\left\{u\in H^2(0,1), u(0)=0, u_x(0)=0\right\},\ \ \|u\|_W^2=\int_0^1|u_{xx}|^2\,dx$$

and the energy space

$$\mathcal{H}=W\times V\times\mathbb{C}\times L^2\Big((0,1)\times(\tau_1,\tau_2)\Big)$$

with the inner product

$$\left\langle\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix},\begin{pmatrix}u^*\\v^*\\\eta^*\\z^*\end{pmatrix}\right\rangle_{\mathcal{H}}=\int_0^1 u_{xx}\overline{u_{xx}^*}\,dx+\int_0^1(v\overline{v^*}+\gamma v_x\overline{v_x^*})\,dx+\eta\overline{\eta^*}+\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)|z|^2dsd\rho.$$

Let u, η and z be smooth solutions of the system. Then multiplying the first equation of the system by $\overline{\Phi}\in W$ and integrating by part on $(0,1)$, we get

$$\int_0^1 u_{tt}\overline{\Phi}-\gamma u_{xxtt}\overline{\Phi}\,dx+\int_0^1 u_{xxxx}\overline{\Phi}\,dx=0 \tag{2.3}$$

Setting

$$I = -\int_0^1 \gamma u_{xxtt}\overline{\Phi}\,dx + \int_0^1 u_{xxxx}\overline{\Phi}\,dx$$

We obtain that

$$\begin{aligned}
I &= \gamma\int_0^1 u_{ttx}\overline{\Phi_x}\,dx - \gamma u_{ttx}(1)\overline{\Phi(1)} + \gamma u_{ttx}(0)\overline{\Phi(0)} + \int_0^1 u_{xx}\overline{\Phi_{xx}}\,dx \\
&+ u_{xxx}(1)\overline{\Phi(1)} - u_{xxx}(0)\overline{\Phi(0)} - u_{xx}(1)\overline{\Phi_x(1)} + u_{xx}(0)\overline{\Phi_x(0)} \\
&= \gamma\int_0^1 u_{ttx}\overline{\Phi_x}\,dx + \int_0^1 u_{xx}\overline{\Phi_{xx}}\,dx + \eta\overline{\Phi_x(1)}
\end{aligned}$$

Now the relation 2.3 becomes

$$\int_0^1 u_{tt}\overline{\Phi}dx + \gamma\int_0^1 u_{ttx}\overline{\Phi_x}\,dx + \int_0^1 u_{xx}\overline{\Phi_{xx}}\,dx + \eta\overline{\Phi_x(1)} = 0 \tag{2.4}$$

Now we define the linear operators $A \in \mathcal{L}(W, W')$, $B \in \mathcal{L}(\mathbb{R}, V')$, $C \in \mathcal{L}(V, V')$, by the following way

$$< Au, \Phi >_{W'\times W} = \int_0^1 u_{xx}\overline{\Phi_{xx}}dx, \ \ \forall\, u, \Phi \in W$$

$$< B\eta, \Phi >_{W'\times W} = \eta\overline{\Phi_x(1)}, \ \ \forall\, \eta \in \mathbb{R}, \forall \Phi \in W$$

$$< Cu, \Phi >_{V'\times V} = \int_0^1 (u\overline{\Phi} + \gamma u_x\overline{\Phi_x})dx, \ \ \forall\, u, \Phi \in W$$

Then by means of the Lax-Milgram theorem, the operator A (resp. C) is the canonical isomorphism of W (resp. V) onto W' (resp. V'). Then we can formulate the variational equation 2.4 as :

$$Cu_{tt} + Au + B\eta = 0, \ \ in\ W'.$$

If we assume that $Ay + B\eta \in \ V'$, then we obtain that :

$$u_{tt} + C^{-1}(Au + B\eta) = 0, \ \ in\ V$$

If we denote by

$$\mathcal{U} = \Big(u, u_t, \eta, z\Big)^{\mathsf{T}},$$

one has

$$\mathcal{U}_t = (u_t, u_{tt}, \eta_t, z_t)^{\mathsf{T}} = \Big(u_t, -C^{-1}(Au + B\eta), u_{xt}(1) - \beta_1\eta - \int_{\tau_1}^{\tau_2} \beta_2(s)z(1,t,s)ds, -s^{-1}z_\rho\Big)^{\mathsf{T}}.$$

Therefore problem (2.2) can be rewritten as:

$$\begin{cases} \mathcal{U}_t = \mathcal{A}\mathcal{U} \\ \mathcal{U}(0) = (u_0, u_1, \eta_0, f_0(., -\rho s)^{\mathsf{T}}, \end{cases} \tag{2.5}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\,(u, v, \eta, z)^{\mathsf{T}} = \Big(u_t, -C^{-1}(Au + B\eta), u_{xt}(1) - \beta_1\eta - \int_{\tau_1}^{\tau_2} \beta_2(s)z(1,t,s)ds, -s^{-1}z_\rho\Big)^{\mathsf{T}},$$

with domain

$$\mathcal{D}(\mathcal{A}) = \Big\{ \, (u, v, \eta, z)^{\mathsf{T}} \in \mathcal{H}, v \in W, Au + B\eta \in V' \text{ and } z \in H^1\Big((0,1)\times(\tau_1,\tau_2)\Big) \,\big|\, z(0) = \eta \ \Big\},$$

As in [19] let's prove the following lemma.

Lemma 2.1. *Let* $(u,v,\eta,z)^{\mathsf{T}} \in \mathcal{H}$. *Then* $(u,v,\eta,z)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$ *if and only if* $u \in W \cap H^3(0,1)$, $v \in W$, $z \in H^1\Big((0,1)\times(\tau_1,\tau_2)\Big)$ *and* $z(0)=\eta$ *such as*
$u_{xxx}(1) + \gamma\Big[C^{-1}(Au+B\eta)\Big]_x(1) = 0$;
$u_{xx}(1) + \eta = 0$.

Proof. The sufficiency is obvious. Indeed let $(u,v,\eta,z)^{\mathsf{T}} \in \mathcal{H}$.
Assume $u \in W \cap H^3(0,1)$, $v \in W$, $z \in H^1\Big((0,1)\times(\tau_1,\tau_2)\Big)$ and $z(0)=\eta$ such as
$u_{xxx}(1) + \gamma\Big[C^{-1}(Au+B\eta)\Big]_x(1) = 0$ and $u_{xx}(1)+\eta = 0$.

We know

$z \in H^1\Big((0,1)\times(\tau_1,\tau_2)\Big)$ and $z(0)=\eta$;
$u \in W \cap H^3(0,1) \Rightarrow u \in W$;
As $W \subset V$, $v \in W \Rightarrow v \in V$.
Moreover, if $u_{xxx}(1) + \gamma\Big[C^{-1}(Au+B\eta)\Big]_x(1) = 0$, this implies that the equation is well posed and this necessarily leads to
$Au + B\eta \in V'$.
So $(u,v,\eta,z)^{\mathsf{T}} \in D(\mathcal{A})$

To prove the necessity, let $(u,v,\eta,z)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$ and
$\mathcal{A}\,(u,v,\eta,z)^{\mathsf{T}} = (g,k,h,q)^{\mathsf{T}}$. Then we obtain

$$\begin{cases} v = g \in W \\ -C^{-1}(Au+B\eta) = k \\ v_x(1) - \beta_1\eta - \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) z(1,t,s)ds = h \\ -s^{-1}z_\rho = q \in L^2\Big((0,1)\times(\tau_1,\tau_2)\Big). \end{cases} \tag{2.6}$$

If the relation $z(0)=\eta$ is obvious, we obtain from the first and last equations of the system (2.6) that $v \in W$, and then $z \in H^1\Big((0,1)\times(\tau_1,\tau_2)\Big)$.
Then since $k \in V$ and $C: V \longrightarrow V'$ is an isomorphism, so the equation $(2.6)_2$ can be rewritten as

$Au + B\eta = -Ck$ in $V' \subset W'$

So for all $\psi \in W$ we have

$$\int_0^1 u_{xx}\overline{\psi_{xx}}\,dx + \eta\overline{\psi_x}(1) = -\int_0^1 (k\overline{\psi} + \gamma k_x\overline{\psi_x})dx \tag{2.7}$$

This means

$$\int_0^1 u_{xx}\overline{\psi_{xx}}\,dx + \eta\overline{\psi_x}(1) + \int_0^1 (k\overline{\psi} + \gamma k_x\overline{\psi_x})dx = 0 \tag{2.8}$$

On the one hand, let's take $\phi \in C_0^\infty(0,1)$ and take $\psi = \displaystyle\int_0^x \phi(s)ds$.
We know $\psi_x = \phi$ and $\psi_{xx} = \phi_x$
By replacing in (2.8) we obtain

$$\int_0^1 u_{xx}\overline{\phi_x}\,dx + \eta\overline{\phi}(1) + \int_0^1 \Big[k\Big(\int_0^x \overline{\phi(s)}ds\Big)\Big]dx + \int_0^1 \gamma k_x\overline{\phi}dx = 0 \tag{2.9}$$

Since $\phi \in C_0^\infty(0,1)$ then $\overline{\phi}(1)=0$, so we get

$$\int_0^1 u_{xx}\overline{\phi_x}\,dx+\int_0^1\Big[k\Big(\int_0^x\overline{\phi(s)}ds\Big)\Big]dx+\int_0^1\gamma k_x\overline{\phi}dx=0 \tag{2.10}$$

In integration by parts we have

$$\Big[u_{xx}\overline{\phi}\Big]_0^1-\int_0^1 u_{xxx}\overline{\phi}\,dx+\Big[\Big(\int_1^x k(s)ds\Big).\Big(\int_0^x\overline{\phi(s)}ds\Big)\Big]_0^1-\int_0^1\Big(\int_1^x k(s)ds\Big)\overline{\phi(x)}dx$$
$$+\int_0^1\gamma k_x\overline{\phi}dx=0 \tag{2.11}$$

But $\Big[\Big(\int_1^x k(s)ds\Big).\Big(\int_0^x\overline{\phi(s)}ds\Big)\Big]_0^1=\Big[u_{xx}\overline{\phi}\Big]_0^1=0$

Consequently, the (2.11) equation can be rewritten

$$\int_0^1 u_{xxx}\overline{\phi(x)}\,dx-\int_0^1\Big(\int_1^x k(s)ds\Big)\overline{\phi(x)}dx+\int_0^1\gamma k_x\overline{\phi(x)}dx=0$$

By inverting the 1 and x terminals in $\displaystyle\int_1^x k(s)ds$ we have

$$\int_0^1 u_{xxx}\overline{\phi(x)}\,dx=-\int_0^1\Big[\Big(\int_x^1 k(s)ds\Big)dx+\gamma k_x\Big]\overline{\phi(x)}dx,\forall\phi\in W$$

However

$$u_{xxx}=\int_x^1 k(s)ds+\gamma k_x\ pp\ in\ L^2(0,1) \tag{2.12}$$

This leads to $u\in H^3(0,1)\cap W$.
In particular, (2.12) allows us to write

$$u_{xxx}(1)-\gamma k_x(1)=0 \tag{2.13}$$

while $k_x(1)=-\Big[C^{-1}(Au+B\eta)\Big]_x(1)$
From which we finally obtain

$$u_{xxx}(1)+\gamma\Big[C^{-1}(Au+B\eta)\Big]_x(1)=0 \tag{2.14}$$

On the other hand, for any $\phi\in V$ such that $\phi(1)=1$, let's pose $\psi=\displaystyle\int_0^x\phi(s)ds$.
Based on the previous calculations, we have

$$\int_0^1 u_{xx}\overline{\phi_x(x)}\,dx+\eta+\int_0^1\Big[\int_x^1 k(s)ds+\gamma k_x\Big]\overline{\phi(x)}dx=0 \tag{2.15}$$

From (2.12) we have $\displaystyle\int_x^1 k(s)ds+\gamma k_x=u_{xxx}$
By replacing in (2.15) we obtain

$$\int_0^1 u_{xx}\overline{\phi_x(x)}\,dx + \eta + \int_0^1 u_{xxx}\overline{\phi(x)}dx = 0$$

By integration by parts we have

$$u_{xx}(1)\overline{\phi(1)} - u_{xx}(0)\overline{\phi(0)} - \int_0^1 u_{xxx}\overline{\phi}\,dx + \eta + \int_0^1 u_{xxx}\overline{\phi(x)}dx = 0$$

This implies that

$$u_{xx}(1)\overline{\phi(1)} - u_{xx}(0)\overline{\phi(0)} + \eta = 0$$

Since $\overline{\phi(1)} = 1$ and $\overline{\phi(0)} = 0$, we finally obtain

$$u_{xx}(1) + \eta = 0 \tag{2.16}$$

The neccessity is also proved.

■

We can now state the following existence results.

Theorem 2.2.
Assume that (1.2) *holds. Then for any datum* $U_0 = (u_0, u_1, \eta_0, f_0)$ *belongs to* $\mathcal{H}$*, the problem (1.1) has one and only one weak solution* $U = (u, u_t, \eta, z)$ *verifying:*

$$\begin{cases} u \in C\left([0,\infty), V\right) \cap C^1\left([0,\infty), L^2(0,1)\right) \\ \eta \in C\left([0,\infty)\right) \end{cases} \tag{2.17}$$

Moreover, if $U_0 = (u_0, u_1, \eta_0, f_0)$ *belongs to* $\mathcal{D}(\mathcal{A})$*, then problem (1.1) has one and only one strong solution* $U = (u, u_t, \eta, z)$ *which satisfies*

$$\begin{cases} u \in C\left([0,\infty), H^2(0,1) \cap V\right) \cap C^1\left([0,\infty), V\right) \cap C^2\left([0,\infty), L^2(0,1)\right) \\ \eta \in C^1\left([0,\infty)\right). \end{cases} \tag{2.18}$$

Proof. We have

$$\begin{aligned}
\left\langle \mathcal{A}\begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} v \\ -C^{-1}(Au + B\eta) \\ v_x(1) - \beta_1\eta - \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) z(1,t,s)ds \\ -s^{-1} z_\rho \end{pmatrix}, \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} \\
&= (v, u)_{W\times W} + (-C^{-1}(Au + B\eta), v)_{V\times V} \\
&+ \left(v_x(1) - \beta_1\eta - \int_{\tau_1}^{\tau_2} \beta_2(s) z(1,t,s)ds\right).\overline{\eta} \\
&- \int_0^1 \int_{\tau_1}^{\tau_2} \beta_2(s) z(\rho)\overline{z_\rho(\rho)}\,ds\,d\rho. \\
&= < Av, u >_{W'\times W} + < -(Au + B\eta), v >_{V'\times V} + v_x(1)\overline{\eta} \\
&- \int_{\tau_1}^{\tau_2} \beta_2(s) z(1,t,s)ds\overline{\eta} - \beta_1|\eta|^2 - \int_0^1 \int_{\tau_1}^{\tau_2} \beta_2(s) z(\rho)\overline{z_\rho(\rho)}\,ds\,d\rho.
\end{aligned}$$

Since $(u,v,\eta,z)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$, then $Au+B\eta \in V'$ and $v \in W$ then we have

$$
\begin{aligned}
< -(Au+B\eta), v >_{V'\times V} &= < -(Au+B\eta), v >_{W'\times W} \\
&= - < Au, v >_{W'\times W} - < B\eta, v >_{W'\times W} \\
&= - < Au, v >_{W'\times W} - \eta\overline{v_x(1)}.
\end{aligned}
$$

We can deduce

$$
\begin{aligned}
\Re\left\langle \mathcal{A}\begin{pmatrix} u\\ v\\ \eta\\ z\end{pmatrix}, \begin{pmatrix} u\\ v\\ \eta\\ z\end{pmatrix}\right\rangle_{\mathcal{H}} &= \Re\Big(< Av, u >_{W'\times W} - < Au, v >_{W'\times W} + v_x(1)\overline{\eta} - \eta\overline{v_x(1)}\Big) \\
&\quad -\Re\left(\int_{\tau_1}^{\tau_2}\beta_2(s)z(1,t,s)ds)\overline{\eta}\right) - \frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1,t,s)|^2ds \\
&\quad +\frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(0,t,s)|^2ds - \beta_1|\eta|^2 \\
&= -\Re\left(\int_{\tau_1}^{\tau_2}\beta_2(s)z(1,t,s)ds)\overline{\eta}\right) - \frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1,t,s)|^2ds \\
&\quad +\frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(0,t,s)|^2ds - \beta_1|\eta|^2 \\
&\le \frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1,t,s)|^2ds + \frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|\eta|^2ds - \beta_1|\eta|^2 \\
&\quad -\frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1,t,s)|^2ds + \frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(0,t,s)|^2ds \\
&\le \frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|\eta|^2ds - \beta_1|\eta|^2 + \frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)|\eta|^2ds \\
&\le \Big(-\beta_1 + \int_{\tau_1}^{\tau_2}\beta_2(s)ds\Big)|\eta|^2
\end{aligned}
$$

and

$$
\Re\left\langle \mathcal{A}\begin{pmatrix} u\\ v\\ \eta\\ z\end{pmatrix}, \begin{pmatrix} u\\ v\\ \eta\\ z\end{pmatrix}\right\rangle_{\mathcal{H}} \le \Big(-\beta_1 + \int_{\tau_1}^{\tau_2}\beta_2(s)ds\Big)|\eta|^2
$$

Now the relation (1.2) allows to conclude that

$$
\Re\left\langle \mathcal{A}\begin{pmatrix} u\\ v\\ \eta\\ z\end{pmatrix}, \begin{pmatrix} u\\ v\\ \eta\\ z\end{pmatrix}\right\rangle_{\mathcal{H}} \le 0
$$

which implies that the operator $\mathcal{A}$ is dissipative.
Let us prove that the operator $\lambda I - \mathcal{A}$ is surjective for at least one $\lambda > 0$.
For $(f,g,h,k)^{\mathsf{T}} \in \mathcal{H}$, we look for $(u,v,\eta,z)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$ solution of

$$
\begin{cases}
\lambda u - v = f & \text{in }]0,1[\\
\lambda v + C^{-1}(Au+B\eta) = g & \text{in } V' \\
\lambda\eta - v_x(1) + \beta_1\eta + \displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)z(1,t,s)ds = h & \\
\lambda z + s^{-1}z_\rho = k & \text{in }]0,1[.
\end{cases}
\tag{2.19}
$$

Suppose that we have found u with the appropriate regularity. It means that we have also found η. Then $v = \lambda u - f$ and we can determine z by solving the system

$$\begin{cases} s^{-1}z_\rho + \lambda z = k \ \text{ in }]0,1[\\ z(0) = \eta. \end{cases} \tag{2.20}$$

We obtain

$$z(\rho) = \eta e^{-\lambda s\rho} + se^{-\lambda s\rho}\int_0^\rho k(\sigma)e^{\lambda s\sigma}\,d\sigma.$$

In particular

$$z(1) = \eta e^{-\lambda s} + \tau e^{-\lambda s}\int_0^1 k(\sigma)e^{\lambda s\sigma}\,d\sigma.$$

The function u verifies now

$$\begin{cases} \lambda^2 Cu + Au = C(g+\lambda f) - B\eta \ \text{ in } V' \\ u(0) = 0 \\ u_x(0) = 0 \end{cases} \tag{2.21}$$

By using Lax-Milgram's Lemma, the problem (2.21) admits a unique weak solution. Indeed multiplying the first equation by $v \in V$ and by integrating formally by parts we get

$$a(u,v) = F(v), \forall\, v \in V, \tag{2.22}$$

where the bilinear and continuous form a is given by

$$a(u,v) = \int_0^1 \left(\lambda^2\gamma u_x v_x + \lambda^2 uv + u_{xx}v_{xx}\right)dx \quad \forall\, u,v \in V,$$

while the linear form F is

$$F(v) = \int_0^1 (g+\lambda f)v + \gamma(g+\lambda f)_x v_x\,dx - \eta v_x(1), \quad \forall\, v \in V.$$

Since a is clearly strongly coercive on V and F is continuous on V, by Lax-Milgram's Lemma, problem (2.21) admits a unique solution $u \in V$. By taking test functions $v \in \mathcal{D}(0;1)$, we recover the first identity of (2.21). This guarantees that u belongs to $H^2(0,1)$. Using now Green's formula, we see that u satisfies the third identity of (2.21).

Finally, we define η and v by setting

$$v = \lambda u - f \text{ and } \eta = \frac{v_x(1) - \displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)z(1,t,s)ds + h}{\beta_1 + \lambda}$$

This shows that the operator $\mathcal{A}$ is m-dissipative on $\mathcal{H}$ and it generates a $\mathcal{C}_0$-semigroup of contractions in $\mathcal{H}$, under Lumer-Phillips theorem. So, we have found $(u,v,\eta,z)^T \in \mathcal{D}(\mathcal{A})$ which verifies (2.21). The proof ends by using the Hille-Yosida theorem. ∎

3. Strong stability

The main results of this section reads as follows.

Theorem 3.1.
The C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is strongly stable on the energy space $\mathcal{H}$, that is for any $U_0 \in \mathcal{H}$,

$$\lim_{t\longrightarrow 0}\left\|e^{t\mathcal{A}}U_0\right\|_{\mathcal{H}} = 0.$$

Proof. We use the spectral decomposition theory of Sz-Nagy-Foias and Foguel [3, 6, 18]. According this theory, since the resolvent of $\mathcal{A}$ is compact, it suffices to establish that $\mathcal{A}$ has no eigenvalue on the imaginary axis. For our purpose, it is easy to prove that the resolvent of the operator $\mathcal{A}$ defined in (2.5) is compact. We are ready now to achieve the proof of theorem 3.1 with the following lemma.

Lemma 3.2.
There is no eigenvalue of $\mathcal{A}$ on the imaginary axis, that is

$$i\mathbb{R} \subset \rho(\mathcal{A}).$$

Proof. By contradiction argument, we assume that there exists at least one $i\lambda \in \sigma(\mathcal{A})$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$ on the imaginary axis. Let $U = (u, v, \eta, z)^{\mathsf{T}} \in D(\mathcal{A})$ be the corresponding normalized eigenvector, that is verifying $\|U\| = 1$ and

$$\mathcal{A}(u, v, \eta, z)^{\mathsf{T}} = i\lambda(u, v, \eta, z)^{\mathsf{T}}, \tag{3.1}$$

which is equivalent to

$$\begin{cases} v - i\lambda u = 0 & \text{in }]0,1[\\ -C^{-1}(Au + B\eta) - i\lambda v = 0 & \text{in } V' \\ v_x(1) - \beta_1\eta - \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) z(1,t,s)ds - i\lambda\eta = 0 & \\ s^{-1} z_\rho + i\lambda z = 0 & \text{in }]0,1[. \end{cases} \tag{3.2}$$

Recalling the dissipativity of $\mathcal{A}$ and setting

$$\Lambda_1 = \beta - \int_{\tau_1}^{\tau_2} \beta_2(s)ds \tag{3.3}$$

in the proof of theorem 2.2, it follows that

$$0 = \Re e \left\langle \mathcal{A}(u, v, \eta, z)^{\mathsf{T}}, (u, v, \eta, z)^{\mathsf{T}} \right\rangle_{\mathcal{H}} \leq -\Lambda\, |\eta|^2 \tag{3.4}$$

So we deduce that $\eta = z = 0$.

Now (3.2) becomes

$$\begin{cases} v - i\lambda u = 0 \quad \text{in } (0,1) \\ C^{-1}Au + i\lambda v = 0 \quad \text{in } (0,1) \\ v_x(1,.) = 0. \end{cases} \tag{3.5}$$

From the first equation of (3.5) we deduce that

$$u(1) = 0$$

Setting $v = i\lambda u$, it remains to find $u \in V$ which verifies

$$\begin{cases} Au - \lambda^2 Cu = 0 \quad \text{in } \ (0,1) \\ u_x(1) = 0 \\ u(1) = 0. \end{cases} \tag{3.6}$$

By Cauchy-Kowalevski theorem, there exists a nonempty neighbourhood $\mathcal{O}$ of 1 such that $u = 0$ in $\mathcal{O} \cap (0,1)$. Then the unicity theorem of Holmgren (see [7]) allows to conclude that

$$u = 0, \quad \text{on } (0,1). \tag{3.7}$$

We deduce that $(u, v, \eta, z)^{\mathsf{T}} = (0,0,0,0)^{\mathsf{T}}$ which contradicts the fact that $\|U\| = 1$. We conclude that $\mathcal{A}$ has no eigenvalue on the imaginary axis. ■

As the conditions of the spectral decomposition theory of Sz-Nagy-Foias and Foguel are full satisfied, the proof of theorem 3.1 is thus completed. ■

4. Polynomial stability

In this section, we shall analyze the rational decays rate in the form t^{-1} of the energy of system. For that purpose we recall first the following result due to Borichev and Tomilov [4].

Lemma 4.1.
Let $\mathbf{A}$ be the generator of a C_0-semigroup of bounded operators on a Hilbert space $\mathbf{X}$ such that $i\mathbb{R} \subset \rho(\mathbf{A})$. Then we have the polynomial decay

$$\left\|e^{t\mathbf{A}}U_0\right\| \leq \frac{C}{t^{1/\theta}}\left\|U_0\right\|,\ t>0,$$

if and only if

$$\limsup_{|\lambda|\to+\infty}\frac{1}{|\lambda|^{\theta}}\left\|(i\lambda-\mathbf{A})^{-1}\right\|<\infty.$$

The main result of this section is the following theorem

Theorem 4.2.
The semigroup of system (1.1) *decays polynomially as*

$$\left\|e^{t\mathcal{A}}U_0\right\| \leq \frac{C}{t}\left\|U_0\right\|,\ \forall\, U_0\in\mathcal{D}(\mathcal{A}),\ \forall\, t>0. \tag{4.1}$$

Proof. It suffices to show following the results in [10, 20] and the above theorem, that for any $U=(u,v,\eta,z)^{\mathsf{T}}\in\mathcal{D}(\mathcal{A})$ and
$F=(f,g,h,k)^{\mathsf{T}}\in\mathcal{H}$, the solution of

$$(i\lambda I-\mathcal{A})\,U=F \tag{4.2}$$

verifies

$$\|U\|_{\mathcal{H}}\leq C\lambda\|F\|_{\mathcal{H}}; \tag{4.3}$$

where $\lambda>0$ and $C>0$.

Problem (1.1) without delay is the following one

$$\begin{cases} u_{tt}(x,t)-\gamma u_{xxtt}(x,t)+u_{xxxx}(x,t)=0 \text{ in }]0,1[\times(0,+\infty)\\ u(0,t)=u_x(0,t)=0\\ u_{xx}(1,t)+\eta(t)=0,\\ u_{xxx}(1,t)-\gamma u_{xtt}(1,t)=0\ \forall\, t\in(0,+\infty)\\ \eta_t(t)-u_{xt}(1,t)+\beta_1\eta(t)=0\ \ \forall\, t\in(0,+\infty)\\ u(\cdot,0)=u_0,\quad u_t(\cdot,0)=u_1 \text{ in }]0,1[,\quad \eta(0)=\eta_0\in\mathbb{C}\\ \eta(t-\tau)=f_0(t-\tau)\ \forall\, t\in(0,\tau), \end{cases}$$

which is well-posed in

$$\mathcal{H}_0:=W\times V\times\mathbb{C} \tag{4.4}$$

endowed with the norm

$$\left\|(u,v,\eta)^{\mathsf{T}}\right\|_{\mathcal{H}_0}^2 := \|u_{xx}\|_{L^2(0,1)}^2 + \|v\|_{L^2(0,1)}^2 + \gamma\,\|v_x\|_{L^2(0,1)}^2 + |\eta|^2. \tag{4.5}$$

The generator of its semigroup is $\mathcal{A}_0$ defined by

$$\mathcal{A}_0\,(u,v,\eta)^{\mathsf{T}} := \left(v, -C^{-1}(Au+B\eta), v_x(1)-\beta_1\eta\right)^{\mathsf{T}} \tag{4.6}$$

with domain

$$\mathcal{D}(\mathcal{A}_0) = \left\{\,(u,v,\eta)^{\mathsf{T}} \in \mathcal{H}, v\in W, Au+B\eta \in V'\,\right\}, \tag{4.7}$$

Thanks to [9], the operator $\mathcal{A}_0$ generates a polynomial stable semigroup with optimal decay rate t^{-1}. Therefore the solution $(u^*,v^*,\eta^*)^{\mathsf{T}}$ of

$$(i\lambda I-\mathcal{A}_0)\begin{pmatrix}u^*\\v^*\\\eta^*\end{pmatrix} = \begin{pmatrix}u\\v\\\eta\end{pmatrix} \tag{4.8}$$

verifies

$$\left\|(u^*,v^*,\eta^*)^{\mathsf{T}}\right\|_{\mathcal{H}_0} \le C_0\lambda\left\|(u,v,\eta)^{\mathsf{T}}\right\|_{\mathcal{H}_0} \tag{4.9}$$

where C_0 is a positive constant.

On the other hand the system (4.8) can be rewritten as

$$\begin{cases} i\lambda u^* - v^* = u\\ i\lambda v^* + C^{-1}(Au^*+B\eta^*) = v\\ i\lambda\eta^* - v_x^*(1)+\beta_1\eta^* = \eta. \end{cases} \tag{4.10}$$

Let $\alpha \in \mathbb{R}$, with the help of integrations by parts and using (4.10) we have

$$
\begin{aligned}
\left\langle (i\lambda I-\mathcal{A})\begin{pmatrix} u\\ v\\ \eta\\ z\end{pmatrix},\begin{pmatrix} u^*\\ v^*\\ \eta^*\\ \alpha z\end{pmatrix}\right\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} i\lambda u - v\\ i\lambda v + C^{-1}(Au+B\eta)\\ i\lambda\eta - v_x(1)+\beta_1\eta+\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\\ i\lambda z + s^{-1}z_\rho\end{pmatrix},\begin{pmatrix} u^*\\ v^*\\ \eta^*\\ \alpha z\end{pmatrix}\right\rangle_{\mathcal{H}}\\
&= \Big(i\lambda u - v, u^*\Big)_{W\times W} + \Big(i\lambda v + C^{-1}(Au+B\eta), v^*\Big)_{V\times V}\\
&\quad +\Big(i\lambda\eta - v_x(1)+\beta_1\eta+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\Big)\overline{\eta^*}\\
&\quad +\alpha\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)(i\lambda z + s^{-1}z_\rho)\overline{z}\,ds\,d\rho\\
&= \int_0^1 (i\lambda u - v)_{xx}\,\overline{u^*_{xx}}dx + \int_0^1 \big(i\lambda v + C^{-1}(Au+B\eta)\big)\,\overline{v^*}dx\\
&\quad +\gamma\int_0^1 \big(i\lambda v + C^{-1}(Au+B\eta)\big)_x\,\overline{v^*}_x dx\\
&\quad +\left(i\lambda\eta - v_x(1)+\beta_1\eta+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\right)\overline{\eta^*} + i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)z\overline{z}\,ds\,d\rho\\
&\quad +\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= i\lambda\int_0^1 u_{xx}\overline{u^*_{xx}}dx - \int_0^1 v_{xx}\overline{u^*_{xx}}dx + i\lambda\int_0^1 v\overline{v^*}dx + i\lambda\gamma\int_0^1 v_x\overline{v^*_x}dx\\
&\quad +\int_0^1 C^{-1}(Au+B\eta)\overline{v^*} + \gamma C^{-1}(Au+B\eta)\overline{v^*}dx\\
&\quad -\overline{\Big[i\lambda\eta^* - v^*_x(1)+\beta_1\eta^*\Big]}\eta + \int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds.\overline{\eta^*} - \overline{v^*_x(1)}\eta - v_x(1)\overline{\eta^*} + 2\beta_1\eta\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)|z|^2\,ds\,d\rho + \alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)z\overline{z}\,ds\,d\rho + \alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= i\lambda\int_0^1 u_{xx}\overline{u^*_{xx}}dx - \int_0^1 v_{xx}\overline{u^*_{xx}}dx + i\lambda\int_0^1 v\overline{v^*}dx + i\lambda\gamma\int_0^1 v_x\overline{v^*_x}dx\\
&\quad -|\eta|^2 + \int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*} - \overline{v^*_x(1)}\eta - v_x(1)\overline{\eta^*} + 2\beta_1\eta\overline{\eta^*}\\
&\quad + < CC^{-1}(Au+B\eta), v^* >_{V'\times V}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)z\overline{z}\,ds\,d\rho + \alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= i\lambda\int_0^1 u_{xx}\overline{u^*_{xx}}dx - \int_0^1 v_{xx}\overline{u^*_{xx}}dx + i\lambda\int_0^1 v\overline{v^*}dx + i\lambda\gamma\int_0^1 v_x\overline{v^*_x}dx\\
&\quad -|\eta|^2 + \int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*} - \overline{v^*_x(1)}\eta - v_x(1)\overline{\eta^*} + 2\beta_1\eta\overline{\eta^*}\\
&\quad + < Au+B\eta, v^* >_{V'\times V}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)z\overline{z}\,ds\,d\rho + \alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho
\end{aligned}
$$

$$
\begin{aligned}
\left\langle (i\lambda I-\mathcal{A})\begin{pmatrix}u\\ v\\ \eta\\ z\end{pmatrix},\begin{pmatrix}u^*\\ v^*\\ \eta^*\\ \alpha z\end{pmatrix}\right\rangle_{\mathcal{H}} &= i\lambda\int_0^1 u_{xx}\overline{u_{xx}^*}dx-\int_0^1 v_{xx}\overline{u_{xx}^*}dx+i\lambda\int_0^1 v\overline{v^*}dx+i\lambda\gamma\int_0^1 v_x\overline{v_x^*}dx\\
&\quad -|\eta|^2+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}-\overline{v_x^*(1)}\eta-v_x(1)\overline{\eta^*}+2\beta_1\eta\overline{\eta^*}\\
&\quad +<Au,v^*>_{V'\times V}+<B\eta,v^*>_{V'\times V}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= i\lambda\int_0^1 u_{xx}\overline{u_{xx}^*}dx-\int_0^1 v_{xx}\overline{u_{xx}^*}dx+i\lambda\int_0^1 v\overline{v^*}dx+i\lambda\gamma\int_0^1 v_x\overline{v_x^*}dx\\
&\quad -|\eta|^2+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}-\overline{v_x^*(1)}\eta-v_x(1)\overline{\eta^*}+2\beta_1\eta\overline{\eta^*}\\
&\quad +\int_0^1 u_{xx}\overline{v_{xx}^*}dx+\eta\overline{v_x^*(1)}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= -\int_0^1 u_{xx}\overline{(i\lambda u^*-v^*)_{xx}}dx-\int_0^1 v\overline{(i\lambda v^*)}+\gamma v_x\overline{(i\lambda v_x^*)}dx\\
&\quad -\int_0^1 v_{xx}\overline{u_{xx}^*}dx-|\eta|^2+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}-v_x(1)\overline{\eta^*}+2\beta_1\eta\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= -\int_0^1 u_{xx}\overline{u_{xx}}dx-<Cv,i\lambda v^*>_{V'\times V}-\overline{<Au^*,v>_{V'\times V}}\\
&\quad -|\eta|^2+\beta_2 z(1)\overline{\eta^*}-\overline{<B\eta^*,v>_{V'\times V}}+2\beta_1\eta\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= -\|u_{xx}\|^2_{L^2(0,1)}-(v,i\lambda v^*)_{V\times V}-\overline{(C^{-1}Au^*,v)_{V\times V}}\\
&\quad -|\eta|^2+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}-\overline{(C^{-1}B\eta^*,v)_{V\times V}}+2\beta_1\eta\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= -\|u_{xx}\|^2_{L^2(0,1)}-(v,i\lambda v^*)_{V\times V}-(v,C^{-1}Au^*)_{V\times V}\\
&\quad -|\eta|^2+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}-(v,C^{-1}B\eta^*)_{V\times V}+2\beta_1\eta\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= -\|u_{xx}\|^2_{L^2(0,1)}-\Big(v,C^{-1}(Au^*+B\eta^*)+i\lambda v^*\Big)_{V\times V}-|\eta|^2\\
&\quad +\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}+2\beta_1\eta\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho
\end{aligned}
$$

$$
\begin{aligned}
\left\langle (i\lambda I-\mathcal{A})\begin{pmatrix}u\\v\\\eta\\z\end{pmatrix},\begin{pmatrix}u^*\\v^*\\\eta^*\\\alpha z\end{pmatrix}\right\rangle_{\mathcal{H}} &= -\|u_{xx}\|^2_{L^2(0,1)}-(v,v)_{V\times V}-|\eta|^2+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}+2\beta_1\eta\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= -\|u_{xx}\|^2_{L^2(0,1)}-\|v\|^2_{L^2(0,1)}-\gamma\|v_x\|^2_{L^2(0,1)}-|\eta|^2\\
&\quad +2\beta_1\eta\overline{\eta^*}+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\\
&= -\|(u,v,\eta)\|^2_{\mathcal{H}_0}+2\beta_1\eta\overline{\eta^*}+\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds.\overline{\eta^*}\\
&\quad +i\lambda\alpha\int_0^1\int_{\tau_1}^{\tau_2}s\beta_2(s)z\overline{z}\,ds\,d\rho+\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho
\end{aligned}
$$

So

$$
\begin{aligned}
\left\|(u,v,\eta)^{\mathsf{T}}\right\|^2_{\mathcal{H}_0} &= \Re\left\langle F,\begin{pmatrix}u^*\\v^*\\\eta^*\\\alpha z\end{pmatrix}\right\rangle_{\mathcal{H}}+\Re\Big(2\beta_1\eta\overline{\eta^*}\Big)+\Re\Big(\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}\Big)\\
&\quad +\Re\Big(\alpha\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\Big) \qquad (4.11)
\end{aligned}
$$

Take $\alpha=\dfrac{-1}{\varepsilon}$ with $\varepsilon>0$.Then (4.11) becomes

$$
\begin{aligned}
\left\|(u,v,\eta)^{\mathsf{T}}\right\|^2_{\mathcal{H}_0} &= \Re\left\langle F,\begin{pmatrix}u^*\\v^*\\\eta^*\\\frac{-1}{\varepsilon}z\end{pmatrix}\right\rangle_{\mathcal{H}}+\Re\Big(2\beta_1\eta\overline{\eta^*}\Big)+\Re\Big(\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}\Big)\\
&\quad -\Re\Big(\frac{1}{\varepsilon}\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\Big) \qquad (4.12)
\end{aligned}
$$

We have by Young's inegality

$$
\begin{aligned}
\Re\Big(2\beta_1\eta\overline{\eta^*}\Big) &\le 2\beta_1|\eta|.|\eta^*|\\
&\le \frac{\beta_1^2}{\varepsilon}|\eta|^2+\varepsilon|\eta^*|^2 \qquad (4.13)
\end{aligned}
$$

Then by Fubbini

$$
\begin{aligned}
-\Re\Big(\frac{1}{\varepsilon}\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)z_\rho\overline{z}\,ds\,d\rho\Big) &= -\Re\Big(\frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)\Big[|z|^2\Big]_0^1 ds\Big)\\
&= -\frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1)|^2\,ds+\frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(0)|^2\,ds\\
&= -\frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1)|^2\,ds+\frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)\,ds.|\eta|^2 \qquad (4.14)
\end{aligned}
$$

Moreover, by the Cauchy-Schwarz inequality

$$
\begin{aligned}
\Re\left\langle F,\begin{pmatrix}u^*\\ v^*\\ \eta^*\\ \alpha z\end{pmatrix}\right\rangle_{\mathcal{H}} &\le \|F\|_{\mathcal{H}}.\|(u^*,v^*,\eta^*)^{\mathsf{T}}\|_{\mathcal{H}_0}+\frac{1}{\varepsilon}\|F\|_{\mathcal{H}}.\|0,0,0,z)^{\mathsf{T}}\|_{\mathcal{H}}\\
&\le \|F\|_{\mathcal{H}}.\|(u^*,v^*,\eta^*)^{\mathsf{T}}\|_{\mathcal{H}_0}+\frac{1}{\varepsilon}\|F\|_{\mathcal{H}}.\|u,v,\eta,z)^{\mathsf{T}}\|_{\mathcal{H}}\\
&\le C_0\lambda\|F\|_{\mathcal{H}}.\|(u,v,\eta)^{\mathsf{T}}\|_{\mathcal{H}_0}+\frac{1}{\varepsilon}\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} \qquad (4.15)
\end{aligned}
$$

Finally, Young's inequality gives us

$$
\Re\Big(\int_{\tau_1}^{\tau_2}\beta_2(s)z(1)ds\overline{\eta^*}\Big)\le\frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1)|^2ds+\frac{\varepsilon}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)ds.|\eta^*|^2 \qquad (4.16)
$$

Summing (4.13),(4.14),(4.15) and (4.16) we have

$$
\begin{aligned}
\Big\|(u,v,\eta)^{\mathsf{T}}\Big\|_{\mathcal{H}_0}^2 \le\ & \frac{\beta_1^2}{\varepsilon}|\eta|^2+\varepsilon|\eta^*|^2-\frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1)|^2\,ds+\frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)\,ds.|\eta|^2\\
&+C_0\lambda\|F\|_{\mathcal{H}}.\|(u,v,\eta)^{\mathsf{T}}\|_{\mathcal{H}_0}+\frac{1}{\varepsilon}\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}}+\frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)|z(1)|^2ds\\
&+\frac{\varepsilon}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)ds|\eta^*|^2\\
\le\ & \Big(\frac{\beta_1^2}{\varepsilon}+\frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)\,ds\Big)|\eta|^2+\varepsilon\Big(1+\frac{1}{2}\int_{\tau_1}^{\tau_2}\beta_2(s)ds\Big)|\eta^*|^2\\
&+\Big(C_0\lambda+\frac{1}{\varepsilon}\Big)\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} \qquad (4.17)
\end{aligned}
$$

Using the fact that $\mathcal{A}$ is dissipative and Cauchy-Schwarz inequality we have

$$
\Big(\beta_1-\int_{\tau_1}^{\tau_2}\beta_2(s)ds\Big)|\eta|^2\le\Re\left\langle(i\lambda I-\mathcal{A})\,U,U\right\rangle_{\mathcal{H}}\le\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} \qquad (4.18)
$$

This leads to

$$
|\eta|^2\le\frac{1}{\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} \qquad (4.19)
$$

Note also that (4.9) and the dissipativity of $\mathcal{A}_0$ give

$$\beta_1|\eta^*|^2 \leq \Re\left\langle (i\lambda I - \mathcal{A}_0)\,(u^*,v^*,\eta^*)^{\mathsf{T}}, (u^*,v^*,\eta^*)^{\mathsf{T}}\right\rangle_{\mathcal{H}_0} \tag{4.20}$$

$$\leq \|(u,v,\eta)^{\mathsf{T}}\|_{\mathcal{H}_0}.\|(u^*,v^*,\eta^*)^{\mathsf{T}}\|_{\mathcal{H}_0} \tag{4.21}$$

$$\leq C_0\lambda\|(u,v,\eta)^{\mathsf{T}}\|^2_{\mathcal{H}_0} \tag{4.22}$$

This means that

$$|\eta^*|^2 \leq \frac{C_0\lambda}{\beta_1}\|(u,v,\eta)^{\mathsf{T}}\|^2_{\mathcal{H}_0} \tag{4.23}$$

In other words

$$|\eta^*|^2 \leq \frac{C_0\lambda}{\beta_1}\|U\|^2_{\mathcal{H}} \tag{4.24}$$

Using (4.19) and (4.24) in (4.17) we get

$$\left\|(u,v,\eta)^{\mathsf{T}}\right\|^2_{\mathcal{H}_0} \leq C_1\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} + \varepsilon\lambda C_2\|U\|^2_{\mathcal{H}} + \left(C_0\lambda + \frac{1}{\varepsilon}\right)\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} \tag{4.25}$$

where C_1 and C_2 are constants that do not depend on λ defined by

$$C_1 = \frac{\dfrac{\beta_1^2}{\varepsilon} + \dfrac{1}{2\varepsilon}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)\,ds}{\beta_1 - \displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}$$

and

$$C_2 = \frac{C_0\left(1 + \dfrac{1}{2}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}{\beta_1}$$

Let $\varepsilon = \dfrac{1}{2C_2\lambda}$, so $C_2\lambda\varepsilon = \dfrac{1}{2}$.Hence (4.25) becomes

$$\left\|(u,v,\eta)^{\mathsf{T}}\right\|^2_{\mathcal{H}_0} \leq \left(C_1 + C_3\lambda\right)\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} + \frac{1}{2}\|U\|^2_{\mathcal{H}} \tag{4.26}$$

with $C_3 = C_0 + 2C_2$.
If we add $\displaystyle\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)|z|^2 ds d\rho$ member by member we have

$$\frac{1}{2}\,\|U\|^2_{\mathcal{H}} \leq \left(C_1 + C_3\lambda\right)\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} + \int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)|z|^2 ds d\rho \tag{4.27}$$

Now we need a better estimate for

$$\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)|z|^2 ds d\rho$$

From (4.2) we get

$$\begin{cases} s^{-1}z_\rho + i\lambda z = k \ \text{ in }]0,1[\\ z(0) = \eta. \end{cases} \tag{4.28}$$

We obtain

$$z(\rho) = \eta e^{-i\lambda s\rho} + s\int_0^\rho k(\sigma)e^{i\lambda(\sigma-\rho)}\,d\sigma.$$

By the triangular inequality we have

$$|z(\rho)| \le |\eta| + s\int_0^\rho |k(\sigma)|\,d\sigma.$$

This implies that

$$|z(\rho)|^2 \le |\eta|^2 + 2|\eta|s\int_0^\rho |k(\sigma)|\,d\sigma + s^2\Big(\int_0^\rho |k(\sigma)|\,d\sigma\Big)^2. \tag{4.29}$$

On the one hand, using Cauchy-Schwarz inequality, we have

$$\Big(\int_0^\rho |k(\sigma)|\,d\sigma\Big)^2 \le \Big(\int_0^\rho |k(\sigma)|^2\,d\sigma\Big)\Big(\int_0^\rho d\sigma\Big) \le \int_0^\rho |k(\sigma)|^2\,d\sigma. \tag{4.30}$$

On the other hand, Young's inequality gives us

$$2s|\eta|s\int_0^\rho |k(\sigma)|\,d\sigma \le |\eta|^2 + s^2\Big(\int_0^\rho |k(\sigma)|\,d\sigma\Big)^2 \le |\eta|^2 + s^2\int_0^\rho |k(\sigma)|^2\,d\sigma \tag{4.31}$$

Using (4.30) and (4.31) in (4.29) we get

$$|z(\rho)|^2 \le 2|\eta|^2 + 2s^2\int_0^\rho |k(\sigma)|^2\,d\sigma. \tag{4.32}$$

Let's now integrate (4.32) on $(0,1)\times(\tau_1,\tau_2)$. We have

$$\begin{aligned}
\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)|z(\rho)|^2 ds d\rho &\le 2\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)|\eta|^2 ds d\rho + 2\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)s^3\int_0^\rho |k(\sigma)|^2\,d\sigma ds d\rho\\
&\le 2\int_0^1 d\rho.\int_{\tau_1}^{\tau_2} s\beta_2(s)ds|\eta|^2 + 2\int_{\tau_1}^{\tau_2}\beta_2(s)s^3\int_0^1 |k(\sigma)|^2 ds d\rho\\
&\le 2\tau_2\int_{\tau_1}^{\tau_2}\beta_2(s)ds|\eta|^2 + 2\tau_2^2\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)s|k(\rho,s)|^2 ds d\rho\\
&\le 2\tau_2\beta_1|\eta|^2 + 2\tau_2^2\int_0^1\int_{\tau_1}^{\tau_2}\beta_2(s)s|k(\rho,s)|^2 ds d\rho.
\end{aligned}\tag{4.33}$$

Using (4.19) and the definition of the norm in $\mathcal{H}$ we deduce from (4.33) that

$$\int_0^1\int_{\tau_1}^{\tau_2} s\beta_2(s)|z(\rho)|^2 ds d\rho \le C_4\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} + 2\tau_2^2\|F\|_{\mathcal{H}}^2. \tag{4.34}$$

with

$$C_4 = \frac{2\tau_2\beta_1}{\beta_1 - \displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}$$

Combining (4.27) and (4.34) we get

$$\|U\|_{\mathcal{H}}^2 \le 2\Big(C_1 + C_3\lambda + C_4\Big)\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} + 4\tau_2^2\|F\|_{\mathcal{H}}^2 \tag{4.35}$$

Taking $ambda$ to be sufficiently large, we obtain

$$\|U\|_{\mathcal{H}}^2 \leq C_3\lambda\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} + 4\tau_2^2\|F\|_{\mathcal{H}}^2 \tag{4.36}$$

$$\leq C\Big(\lambda\|F\|_{\mathcal{H}}.\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\Big) \tag{4.37}$$

where $C \geq max\left\{C_3, 4\tau_2^2\right\}$
Hence the result

$$\|U\|_{\mathcal{H}} \leq C\lambda\|F\|_{\mathcal{H}}. \tag{4.38}$$

Therefore $\limsup_{\lambda\to+\infty} \dfrac{1}{\lambda}\left\|(i\lambda - \mathbf{A})^{-1}\right\| < \infty$, whence the semi-group decreases polynomially according to the rate t^{-1}.

■

5. Exponential unstability

In this section, we show that the semigroup generated by the operator $\mathcal{A}$ is not exponentially stable. For that we use the frequency domain approach (see Huang [8] and Pruss [5]), namely the below result.

Lemma 5.1. *A contraction semigroup on a Hilbert space is exponentially stable if and only if*

$$i\mathbb{R} = \{i\lambda, \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A}) \tag{5.1}$$

and

$$\sup_{|\lambda|\to\infty} \|\,(i\lambda I - \mathcal{A})^{-1}\,\| < +\infty. \tag{5.2}$$

$\rho(\mathcal{A})$ denotes the resolvent set of the operator $\mathcal{A}$.

We state on the following result that constitutes the main of this section

Theorem 5.2. *The system* (2.2) *is not exponentially stable on the $\mathcal{H}$ energy space.*

Proof. Following the lemma (5.1), we prove that the condition (5.2) is not satisfied satisfied in the sense that there are sequences $(\lambda_n), (U_n)$ and (F_n) such that

$$(i\lambda_n - \mathcal{A})U_n = F_n; \tag{5.3}$$

$$\|F_n\|_{\mathcal{H}} = O(1); \tag{5.4}$$

$$\lim_{n\to+\infty} \|U_n\|_{\mathcal{H}} = +\infty. \tag{5.5}$$

Note that this technique was used in [15], [2], [16], [17] and in several other articles
Let $U_n = (u^n, v^n, \eta^n, z^n)^T$ et $F_n = (f^{1n}, f^{2n}, f^{3n}, f^{4n})^T$

Assuming that (5.3) is verified, we have

$$\begin{cases} i\lambda_n u^n - v^n = f^{1n} \\ i\lambda_n v^n + C^{-1}(Au^n + B\eta^n) = f^{2n} \\ i\lambda_n \eta^n - v_x^n(1) + \beta_1\eta^n + \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s)z^n(1,t,s)ds = f^{3n} \\ i\lambda_n z^n + s^{-1}z_\rho^n = f^{4n}. \end{cases} \tag{5.6}$$

We are looking for a particular solution defined for $f^{1n} = f^{3n} = f^{4n} = 0$ and $f^{2n}(x) = e^{\frac{1}{\sqrt{\gamma}}x} - e^{\frac{-1}{\sqrt{\gamma}}x}$ solution of the differential equation $-\gamma f_{xx} + f = 0$.
The system becomes

$$\begin{cases} v^n = i\lambda_n u^n \\ -\lambda_n^2 C u^n + A u^n + B\eta^n = C f_{2n} \\ i\lambda_n \eta^n - i\lambda_n u_x^n(1) + \beta_1 \eta^n + \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) z^n(1,t,s) ds = 0 \\ i\lambda_n z^n + s^{-1} z_\rho^n = 0. \end{cases} \tag{5.7}$$

Using the definition of the operators A, B and C, we obtain for any $\Phi \in W$ the following variational formulation

$$\int_0^1 u_{xx}^n \overline{\Phi_{xx}}\, dx - \lambda_n^2 \int_0^1 u^n \overline{\Phi} + \gamma u_x^n \overline{\Phi_x}\, dx + \eta^n \overline{\Phi_x(1)} = \int_0^1 f^{2n}\overline{\Phi} + \gamma f_x^{2n} \overline{\Phi_x} dx \tag{5.8}$$

Integration by parts gives

$$\begin{aligned} &\Big[u_{xx}^n \overline{\Phi_x}\Big]_0^1 - \Big[u_{xxx}^n \overline{\Phi}\Big]_0^1 + \int_0^1 u_{xxxx}^n \overline{\Phi}\, dx - \lambda_n^2 \int_0^1 u^n \overline{\Phi} dx - \lambda_n^2 \gamma \Big[u_x^n \overline{\Phi}\Big]_0^1 \\ &+\lambda_n^2 \gamma \int_0^1 u_{xx}^n \overline{\Phi} dx + \eta^n \overline{\Phi_x(1)} = \int_0^1 f^{2n} \overline{\Phi} dx + \gamma \Big[f_x^{2n} \overline{\Phi}\Big]_0^1 - \gamma \int_0^1 f_{xx}^{2n} \overline{\Phi} dx \end{aligned} \tag{5.9}$$

This leads to

$$\begin{aligned} u_{xx}^n(1)\overline{\Phi_x}(1) - u_{xx}^n(0)\overline{\Phi_x}(0) - u_{xxx}^n(1)\overline{\Phi}(1) + u_{xxx}^n(0)\overline{\Phi}(0) + \int_0^1 u_{xxxx}^n \overline{\Phi}\, dx - \lambda_n^2 \int_0^1 u^n \overline{\Phi} dx \\ -\lambda_n^2 \gamma u_x^n(1)\overline{\Phi}(1) + \lambda_n^2 \gamma u_x^n(0)\overline{\Phi}(0) + \lambda_n^2 \gamma \int_0^1 u_{xx}^n \overline{\Phi} dx + \eta^n \overline{\Phi_x(1)} \\ = \int_0^1 \Big[-\gamma f_{xx}^{2n} + f^{2n}\Big]\overline{\Phi} dx + \gamma f_x^{2n}(1)\overline{\Phi}(1) - \gamma f_x^{2n}(0)\overline{\Phi}(0) \end{aligned} \tag{5.10}$$

Since $\Phi(0) = \Phi_x(0) = 0$ and $-\gamma f_{xx}^{2n} + f^{2n} = 0$, (5.10) can be written as

$$\int_0^1 \Big[u_{xxxx}^n + \lambda_n^2 u_{xx}^n - \lambda_n^2 \gamma u^n\Big]\overline{\Phi}\, dx + \Big[u_{xx}^n(1) + \eta^n\Big]\overline{\Phi_x}(1) - \Big[u_{xxx}^n(1) + \lambda_n^2 \gamma u_x^n(1) + \gamma f_x^{2n}(1)\Big]\overline{\Phi}(1) = 0 \tag{5.11}$$

This is equivalent to the system

$$\begin{cases} u_{xxxx}^n + \lambda_n^2 u_{xx}^n - \lambda_n^2 \gamma u^n = 0; \\ u_{xx}^n(1) + \eta^n = 0; \\ u_{xxx}^n(1) + \lambda_n^2 \gamma u_x^n(1) + \gamma f_x^{2n}(1) = 0; \\ u^n(0) = u_x^n(0) = 0. \end{cases} \tag{5.12}$$

Let's now try to express η^n as a function of u^n. To do this, we'll solve the equation of the (5.7) system, which is

$$i\lambda_n z^n + s^{-1} z_\rho^n = 0 \tag{5.13}$$

The solution of (5.13) is of the form

$$z^n(\rho, s) = C e^{-i\lambda_n s\rho} \tag{5.14}$$

Now $z^n(0) = \eta^n(t)$ so $C := \eta^n(t)$.
Thus (5.14) is written as

$$z^n(\rho, s) = \eta^n(t)e^{-i\lambda_n s\rho} \tag{5.15}$$

When we derive this solution with respect to t and with respect to ρ we obtain the equation

$$\eta^n_t - i\lambda_n \eta^n = 0 \tag{5.16}$$

After integration, we also obtain that (5.16) has the solution
$\eta^n = ke^{i\lambda_n t}$, with $k \in \mathbb{C}$.
Since $\eta^n(0) = \eta^n_0$ we obtain $k = \eta^n_0$, from which $\eta^n = \eta^n_0 e^{i\lambda_n t}$.
Replacing η^n by $\eta^n_0 e^{i\lambda_n t}$ in (5.15) gives us

$$z(\rho) = \eta^n_0 e^{i\lambda_n(t-s\rho)}.$$

In particular

$$z^n(1) = \eta^n_0 e^{i\lambda_n(t-s)} = \eta^n e^{-i\lambda_n s}.$$

From the third equation of (5.7) we finally obtain by replacing $z^n(1)$ by $\eta^n e^{-i\lambda_n s}$

$$\eta^n = \frac{i\lambda_n u^n_x(1)}{i\lambda_n + \beta_1 + \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s)e^{-i\lambda_n s}ds} \tag{5.17}$$

The (5.12) system thus becomes

$$\begin{cases} u^n_{xxxx} + \lambda_n^2 u^n_{xx} - \lambda_n^2 \gamma u^n = 0; \\ u^n_{xx}(1) + \dfrac{i\lambda_n}{i\lambda_n + \beta_1 + \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s)e^{-i\lambda_n s}ds} u^n_x(1) = 0; \\ u^n_{xxx}(1) + \lambda_n^2 \gamma u^n_x(1) + \gamma f^{2n}_x(1) = 0; \\ u^n(0) = u^n_x(0) = 0. \end{cases} \tag{5.18}$$

For the rest of the proof, let's assume, as in article [9]

$$\lambda_n = \frac{n\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} + o(1) \tag{5.19}$$

In other words

$$\lambda_n = \frac{n\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} + l(n) \; with \; \lim_{n\to+\infty} l(n) = 0 \tag{5.20}$$

It is clear that from a certain rank $n \geq n_0, n_0$ very large

$$\frac{i\lambda_n}{i\lambda_n + \beta_1 + \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s)e^{-i\lambda_n s}ds} \approx 1$$

and

$$\begin{aligned} u^n_{xxx}(1) + \lambda_n^2 \gamma u^n_x(1) + \gamma f^{2n}_x(1) &= u^n_{xxx}(1) + \lambda_n^2 \gamma u^n_x(1) + 2\sqrt{\gamma} ch(\frac{1}{\sqrt{\gamma}}) \\ &\approx u^n_{xxx}(1) + \lambda_n^2 \gamma u^n_x(1). \end{aligned}$$

We therefore conclude that when $\lambda_n \longrightarrow +\infty$ the system (5.18) is equivalent to the system

$$\begin{cases} u^n_{xxxx} + \lambda_n^2 u^n_{xx} - \lambda_n^2 \gamma u^n = 0; \\ u^n_{xx}(1) + u^n_x(1) = 0; \\ u^n_{xxx}(1) + \lambda_n^2 \gamma u^n_x(1) = 0; \\ u^n(0) = u^n_x(0) = 0. \end{cases} \tag{5.21}$$

On the one hand, Serge Nicaise and associates have shown in [9] that (5.21) admits a solution verifying

$$\|u^n\|_W \sim n^2 \; et \; \|u^n\|_V \sim n \; when \; n \longrightarrow +\infty$$

This gives us (5.5).

$$\lim_{n\to+\infty} \|U_n\|_{\mathcal{H}} = +\infty.$$

On the other hand, according to the choice of F_n we have

$$\begin{aligned}
\|F_n\|^2_{\mathcal{H}} &= \int_0^1 \Big[f^{2n}(x)\Big]^2 + \gamma\Big[f^{2n}_x(x)\Big]^2 dx \\
&= \int_0^1 \Big[e^{\frac{1}{\sqrt{\gamma}}x} - e^{\frac{-1}{\sqrt{\gamma}}x}\Big]^2 + \Big[e^{\frac{1}{\sqrt{\gamma}}x} + e^{\frac{-1}{\sqrt{\gamma}}x}\Big]^2 dx \\
&= \int_0^1 \Big[e^{\frac{2}{\sqrt{\gamma}}x} - 2 + e^{\frac{-2}{\sqrt{\gamma}}x}\Big] + \Big[e^{\frac{2}{\sqrt{\gamma}}x} + 2 + e^{\frac{-2}{\sqrt{\gamma}}x}\Big]^2 dx \\
&= \Big[\frac{\sqrt{\gamma}}{2}e^{\frac{2}{\sqrt{\gamma}}x} - 2x - \frac{\sqrt{\gamma}}{2}e^{\frac{-2}{\sqrt{\gamma}}x}\Big]_0^1 + \Big[\frac{\sqrt{\gamma}}{2}e^{\frac{2}{\sqrt{\gamma}}x} + 2x - \frac{\sqrt{\gamma}}{2}e^{\frac{-2}{\sqrt{\gamma}}x}\Big]_0^1 \\
&= \Big[\frac{\sqrt{\gamma}}{2}e^{\frac{2}{\sqrt{\gamma}}} - 2 - \frac{\sqrt{\gamma}}{2}e^{\frac{-2}{\sqrt{\gamma}}}\Big] + \Big[\frac{\sqrt{\gamma}}{2}e^{\frac{2}{\sqrt{\gamma}}} + 2 - \frac{\sqrt{\gamma}}{2}e^{\frac{-2}{\sqrt{\gamma}}}\Big] \\
&\quad - \Big[\frac{\sqrt{\gamma}}{2} - \frac{\sqrt{\gamma}}{2}\Big] - \Big[\frac{\sqrt{\gamma}}{2} - \frac{\sqrt{\gamma}}{2}\Big] \\
&= \sqrt{\gamma}\Big(\frac{e^{\frac{2}{\sqrt{\gamma}}} - e^{\frac{-2}{\sqrt{\gamma}}}}{2}\Big) - 2 + \sqrt{\gamma}\Big(\frac{e^{\frac{2}{\sqrt{\gamma}}} - e^{\frac{-2}{\sqrt{\gamma}}}}{2}\Big) + 2 \\
&= 2.sh\Big(\frac{2}{\sqrt{\gamma}}\Big)
\end{aligned}$$

This means that

$$\|F_n\|_{\mathcal{H}} = O(1) \tag{5.22}$$

Finally, we've found sequences (λ_n), (U_n) and (F_n) satisfying $(5.3)-(5.5)$. Consequently, the proof of Theorem (5.2) is complete. ■

Conclusion

In this paper we have studied a Rayleigh-type problem with a distributed delay. We used the tools of functional analysis and semi-group theory to obtain the existence, uniqueness and polynomial decay. However, we have established that this polynomial decay is the best in the sense that it is impossible to have an exponential decay. In the future, we'd like to continue our study by replacing the distributed delay with a variable delay.

6. Acknowledgement

We would like to thank the reviewers who agreed to examine our work. May they be honoured.

References

[1] B.C. Tripathy and B. Hazarika, $\mathcal{I}$-monotonic and $\mathcal{I}$-convergent sequences, *Kyungpook Math. J.,* **51**(2011), 233–239.

[2] Gilbert Bayili, Serge Nicaise, and Roland Silga, Rational energy decay rate for the wave equation with delay term on the dynamical control. *Journal of Mathematical Analysis and Applications*, **495**(1):124693, 2021.

[3] Claude D Benchimol, A note on weak stabilizability of contraction semigroups. *SIAM journal on Control and Optimization*, **16**(3):373–379, 1978.

[4] Alexander Borichev and Yuri Tomilov, Optimal polynomial decay of functions and operator semigroups. *Mathematische Annalen*, **347**:455–478, 2010.

[5] Filippo Dell'Oro and David Seifert, A short elementary proof of the gearhart-pr\" uss theorem for bounded semigroups. *arXiv preprint arXiv:2206.06078*, 2022.

[6] R Foguel, Powers of a contraction in hilbert space. 1963.

[7] Haakan Hedenmalm, On the uniqueness theorem of holmgren. *Mathematische Zeitschrift*, **281**(1-2):357–378, 2015.

[8] Falun Huang, Strong asymptotic stability of linear dynamical systems in Banach spaces. *Journal of Differential Equations*, **104**(2):307–324, 1993.

[9] Denis Mercier, Serge Nicaise, Mohamad Sammoury, and Ali Wehbe, Optimal energy decay rate of rayleigh beam equation with only one dynamic boundary control. *Boletim da Sociedade Paranaense de Matematica*, **35**(3):131–171, 2017.

[10] Higidio Portillo Oquendo and Patricia Sánez Pacheco, Optimal decay for coupled waves with kelvin–voigt damping. *Applied Mathematics Letters*, **67**:16–20, 2017.

[11] Innocent Ouedraogo and Gilbert Bayili, Exponential stability for damped shear beam model and new facts related to the classical timoshenko system with a distributed delay term. *Journal of Mathematics research*, **15**(3):45, 2023.

[12] Innocent Ouedraogo and Gilbert Bayili, Stability of a timoshenko system with constant delay. *International Journal of Applied Mathematics*, **36**(2):253, 2023.

[13] INNOCENT OUEDRAOGO and GILBERT BAYILI, Stability for shear beam model and new facts related to the classical timoshenko system with variable delay. *Journal of Nonlinear Evolution Equations and Applications*, **2024**(3):37–54, 2024.

[14] Innocent Ouedraogo, Seck Cheikh, Silga Roland, and Gilbert Bayili, Lack of exponential stability for rayleigh beam equation of only one dynamical boundary control with delay. *Discussiones Mathematicae Differential Inclusions Control and Optimization*, **44**(1):51–75, 09 2024.

[15] Roland SiILGA and Gilbert BAYILI, Stabilization for 1d wave equation with delay term on the dynamical control. *Journal de Mathématiques Pures et Appliquées de Ouagadougou (JMPAO)*,**1** (01), 2022.

[16] Roland Silga and Gilbert Bayili, Polynomial stability of the wave equation with distributed delay term on the dynamical control. *Nonautonomous Dynamical Systems*, **8**(1):207–227, 2021.

[17] Roland Silga, Bila Adolphe Kyelem, and Gilbert Bayili, Indirect boundary stabilization with distributed delay of coupled multi-dimensional wave equations. *Annals of the University of Craiova-Mathematics and Computer Science Series*, **49**(1):15–34, 2022.

[18] Béla Szőkefalvi-Nagy and Ciprian Foia, *Analyse harmonique des opérateurs de l'espace de Hilbert*. Akademiai Kiado, 1967.

[19] Jun-min Wang, Gen-qi Xu, and Siu-Pang Yung, Exponential stability of variable coefficients rayleigh beams under boundary feedback controls: a riesz basis approach. *Systems & control letters*, **51**(1):33–50, 2004.

[20] Ali Wehbe, Rational energy decay rate for a wave equation with dynamical control. *Applied Mathematics Letters*, **16**(3):357–364, 2003.

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **12(04)**(2024), 412–436.
http://doi.org/10.26637/mjm1204/005

Exponential stability of a porous thermoelastic system with Gurtin Pipkin thermal law and distributed delay time

C. BOULKHELOUA[1], H. E. KHOCHEMANE*[2], L. BOUZETTOUTA[1]

[1] *Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS), University of 20 August 1955 Skikda, Algeria.*
[2] *Ecole normale supérieure d'enseignement technologique-Skikda-Algeria.*

Received 24 April 2024; Accepted 20 September 2024

Abstract. In this paper, we consider a one-dimensional porous thermoelastic system with herditary heat conduction and a distributed delay time acting only on the porous equation, where the heat conduction is given by Gurtin Pipkin law. Existence and uniqueness of solution are obtained by the use of Hille-Yosida theorem. Then, based on the energy method as well as by constructing a suitable Lyapunov functional, we prove under some assumptions on the derivative of the heat-flux kernel, that the solution of the system decays exponentially without any assumptions on the wave speeds.

AMS Subject Classifications: 35B40, 47D03, 74D05, 74F05.

Keywords: Porous thermo-elastic system, semigroup theory, exponential stability, Gurtin Pipkin law, energy method, distributed delay time.

Contents

1. Introduction

In this paper we are concerned by the following porous thermoelastic system with distributed delay time

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x - \gamma_1 u_t - \int_{\tau_1}^{\tau_2} \gamma_2(\sigma) u_t(x, t-\sigma)\, d\sigma & in\ (0,\pi)\times(0,\infty) \\ J\varphi_{tt} = \alpha\varphi_{xx} - b u_x - \xi\varphi + \delta\theta - \tau\varphi_t & in\ (0,\pi)\times(0,\infty) \\ c\theta_t = -q_x - \beta u_{xt} - \delta\varphi_t & in\ (0,\pi)\times(0,\infty) \end{cases} \tag{1.1}$$

***Corresponding author.** Email address: **khochmanehoussem@hotmail.com ; c.boulkheloua@univ-skikda.dz** (Houssem Eddine Khochemane)

with the boundary conditions and the initial data

$$\begin{cases} u(0,t) = u(\pi,t) = \varphi_x(0,t) = \varphi_x(\pi,t) = \theta_x(0,t) = \theta_x(\pi,t) = 0 \ , \ t > 0 \\ u(x,0) = u_0(x) \quad , \quad \varphi(x,0) = \varphi_0(x) \quad , \quad \theta(x,0) = \theta_0(x) \quad , \quad x \in (0,\pi) \\ u_t(x,0) = u_1(x) \quad , \quad \varphi_t(x,0) = \varphi_1(x) \quad , \quad x \in (0,\pi) \\ u_t(x,-t) = f_0(x,t) \quad , \quad x \in (0,\pi) \quad , \quad t \in (0,\tau_2) \end{cases} \tag{1.2}$$

where $u = u(x,t)$ is the transversal displacement, $\varphi = \varphi(x,t)$ is the volume fraction, $\theta = \theta(x,t)$ is the temperature variation from an equilibrium reference value and $q = q(x,t)$ is the heat flux. The coefficients $\rho_1,\ J,\ c,\ \mu,\ \alpha,\ b,\ \xi,\ \tau,\ \gamma_1$ are positive constitutive constants such that

$$\mu\,\xi > b^2 \tag{1.3}$$

The coefficient β and δ are the coupling constants that are different from zero but their signs does not matter in the analysis. The term $\int\limits_{\tau_1}^{\tau_2} \gamma_2(\sigma)\, u_t(x,t-\sigma)\, d\sigma$ is a distributed delay that acting only on the porous equation and $\gamma_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function, where τ_1 and τ_2 are two real numbers satisfying $0 \leq \tau_1 < \tau_2$. The initial data $u_0,\ u_1,\ \varphi_0,\ \varphi_1,\ \theta_0,\ f_0$ belongs to the suitable functional space.
In order to determine system (1.1)-(1.2), an additional equation relating q and θ must be used.

Over the years, many scientists and researchers have come up with theories about thermoelasticity. In the classical model of heat diffusion or what is known as the classical theory of thermoelasticity, heat flow obeys Fourier's law of thermal conductivity, which states that heat flow is proportional to a temperature gradient. The thermal conductivity equation is given by Fourier's law as

$$q = -\kappa\theta_x \tag{1.4}$$

where $\kappa > 0$ represents the coefficient of thermal conductivity of the material.

In the last three decades much has been written on the analysis of the longtime behavior of porous thermoelastic systems. Casas and Quintanilla [3] proved the exponential decay of the solution of the following system

$$\begin{cases} \rho\, u_{tt} = \mu\, u_{xx} + b\,\varphi_x - \beta\,\theta_x & in\ (0,\pi) \times (0,\infty) \\ J\,\varphi_{tt} = \alpha\,\varphi_{xx} - b\,u_x - \xi\,\varphi + \delta\,\theta - \tau\,\varphi_t & in\ (0,\pi) \times (0,\infty) \\ c\,\theta_t = \kappa\,\theta_{xx} - \beta\,u_{xt} - \delta\,\varphi_t & in\ (0,\pi) \times (0,\infty) \end{cases}$$

In [24, 30] Quintanilla and co-authors showed the slow decay for the solution of the above system when the frictional damping is removed ($\tau = 0$) or replaced by a viscoelastic damping. Moreover, in [30] they established a polynomial rate of decay provided that $\delta\,(\beta\, b - \delta\,\mu) > 0$.

Closely to the porous thermoelastic systems, Muñoz Rivera and Racke [31] studied the Timoshenko type system

$$\begin{cases} \rho_1\,\varphi_{tt} = k(\varphi_x + \psi)_x \\ \rho_1\,\psi_{tt} = b\,\psi_{xx} - k\,(\varphi_x + \psi) + \gamma\,\theta_x \\ c\,\theta_t = \kappa\,\theta_{xx} - \gamma\,\psi_{tx} \end{cases}$$

with different boundary conditions, where ψ represents the rotation angle of the filament, they proved that the solution of the system is exponentially stable in the case of the wave speeds are equal.

It should be noted that Fourier's thermal conductivity equation is an equation of parabolic type, which leads to the physical contradiction of the infinite speed of heat diffusion, in other words any thermal disturbance at a point will instantly transfer to other parts of the body. To overcome this paradox, other theories of thermoelasticity have emerged.

Green and Naghdi [14, 15] proposed a way to eliminate the paradox of infinite velocities, they used an analogy between the concepts and equations of purely thermal theories and purely mechanical theories and came up with three types of constitutive equations for heat flow in a fixed solid cohesive material classified as type I , type II and type III , where Type I leads to the usual thermal conductivity according to Fourier's law. In type II and type III theories, the constitutive equations for the heat flux are given by

$$q = -f_1\,\psi_x \quad , \quad q = -f_1\,\psi_x - f_2\,\theta_x \tag{1.5}$$

respectively, where

$$\psi = \theta_0\,(x) + \int_0^t \theta\,(x,\tau)\,d\tau$$

is the thermal displacement and f_1 , f_2 are positive constants.

In the framework of Green and Naghdi theory, Quintanilla and co-workers [21, 29] considered the following porous thermoelastic system

$$\begin{cases} \rho_1\,u_{tt} = \mu\,u_{xx} + \gamma\,\phi_x - \beta\,\psi_{tx} \\ J\,\phi_{tt} = b\,\phi_{xx} + m\ \psi_{xx} - \xi\phi + d\,\psi_t - \gamma\,u_x - \tau\,\phi_t \\ a\,\psi_{tt} = k\,\psi_{xx} + m\ \phi_{xx} - d\,\psi_t - \beta\,u_{tx} + k^*\,\theta_{xx} \end{cases}$$

where $(x,t) \in (0,\pi) \times (0,\infty)$ with coefficient satisfy $\mu\,\xi > \gamma^2$ and $b\,k > m^2$.Precisely. Leseduarte et al [21] examined the type II case $(k^* = 0)$ with $(\tau \neq 0)$ and Miranville and Quintanilla [29] considered the type III case $(k^* \neq 0)$ with $(\tau = 0)$. Both have proven that the solution is exponentially stable.

In [11, 19, 25, 27, 28] the authors were considered Timoshenko systems with thermoelastic dissipation of type III, the exponential stability was obtained provided that the wave speeds associated to the hyperbolic part of the system are equal. Otherwise, the solution decays polynomially.

In [22] Lord and Shulman propose a second theory to overcome the paradox of infinite velocity due to Fourier's law, They suggest to replace Fourier's law with the following Cattaneo's law of heat conduction

$$\tau_0\,q_t + q + \kappa\,\theta_x = 0 \tag{1.6}$$

where τ_0 is a positive constant represents the time lag in the response of the heat flux to the temperature gradient and is referred to as the thermal relaxation time.

In accordance with this theory, a hyperbolic system was obtained, and as a result, the heat spreads with a finite speed and a new component of the wave speed appears. The heat is transferred by the process of wave propagation rather than the usual diffusion, and this process is known as the second sound, making the first sound the usual sound.

Fernandez Sare and Racke [12] considered a Timoshenko system coupled with the heat equation modeled by Cattaneo's law, they prove that the solution of the system losses the exponential stability in the case of equal wave speeds.

By introducing a new stability number χ_0 that links all the wave speeds (three), Santos et al [35] refined the results found in [12] and demonstrated the exponential stability of the solution in the case of $\chi_0 = 0$ where

$$\chi_0 = \left(\tau - \frac{\kappa\,\rho_1}{\rho_3}\right)\left(\rho_2 - \frac{b\,\rho_1}{\kappa}\right) - \frac{\tau\,\delta^2\,\rho_1}{\kappa\,\rho_3}$$

In the setting of hyperbolic type porous thermoelastic systems, Han and Xu [17] considered the non uniform porous system with second sound thermoelasticity

$$\begin{cases} \rho\,(x)\,u_{tt} = [\mu\,(x)\,u_x\,(x)]_x + [b\,(x)\,\phi\,(x)]_x - [\beta\,(x)\,\theta\,(x)]_x \\ J\,(x)\,\phi_{tt} = [\alpha\,(x)\,\phi_x\,(x)]_x - b\,(x)\,u_x\,(x) - \xi\,(x)\,\phi\,(x) + m\,(x)\,\theta\,(x) - \tau\,(x)\,\phi_t\,(x) \\ c\,(x)\,\theta_t\,(x) = -q_x\,(x) - \beta\,(x)\,u_{tx}\,(x) - m\,(x)\,\phi_t\,(x) \\ q_t\,(x) + \delta q\,(x) + \eta\theta_x\,(x) = 0 \end{cases} \tag{1.7}$$

where $\rho,\ \mu,\ J,\ \alpha,\ b,\ \xi$ and τ are positive function in $[0,1]$ and $\mu(x)\,\xi(x) > (b(x))^2$ for any $x \in [0,1]$, they have used the spectral method and proved that the solution decays exponentialy. Messaoudi and Fareh [26] studied the uniform case of (1.7). they used the multiplier method and established an exponential stability result. Fareh and Messaoudi [9] examined the solution of the following system

$$\begin{cases} \rho\, u_{tt} - \mu\, u_{xx} + b\,\phi_x = 0 \\ J\,\phi_{tt} - \alpha\,\phi_{xx} + b\,u_x + \xi\,\phi + \beta\,\theta = 0 \\ c\,\theta_t + q_x + \beta\,\phi_{tx} + \delta\,\theta = 0 \\ \tau_0\, q_t + q + \kappa\,\theta_x = 0 \end{cases}$$

in the case when $\mu\,\xi = b^2$. They introduce the stability number

$$\chi = \beta^2 - \left(\frac{c\,\alpha\,\mu}{\rho} - \frac{\kappa\,\alpha}{\tau_0}\right)\left(\frac{J}{\alpha} - \frac{\rho}{\mu}\right)$$

and showed that the solution is exponentially stable if and only if $\chi = 0$.

It is important to note that the second sound and type III theories cannot adequately explain the memory effect that predominates in specific materials, especially at low temperatures. As a result, a more general fundamental assumption about heat flow to thermal memory is required. In [16] Gurtin and Pipkin prosed that heat flux depends on the integrated history of the weighted temperature gradient against a relaxation function called the heat flux kernel. They developed a general nonlinear theory in which thermal disturbances propagate with a finite speed. According to this theory, the linear constitutive equation for q is given as follows

$$q = -\int_{-\infty}^{t} k\,(t-s)\,\theta_x\,(x,s)\,ds \tag{1.8}$$

where $k\,(s)$ is the heat conductivity relaxation kernal. The presence of the convolution term (1.8) renders the porous system coupled with the heat equation into a fully hyperbolic system, this allows the heat to propagate with finite speed and admits to describe the memory effect of heat conduction. We note that many different constitutive models arise from different choices for $k\,(s)$, in particular, if we take $k\,(s) = \kappa\,\delta\,(s)$, where δ is the Dirac mass weighted at 0, then (1.8) reduced to the Fourier's law (1.4), and if we choose

$$k\,(s) = \frac{\kappa}{\tau_0}\,e^{-\frac{s}{\tau_0}} \quad , \quad \tau_0 > 0$$

we obtained Cattaneo's law (1.6). So (1.8) is a generalized from Fourier's and Cattaneo's law.

In [6] Dell'Oro an Pata extended the result of [35] to the following system

$$\begin{cases} \rho_1\,\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0 \\ \rho_2\,\psi_{tt} - b\,\psi_{xx} + \kappa\,(\varphi_x + \psi) + \delta\,\theta_x = 0 \\ \rho_3\,\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^{\infty} k\,(s)\,\theta_{xx}\,(t-s)\,ds + \delta\,\psi_{tx} = 0 \end{cases}$$

they introduce hte stability number

$$\chi_k = \left(\frac{\rho_1}{\rho_3\,\kappa} - \frac{\beta}{k\,(0)}\right)\left(\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right) - \frac{\beta}{k\,(0)}\frac{\rho_1\,\delta^2}{\rho_3\,\kappa\,b}$$

and proved in the case of $\chi_k = 0$ that the solution of the system is exponentially stable. For other models with Gurtin-Pipkin composition, we refer the readers to [2, 4, 5, 8, 10, 18, 33].

In the present paper we consider the porous thermoelastic system (1.1)-(1.2) coupled with the heat equation via the constitutive equation (1.8) and establish an exponential stability result without any restriction on the coefficients. We note that our work is an extension of the results obtained in [7].

The rest of this article is organized as follows: In section 2, we introduce some transformations and state the assumptions needed in our work. In section 3, we use the semigroupe method to prove the well-posedness of problem. Finally, in section 4, we state and prove our stability results. We use c_0 throughout this paper to denote a geniric positive constant.

2. Preliminaries

We note that the presence of the convolution term in the constitutive equation for q renders the family operators mapping the initial value $(u_0,\ u_1,\ \varphi_0,\ \varphi_1,\ \theta_0,\ f_0)$ into the solution (u, φ, θ) not match the semigroup properties. This is due to the fact that the solution value of θ at time t depends on the whole function up to time t. In order to overcome this difficulty we introduce the new variables

$$\theta^t(x,s) = \theta(x,t-s) \quad , \quad s \geq 0 \tag{2.1}$$

and

$$\eta(x,s) = \eta^t(x,s) = \int_0^s \theta^t(x,\lambda)\, d\lambda \quad , \quad s \geq 0 \tag{2.2}$$

which denote the past history and the summed past history of θ up to t, respectively.

Clearly, $\eta^t(x,s)$ satisfies the following conditions

$$\eta_x(0,s) = \eta_x(\pi,s) = 0 \quad , \quad s \geq 0 \ , \ t > 0$$

$$\eta^0(x,s) = \eta_0(x,s) \quad , \quad x \in (0,\pi) \quad , \quad s \geq 0$$

$$\eta(x,0) = \lim_{s\to 0} \eta^t(x,s) = 0 \quad , \quad x \in (0,\pi) \ , \ t > 0$$

and it's easy to prove that

$$\eta_t(x,s) = \theta - \eta_s(x,s) \qquad in\ \ (0,\pi)\times(0,\infty)\times(0,\infty) \tag{2.3}$$

Moreover, we assume that $\lim_{s\to\infty} k(s) = 0$ then a simple computations give us

$$q = -\int_{-\infty}^{t} k(t-s)\,\theta_x(x,s)\, ds = \int_0^\infty k'(s)\,\eta_x^t(x,s)\, ds$$

setting $\kappa(s) = -k'(s)$, system (1.1)-(1.2) and equation (2.2) become

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x - \gamma_1 u_t - \int_{\tau_1}^{\tau_2} \gamma_2(\sigma)\, u_t(x,t-\sigma)\, d\sigma & in\ (0,\pi)\times(0,\infty) \\ J\varphi_{tt} = \alpha\varphi_{xx} - b u_x - \xi\varphi + \delta\theta - \tau\varphi_t & in\ (0,\pi)\times(0,\infty) \\ c\,\theta_t = \int_0^\infty \kappa(s)\,\eta_{xx}^t(x,s)\, ds - \beta u_{xt} - \delta\varphi_t & in\ (0,\pi)\times(0,\infty) \\ \eta_t(x,s) = \theta - \eta_s(x,s) & in\ \ (0,\pi)\times(0,\infty)\times(0,\infty) \end{cases} \tag{2.4}$$

with the boundary conditions and the initial data

$$\begin{cases} u(0,t) = u(\pi,t) = \varphi_x(0,t) = \varphi_x(\pi,t) = \theta_x(0,t) = \theta_x(\pi,t) = 0 \ , \ t > 0 \\ \eta_x(0,s) = \eta_x(\pi,s) = 0 \quad , \quad s \geq 0 \ , \ t > 0 \\ u(x,0) = u_0(x) \quad , \quad \varphi(x,0) = \varphi_0(x) \quad , \quad \theta(x,0) = \theta_0(x) \quad , \quad x \in (0,\pi) \\ u_t(x,0) = u_1(x) \quad , \quad \varphi_t(x,0) = \varphi_1(x) \quad , \quad x \in (0,\pi) \\ \eta^0(x,s) = \eta_0(x,s) \quad , \quad x \in (0,\pi) \quad , \quad s \geq 0 \\ \eta(x,0) = 0 \quad , \quad x \in (0,\pi) \ , \ t > 0 \\ u_t(x,-t) = f_0(x,t) \quad , \quad x \in (0,\pi) \quad , \quad t \in (0,\tau_2) \end{cases} \tag{2.5}$$

Conserning the memory kernel κ, we assume the following set of hypotheses:

$(H1) : \kappa \in C(IR^+) \cap L^1(IR^+)$

$(H2) : \kappa(s) \geq 0 \ , \ \kappa'(s) \leq 0 \ , \ \forall s \geq 0$

$(H3) : \kappa(0) > 0$

$(H4) : \int\limits_0^\infty \kappa(s)\, ds = \kappa_0 = k(0)$

$(H5) : \int\limits_0^\infty s\, \kappa(s)\, ds = 1$

$(H6) : \exists r > 0 \ ; \ \kappa'(s) \leq -r\, \kappa(s) \ , \ \forall s \geq 0$

$(H7) : \lim\limits_{s \to \infty} \kappa(s) = 0$

concerning the weight of the delay, we only assume that

$$\int\limits_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\, d\sigma < \gamma_1 \tag{2.6}$$

In view of the boundary conditions, our system can have solutions (uniform in the variable x), which do not decay. In other words, it is known that for the problem determined by (2.4)-(2.5) we can always take solutions where φ and θ are constant, for this reason, we impose that

$$\int\limits_0^\pi \varphi_0(x)\, dx = \int\limits_0^\pi \varphi_1(x)\, dx = \int\limits_0^\pi \theta_0(x)\, dx = 0 \tag{2.7}$$

It is worth noting that condition (2.7) is imposed to guarantee that the solution decays. Thus, if we want to avoid this behavior we need to impose condition (2.7). In addition as in [1], to be able to use Poincaré's inequality for φ and θ we perform the following transformation

From $(2.4)_2$ and $(2.4)_3$ respectively we have

$$\begin{cases} J \int\limits_0^\pi \varphi_{tt}\, dx + \tau \int\limits_0^\pi \varphi_t\, dx + \xi \int\limits_0^\pi \varphi\ dx - \delta \int\limits_0^\pi \theta\, dx = 0 \\ c \int\limits_0^\pi \theta_t\, dx + \delta \int\limits_0^\pi \varphi_t\, dx = 0 \end{cases} \tag{2.8}$$

If we take $\psi(t) = \int\limits_0^\pi \varphi\ dx$ and $\vartheta(t) = \int\limits_0^\pi \theta\, dx$, we observe that $\psi(0) = \int\limits_0^\pi \varphi_0\ dx$, $\psi'(0) = \int\limits_0^\pi \varphi_1\ dx$ and

$\vartheta(0) = \int_0^\pi \theta_0 \, dx$. Moreover, (ψ, ϑ) is a solution of the following initial value system

$$\begin{cases} J\,\psi'' + \tau\,\psi' + \xi\,\psi - \delta\,\vartheta = 0 \quad , \quad \forall t \geq 0 \\ c\,\vartheta' + \delta\,\psi' = 0 \qquad , \qquad \forall t \geq 0 \\ \psi(0) = \int_0^\pi \varphi_0 \; dx = 0 \\ \psi'(0) = \int_0^\pi \varphi_1 \; dx = 0 \\ \vartheta(0) = \int_0^\pi \theta_0 \, dx = 0 \end{cases}$$

The solution of system is $\psi(t) = \vartheta(t) = 0 \quad , \quad \forall t \geq 0$

Consequently

$$\int_0^\pi \varphi(x,t)\, dx = \int_0^\pi \theta(x,t) \; dx = 0 \quad , \quad \forall t \geq 0$$

Further more, from (2.2) we get

$$\int_0^\pi \eta(x,s) \; dx = 0 \quad , \quad \forall t \geq 0 \quad , \quad \forall s \geq 0$$

3. Well-posedness

In this section, we give the existence and uniqueness of solutions for the system (2.4)-(2.5) using semigroup theory.

First, we introduce as in [32], new dependent variable

$$z(x,\rho,\sigma,t) = u_t(x, t - \rho\,\sigma) \qquad in \; (0,\pi) \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty) \tag{3.1}$$

A simple differentiation shows that z satisfies

$$\sigma\, z_t(x,\rho,\sigma,t) + z_\rho(x,\rho,\sigma,t) = 0 \quad in \; (0,\pi) \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty) \tag{3.2}$$

Hence problem (2.4) takes the form:

$$\begin{cases} \rho_1\, u_{tt} = \mu\, u_{xx} + b\,\varphi_x - \beta\,\theta_x - \gamma_1\, u_t - \int_{\tau_1}^{\tau_2} \gamma_2(\sigma)\, z(x,1,\sigma,t)\, d\sigma \quad in \; (0,\pi) \times (0,\infty) \\ J\,\varphi_{tt} = \alpha\,\varphi_{xx} - b\,u_x - \xi\,\varphi + \delta\,\theta - \tau\,\varphi_t \qquad in \; (0,\pi) \times (0,\infty) \\ c\,\theta_t = \int_0^\infty \kappa(s)\, \eta^t_{xx}(x,s)\, ds - \beta\, u_{xt} - \delta\,\varphi_t \qquad in \; (0,\pi) \times (0,\infty) \\ \sigma\, z_t = -z_\rho \qquad in \; (0,\pi) \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty) \\ \eta_t(x,s) = \theta - \eta_s(x,s) \qquad in \; (0,\pi) \times (0,\infty) \times (0,\infty) \end{cases} \tag{3.3}$$

with the boundary and the initial data

$$\begin{cases} u(0,t) = u(\pi,t) = \varphi_x(0,t) = \varphi_x(\pi,t) = \theta_x(0,t) = \theta_x(\pi,t) = 0 \ , \ t > 0 \\ \eta_x(0,s) = \eta_x(\pi,s) = 0 \quad , \quad s \geq 0 \ , \ t > 0 \\ u(x,0) = u_0(x) \quad , \quad \varphi(x,0) = \varphi_0(x) \quad , \quad \theta(x,0) = \theta_0(x) \quad , \quad x \in (0,\pi) \\ u_t(x,0) = u_1(x) \quad , \quad \varphi_t(x,0) = \varphi_1(x) \quad , \quad x \in (0,\pi) \\ \eta^0(x,s) = \eta_0(x,s) \quad , \quad x \in (0,\pi) \quad , \quad s \geq 0 \\ \eta(x,0) = 0 \quad , \quad x \in (0,\pi) \ , \ t > 0 \\ z(x,\rho,\sigma,0) = f_0(x,\rho\,\sigma) \qquad in \ \ (0,\pi) \times (0,1) \times (0,\tau_2) \end{cases} \tag{3.4}$$

Second, we introduce the vector function $U = (u, v, \varphi, \phi, \theta, z, \eta)^T$, with $v = u_t$, and $\phi = \varphi_t$.

We consider the following Hilbert spaces:

$$\begin{aligned} L^2_*(0,\pi) &= \left\{ w \in L^2(0,\pi) \ \ , \ \ \int_0^\pi w(x)\,dx = 0 \right\}, \\ H^1_*(0,\pi) &= H^1(0,\pi) \cap L^2_*(0,\pi), \\ H^2_*(0,\pi) &= \left\{ w \in H^2(0,\pi) \ ; \ w_x(0) = w_x(\pi) = 0 \right\} \end{aligned}$$

Furthermore, we introduce the weight Hilbert spaces

$$\mathcal{M}_1 = L^2_\kappa\left((0,\infty); H^1_*(0,\pi)\right) = \left\{ w : R_+ \to H^1_*(0,\pi) \ ; \ \int_0^\infty \kappa(s) \left\| w_x(s) \right\|_2^2 ds < \infty \right\}$$

and

$$\mathcal{H} = H^1_\kappa\left((0,\infty) \ , \ H^1_*(0,\pi)\right) = \{\eta \ / \ \eta, \eta_s \in \mathcal{M}_1\}$$

We define the enegy space by

$$\begin{aligned} \mathbb{H} =& H^1_0(0,\pi) \times L^2(0,\pi) \times H^1_*(0,\pi) \times L^2_*(0,\pi) \times H^1_*(0,\pi) \\ & \times L^2\left((0,\pi) \times (0,1) \times (\tau_1,\tau_2)\right) \times \mathcal{M}_1 \end{aligned}$$

Then $\mathbb{H}$, along with the inner product

$$\begin{aligned} \left\langle U, \tilde{U} \right\rangle_{\mathbb{H}} =& \rho_1 \int_0^\pi v\tilde{v}dx + J \int_0^\pi \phi\tilde{\phi}dx + c \int_0^\pi \theta\tilde{\theta}dx + \alpha \int_0^\pi \varphi_x\tilde{\varphi}_x dx \\ &+ \frac{\mu}{2} \int_0^\pi \left(u_x + \frac{b}{\mu}\varphi\right)\left(\tilde{u}_x + \frac{b}{\mu}\tilde{\varphi}\right) dx + \frac{1}{2}\left(\mu - \frac{b^2}{\xi}\right) \int_0^\pi u_x\,\tilde{u}_x\,dx \\ &+ \frac{\xi}{2} \int_0^\pi \left(\varphi + \frac{b}{\xi}u_x\right)\left(\tilde{\varphi} + \frac{b}{\xi}\tilde{u}_x\right) dx + \frac{1}{2}\left(\xi - \frac{b^2}{\mu}\right) \int_0^\pi \varphi\,\tilde{\varphi}\,dx \\ &+ \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} \sigma\,\left|\gamma_2(\sigma)\right|\,z\,\tilde{z}\,d\sigma\,d\rho\,dx + \int_0^\infty \kappa(s) \int_0^\pi \eta_x\,\tilde{\eta}_x\,dx\,ds \end{aligned} \tag{3.5}$$

is a Hilbert space for any $U = (u, v, \varphi, \phi, \theta, z, \eta)^T \in \mathbb{H}$ and $U = (\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{\phi}, \tilde{\theta}, \tilde{z}, \tilde{\eta})^T \in \mathbb{H}$.

The system (3.3)-(3.4) can be rewritten as follows:

$$\begin{cases} \dfrac{d\,U(t)}{dt} = \mathcal{A}\,U(t) \ , \ t > 0, \\ U(x,0) = U_0(x) = (u_0, u_1, \varphi_0, \varphi_1, \theta_0, f_0, \eta_0)^T, \end{cases}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} v \\ \dfrac{\mu}{\rho_1} u_{xx} + \dfrac{b}{\rho_1}\varphi_x - \dfrac{\beta}{\rho_1}\theta_x - \frac{\gamma_1}{\rho_1}\ v - \dfrac{1}{\rho_1}\displaystyle\int_{\tau_1}^{\tau_2} \gamma_2(\sigma)\ z(x,\ 1,\ \sigma,\ t)\ d\sigma \\ \phi \\ \dfrac{\alpha}{J}\varphi_{xx} - \dfrac{b}{J}u_x - \dfrac{\xi}{J}\varphi + \dfrac{\delta}{J}\theta - \dfrac{\tau}{J}\phi \\ \dfrac{1}{c}\displaystyle\int_0^\infty \kappa(s)\ n_{xx}(x,\ s)\ ds - \dfrac{\beta}{c}v_x - \dfrac{\delta}{c}\phi \\ -\dfrac{1}{\sigma}z_\rho \\ \theta - \eta_s \end{pmatrix}.$$

The domain of $\mathcal{A}$ is given by

$$\begin{aligned} D(\mathcal{A}) = \big\{ & U \in \mathbb{H} / u \in H^2(0,\pi) \cap H_0^1(0,\pi)\ ;\ \varphi,\theta \in H_*^2(0,\pi) \cap H_*^1(0,\pi)\ ; \\ & v \in H_0^1(0,\pi)\ ;\ \phi \in H_*^1(0,\pi)\ ;\ z,\ z_\rho \in L^2((0,\pi)\times(0,1)\times(\tau_1,\tau_2))\ ; \\ & \eta \in \mathcal{H}\ ;\ \int_0^\infty \kappa(s)\ n_{xx}(x,\ s)\ ds \in L^2(0,\pi)\ ;\ \eta(x,0) = 0 \big\}. \end{aligned}$$

Now we have the following existence and uniqueness result

Theorem 3.1. *Let $U_0 \in \mathbb{H}$ and assume that* (1.3) *holds. Then, there exists a unique solution $U \in C(\mathbb{R}_+, \mathbb{H})$ for problem* (3.3)-(3.4). *Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathbb{H}).$$

Proof. We use the semi-group approach. So we prove that $\mathcal{A}$ is a maximal dissipative operator.

First, we prove that $\mathcal{A}$ is dissipative. Let $U \in D(\mathcal{A})$, then we have

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathbb{H}} = & -\gamma_1 \int_0^\pi v^2\,dx - \tau \int_0^\pi \phi^2\,dx + \int_0^\pi\int_0^\infty \kappa(s)\ \eta_s\,\eta_{xx}\,ds\,dx \\ & - \int_0^\pi v \int_{\tau_1}^{\tau_2} \gamma_2(\sigma)\,z(x,\ 1,\ \sigma,\ t)\,d\sigma\,dx - \int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\ z_\rho\,z\,d\sigma\,d\rho\,dx \end{aligned} \tag{3.6}$$

Using integration by parts and the fact that $z(x,0,t) = v(x,t)$, the last term in the right-hand side of (3.6) gives

$$\begin{aligned} -\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\ z_\rho\,z\,d\sigma\,d\rho\,dx = & -\frac{1}{2}\int_0^\pi\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\,z^2(x,\ 1,\ \sigma,\ t)\,d\sigma\,dx \\ & + \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\,d\sigma\right)\int_0^\pi v^2\,dx \end{aligned}$$

Also, using Young's inequality we get

$$
\begin{aligned}
&-\int_0^\pi v\int_{\tau_1}^{\tau_2}\gamma(\sigma)\,z(x,\,1,\,\sigma,\,t)\,d\sigma\,dx\\
&\leq \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\gamma(\sigma)|\,d\sigma\right)\int_0^\pi v^2\,dx+\frac{1}{2}\int_0^\pi\int_{\tau_1}^{\tau_2}|\gamma(\sigma)|\,z^2(x,\,1,\,\sigma,\,t)\,d\sigma\,dx
\end{aligned}
$$

Furthermore, using integation by part and bringing in mind $(H7)$ we have

$$
\int_0^\pi\int_0^\infty \kappa(s)\,\eta_s\,\eta_{xx}\,ds\,dx=\frac{1}{2}\int_0^\infty \kappa'(s)\int_0^\pi n_x^2\,dx\,ds
$$

Consequently, using $(H2)$,(3.6) and (2.6) yields

$$
\begin{aligned}
\langle \mathcal{A}U,U\rangle_{\mathbb{H}} \leq &-\left(\gamma_1-\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\,d\sigma\right)\int_0^\pi v^2\,dx\\
&+\frac{1}{2}\int_0^\infty \kappa'(s)\int_0^\pi n_x^2\,dx\,ds-\tau\int_0^\pi \phi^2\,dx\leq 0
\end{aligned}
$$

Therefore, the operator $\mathcal{A}$ is dissipative. Next, we prove that the operator $\lambda I-\mathcal{A}$ is surjective. For any $F=(f_1,f_2,f_3,f_4,f_5,f_6,f_7)^T\in\mathbb{H}$, we prove that there exists a unique $U\in D(\mathcal{A})$ such that

$$
(\lambda I-\mathcal{A})\,U=F \tag{3.7}
$$

The problem (3.7) , leads to solve the following system

$$
\begin{cases}
\lambda u-v=f_1\in H_0^1(0,\pi)\\
(\lambda\rho_1+\gamma_1)\,v-\mu\,u_{xx}-b\,\varphi_x+\beta\,\theta_x+\displaystyle\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\;z(x,\;1,\;\sigma,\;t)\;d\sigma\\
=\rho_1\,f_2\in L^2(0,\pi)\\
\lambda\varphi-\phi=f_3\in H_*^1(0,\pi)\\
(\lambda J+\tau)\;\phi-\alpha\;\varphi_{xx}+b\;u_x+\xi\;\varphi-\delta\;\theta=J\,f_4\in L_*^2(0,\pi)\\
\lambda c\,\theta-\displaystyle\int_0^\infty \kappa(s)\,\eta_{xx}(x,s)\,ds+\beta\,v_x+\delta\,\phi=c\,f_5\in H_*^1(0,\pi)\\
\lambda\sigma\,z+z_\rho=\sigma\,f_6\in L^2((0,\pi)\times(0,1)\times(0,\infty))\\
\lambda\eta-\theta+\eta_s=f_7\in\mathcal{M}_1
\end{cases} \tag{3.8}
$$

Suppose u and φ are given with the appropriate regularity. Then, the first and the third equations in (3.8) yield

$$
v=\lambda u-f_1\in H_0^1(0,\pi) \tag{3.9}
$$

and

$$
\phi=\lambda\varphi-f_3\in H_*^1(0,\pi) \tag{3.10}
$$

respectively.

The sixth equation in (3.8) together with (3.9) and the fact that $z(x,0) = v(x,t)$ gives

$$z(x,\rho,\sigma,t) = \lambda u(x,t)\, e^{-\lambda\sigma\rho} - f_1\, e^{-\lambda\sigma\rho} + \sigma\, e^{-\lambda\sigma\rho} \int_0^\rho e^{\lambda\sigma y}\, f_6(x,y,\sigma,t)\, dy \tag{3.11}$$

The last equation in (3.8) under the condition $\eta(0) = 0$ gives

$$\eta(x,s) = \frac{1}{\lambda}\theta(x,t)\left(1 - e^{-\lambda s}\right) + \int_0^s e^{\lambda(w-s)} f_7(w)\, dw \tag{3.12}$$

Using integration by parts, it can easily be shown that the second, fourth and fifth equations in (3.8) satisfy the following:

$$\begin{cases} (\lambda\rho_1 + \gamma_1)\displaystyle\int_0^\pi v\,\tilde{u}\,dx + \mu\int_0^\pi u_x\,\tilde{u}_x\,dx + b\int_0^\pi \varphi\,\tilde{u}_x\,dx - \beta\int_0^\pi \theta\,\tilde{u}_x\,dx \\ + \displaystyle\int_0^\pi \tilde{u}\int_{\tau_1}^{\tau_2} \gamma(\sigma)\, z(x,1,\sigma,t)\, d\sigma\, dx = \rho_1\int_0^\pi f_2\,\tilde{u}\,dx \\ \\ (\lambda J + \tau)\displaystyle\int_0^\pi \phi\,\tilde{\varphi}\,dx + \alpha\int_0^\pi \varphi_x\,\tilde{\varphi}_x\,dx + b\int_0^\pi u_x\,\tilde{\varphi}\,dx + \xi\int_0^\pi \varphi\,\tilde{\varphi}\,dx - \delta\int_0^\pi \theta\,\tilde{\varphi}\,dx \\ = J\displaystyle\int_0^\pi f_4\,\tilde{\varphi}\,dx \\ \\ c\displaystyle\int_0^\pi \theta\,\tilde{\theta}\,dx + \frac{1}{\lambda}\int_0^\pi \tilde{\theta}_x\int_0^\infty \kappa(s)\,\eta_x\,ds\,dx + \frac{\beta}{\lambda}\int_0^\pi v_x\,\tilde{\theta}\,dx + \frac{\delta}{\lambda}\int_0^\pi \phi\,\tilde{\theta}\,dx = \frac{c}{\lambda}\int_0^\pi f_5\,\tilde{\theta}\,dx \end{cases} \tag{3.13}$$

Furthermore, by using (3.9)-(3.12) , we have the following corresponding weak formulation for the second, fourth and fifth equation in (3.8):

Finding $(u,\varphi,\theta) \in H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_*^1(0,\pi)$ such that for all $\left(\tilde{u},\tilde{\varphi},\tilde{\theta}\right) \in H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_*^1(0,\pi)$ the following holds:

$$B\left((u,\varphi,\theta);\left(\tilde{u},\tilde{\varphi},\tilde{\theta}\right)\right) = l\left(\tilde{u},\tilde{\varphi},\tilde{\theta}\right) \tag{3.14}$$

where $B : \left[H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_*^1(0,\pi)\right]^2 \to \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned} B\left((u,\varphi,\theta);\left(\tilde{u},\tilde{\varphi},\tilde{\theta}\right)\right) = {} & \mu_0\int_0^\pi u\,\tilde{u}\,dx + \mu\int_0^\pi u_x\,\tilde{u}_x\,dx + \mu_1\int_0^\pi \varphi\,\tilde{\varphi}\,dx + \alpha\int_0^\pi \varphi_x\,\tilde{\varphi}_x\,dx \\ & + c\int_0^\pi \theta\,\tilde{\theta}\,dx + c_\kappa\int_0^\pi \theta_x\,\tilde{\theta}_x\,dx + b\int_0^\pi \left(\varphi\,\tilde{u}_x + u_x\,\tilde{\varphi}\right)\,dx \\ & + \beta\int_0^\pi \left(u_x\tilde{\theta} - \theta\,\tilde{u}_x\right)\,dx + \delta\int_0^\pi \left(\varphi\,\tilde{\theta} - \theta\,\tilde{\varphi}\right)\,dx \end{aligned}$$

and $l : H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_*^1(0,\pi) \to \mathbb{R}$ is the linear form given by

$$l\left(\tilde{u}, \tilde{\varphi}, \tilde{\theta}\right) = \int_0^\pi g_1\, \tilde{u}\, dx + \int_0^\pi g_2\, \tilde{\varphi}\, dx + \int_0^\pi g_3\, \tilde{\theta}\, dx + \int_0^\pi g_4\, \tilde{\theta}_x\, dx.$$

where

$$\mu_0 = \lambda^2 \rho_1 + \lambda \gamma_1 + \lambda \int_{\tau_1}^{\tau_2} \gamma_2(\sigma)\, e^{-\lambda\,\sigma} d\sigma \quad > 0$$

$$\mu_1 = \lambda^2 J + \xi + \lambda\, \tau \quad > 0$$

$$c_\kappa = \frac{1}{\lambda^2} \int_0^\infty \kappa(s) \left(1 - e^{-\lambda\, s}\right) ds \quad > 0$$

$$g_1 = \frac{\mu_0}{\lambda} f_1 + \rho_1 f_2 - \int_{\tau_1}^{\tau_2} \sigma\, \gamma_2(\sigma)\, e^{-\lambda\,\sigma} \int_0^1 e^{\lambda\,\sigma\, y} f_6(x,y,\sigma,t)\, dy\, d\sigma \ \in L^2(0,\pi)$$

$$g_2 = (\lambda\, J + \tau)\, f_3 + J\, f_4 \ \in L^2(0,\pi)$$

$$g_3 = \frac{\beta}{\lambda} f_{1_x} + \frac{\delta}{\lambda} f_3 + \frac{c}{\lambda} f_5 \ \in L^2(0,\pi)$$

$$g_4 = -\frac{1}{\lambda} \int_0^\infty \kappa(s) \int_0^s e^{\lambda(w-s)} f_{7_x}(w)\, dw\, ds \ \in L^2(0,\pi)$$

Now, for $\mathcal{V} = H_0^1(0,\pi) \times H_*^1(0,\pi) \times H_*^1(0,\pi)$ equipped with the norm

$$\|(u,\varphi,\theta)\|_V^2 = \|u\|_2^2 + \|u_x\|_2^2 + \|\varphi\|_2^2 + \|\varphi_x\|_2^2 + \|\theta\|_2^2 + \|\theta_x\|_2^2$$

we have

$$\begin{aligned} |B((u,\varphi,\theta);(u,\varphi,\theta))| = {} & \mu_0 \int_0^\pi u^2\, dx + \mu \int_0^\pi u_x^2\, dx + \mu_1 \int_0^\pi \varphi^2\, dx + \alpha \int_0^\pi \varphi_x^2\, dx \\ & + c \int_0^\pi \theta^2\, dx + c_\kappa \int_0^\pi \theta_x^2\, dx + 2b \int_0^\pi u_x\, \varphi\, dx \end{aligned}$$

On the other hand , we can write

$$\begin{aligned} \mu\, u_x^2 + \mu_1\, \varphi^2 + 2b\, u_x\, \varphi = {} & \frac{1}{2}\left[\mu\left(u_x + \frac{b}{\mu}\varphi\right)^2 + \mu_1\left(\varphi + \frac{b}{\mu_1} u_x\right)^2\right] \\ & + \frac{1}{2}\left[\left(\mu - \frac{b^2}{\mu_1}\right) u_x^2 + \left(\mu_1 - \frac{b^2}{\mu}\right)\varphi^2\right] \end{aligned}$$

then, using (1.3) we deduce that

$$\mu\, u_x^2 + \mu_1\, \varphi^2 + 2b\, u_x\, \varphi \geq \frac{1}{2}\left[\left(\mu - \frac{b^2}{\mu_1}\right) u_x^2 + \left(\mu_1 - \frac{b^2}{\mu}\right)\varphi^2\right]$$

consiquently

$$|B((u,\varphi,\theta);(u,\varphi,\theta))| \geq M\, \|(u,\varphi,\theta)\|_V^2$$

where $M = min\left\{\frac{1}{2}\left(\mu - \frac{b^2}{\mu_1}\right)\ ;\ \frac{1}{2}\left(\mu_1 - \frac{b^2}{\mu}\right)\ ;\ \alpha\ ;\ c\ ;\ \mu_0\ ;\ c_\kappa\right\}$. Thus, B is coercive. Moreover, we can easily see that B and l are bounded. Consequently, by Lax-Milgram Lemmam we conclude that there exists a unique $(u, \varphi, \theta) \in \mathcal{V}$ which satisfies (3.14).

Substituting u in (3.9) and (3.11), respectively, we obtain

$$v \in\ H_0^1(0,\pi) \quad , \quad z \in L^2\left((0,\pi)\times(0,1)\times(\tau_1,\tau_2)\right)$$

and z in $(3.8)_6$ we find $z_\rho \in L^2\left((0,\pi)\times(0,1)\times(\tau_1,\tau_2)\right)$

then, inserting φ in (3.10) and we get

$$\phi \in\ H_*^1(0,\pi)$$

Similarly, inserting θ in (3.12) and bearing in mind $(3.8)_7$, we obtain

$$\eta \in \mathcal{H} \quad , \quad \eta(x,0) = 0$$

Moreover, if we take $\left(\tilde{\varphi}, \tilde{\theta}\right) \equiv (0,0) \in H_*^1(0,\pi) \times H_*^1(0,\pi)$, then (3.14) reduces to

$$\mu \int_0^\pi u_x\, \tilde{u}_x\, dx + b \int_0^\pi \varphi\, \tilde{u}_x\, dx - \beta \int_0^\pi \theta\, \tilde{u}_x\, dx = -\int_0^\pi \left(\, -g_1 + \mu_0\, u\right) \tilde{u}\, dx \quad , \ \forall \tilde{u} \in H_0^1(0,\pi)$$

That is

$$\mu\, u_{xx} = -g_1 + \mu_0\, u - b\, \varphi_x + \beta\, \theta_x \quad , \quad in\ L^2(0,\pi)$$

which implies

$$u \in H^2(0,\pi) \cap H_0^1(0,\pi)$$

Then, we choose $\left(\tilde{u}, \tilde{\theta}\right) \equiv (0,0) \in H_0^1(0,\pi) \times H_*^1(0,\pi)$, then (3.14) become

$$\alpha \int_0^\pi \varphi_x\, \tilde{\varphi}_x\, dx = -\int_0^\pi \left(\mu_1\, \varphi + b\, u_x - \delta\, \theta - g_2\right) \tilde{\varphi}\, dx \quad , \ \forall \tilde{\varphi} \in H_*^1(0,\pi) \tag{3.15}$$

Here, we can not use the regularity theorem, because $\tilde{\varphi} \in H_*^1(0,\pi)$. So, we take $\psi \in H_0^1(0,\pi)$ and we set

$$\tilde{\varphi}(x) = \psi(x) - \int_0^\pi \psi(x)\ dx$$

It's clear that $\tilde{\varphi} \in H_*^1(0,\pi)$. Then, a substitution in (3.15) leads to

$$\alpha \int_0^\pi \varphi_x\, \psi_x\, dx = -\int_0^\pi r\, \psi\, dx \quad , \ \forall \psi \in H_0^1(0,\pi)$$

where,

$$r = \mu_1\, \varphi + b\, u_x - \delta\, \theta - g_2$$

That is

$$\alpha\, \varphi_{xx} = \mu_1\, \varphi + b\, u_x - \delta\, \theta - g_2 \quad , \ in\, L^2(0,\pi) \tag{3.16}$$

which implies

$$\varphi \in H^2(0,\pi)$$

On the other hand, from (3.15) and using integration by parts we get

$$\alpha\left[\varphi_x\,\tilde{\varphi}\right]_0^{\pi}-\alpha\int_0^{\pi}\varphi_{xx}\,\tilde{\varphi}\,dx+\int_0^{\pi}\left(\mu_1\,\varphi+b\,u_x-\delta\,\theta-g_2\right)\,\tilde{\varphi}\,dx=0\ \ ,\ \ \forall\tilde{\varphi}\in H_*^1\left(0,\pi\right)$$

and from (3.16) we obtain

$$\varphi_x\left(\pi\right)\,\varphi\left(\pi\right)-\varphi_x\left(0\right)\,\varphi\left(0\right)=0$$

Since $\tilde{\varphi}\in H_*^1\left(0,\pi\right)$ is arbitrary then,

$$\varphi_x\left(\pi\right)\,=\varphi_x\left(0\right)\,=0$$

Consequently

$$\varphi\in H_*^2\left(0,\pi\right)\cap H_*^1\left(0,\pi\right)$$

Similary if we take $\left(\tilde{u},\tilde{\varphi}\right)\equiv(0,0)\in H_0^1\left(0,\pi\right)\times H_*^1\left(0,\pi\right)$, we find

$$\theta\in H_*^2\left(0,\pi\right)\cap H_*^1\left(0,\pi\right)$$

Finally, from $(3.8)_5$ we get

$$\int_0^{\infty}\kappa\left(s\right)\eta_{xx}\left(x,s\right)ds\in L^2\left(0,\pi\right)$$

Hence, there exists a unique $U\in D\left(\mathcal{A}\right)$ such that (3.7) is satisfied. Consequently, the operator $\mathcal{A}$ is maximal. With this , we conclude that $\mathcal{A}$ is a maximal dissipative operator. Consequently, $\mathcal{A}$ is the infinitesimal generator of a linear contraction C_0-semigroup on $\mathbb{H}$. Therefore, the well-posedness result follows from the Hille-Yosida theorem. (see [34]) ■

4. Exponential decay

In this section, we state and prove technical lemmas needed for the proof of our stability result.

Lemma 4.1. *Let* $(u,\varphi,\theta,z,\eta)$ *be a solution of* (3.3) - (3.4) *. Then, the energy functional* $E\left(t\right)$*, defined by*

$$\begin{aligned}E\left(t\right)&=\frac{1}{2}\int_0^{\pi}\left(\rho_1\,u_t^2+J\,\varphi_t^2+c\,\theta^2+\mu\,u_x^2+\alpha\,\varphi_x^2+\xi\,\varphi^2+2b\,\varphi\,u_x\right)dx\\&\quad+\frac{1}{2}\int_0^{\pi}\int_0^{1}\int_{\tau_1}^{\tau_2}\sigma\,\left|\gamma_2\left(\sigma\right)\right|\,z^2\left(x,\rho,\sigma,t\right)d\sigma\,d\rho\,dx+\frac{1}{2}\int_0^{\infty}\kappa\left(s\right)\int_0^{\pi}\eta_x^2\left(x,s\right)dx\,ds\end{aligned}\tag{4.1}$$

satisfies

$$E'\left(t\right)\leq-\left(\gamma_1-\int_{\tau_1}^{\tau_2}\left|\gamma_2\left(\sigma\right)\right|d\sigma\right)\int_0^{\pi}u_t^2\,dx+\frac{1}{2}\int_0^{\infty}\kappa'\left(s\right)\int_0^{\pi}\eta_x^2\left(x,s\right)dx\,ds-\tau\int_0^{\pi}\varphi_t^2\,dx\tag{4.2}$$

Proof. Multiplying $(3.3)_1$, $(3.3)_2$, $(3.3)_3$ by u_t , φ_t , θ respectively, integrating over $(0,\pi)$, and Multiplying$(3.3)_4$ by $\left|\gamma_2\left(\sigma\right)\right|\,z$, integrating over $(0,\pi)\times(0,1)\times(\tau_1,\tau_2)$ then,using integration by part and taking into account the boundary conditions and summing them up, we obtain

$$\frac{d}{2\,dt}\left\{\int_0^\pi\left(\rho_1\,u_t^2+J\,\varphi_t^2+c\,\theta^2+\mu\,u_x^2+\alpha\,\varphi_x^2+\xi\,\varphi^2+2b\,\varphi\,u_x\right)dx\right.$$
$$\left.+\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2}\sigma\,|\gamma_2(\sigma)|\,z^2(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx\right\}$$
$$=-\gamma_1\int_0^\pi u_t^2\,dx-\tau\int_0^\pi\varphi_t^2\,dx-\int_0^\pi u_t\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\,z(x,1,\sigma,t)\,d\sigma\,dx$$
$$+\int_0^\pi\theta\int_0^\infty\kappa(s)\,\eta_{xx}(x,s)\,ds\,dx-\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\,z_\rho\,z(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx$$

Using $(3.3)_5$, we obtain

$$\frac{d}{2\,dt}\left\{\int_0^\pi\left(\rho_1\,u_t^2+J\,\varphi_t^2+c\,\theta^2+\mu\,u_x^2+\alpha\,\varphi_x^2+\xi\,\varphi^2+2b\,\varphi\,u_x\right)dx\right.$$
$$\left.+\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2}\sigma\,|\gamma_2(\sigma)|\,z^2(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx\right\}$$
$$=-\int_0^\pi u_t\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\,z(x,1,\sigma,t)\,d\sigma\,dx-\tau\int_0^\pi\varphi_t^2\,dx-\gamma_1\int_0^\pi u_t^2\,dx \tag{4.3}$$
$$-\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\,z_\rho\,z(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx$$
$$+\int_0^\pi\int_0^\infty\kappa(s)\,\eta_t\,\eta_{xx}(x,s)\,ds\,dx+\int_0^\pi\int_0^\infty\kappa(s)\,\eta_s\,\eta_{xx}(x,s)\,ds\,dx \tag{4.4}$$

integrating by part the last two terms of (4.4) we get

$$E(t)=\frac{1}{2}\int_0^\pi\left(\rho_1\,u_t^2+J\,\varphi_t^2+c\,\theta^2+\mu\,u_x^2+\alpha\,\varphi_x^2+\xi\,\varphi^2+2b\,\varphi\,u_x\right)dx$$
$$+\frac{1}{2}\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2}\sigma\,|\gamma_2(\sigma)|\,z^2(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx+\frac{1}{2}\int_0^\infty\kappa(s)\int_0^\pi\eta_x^2(x,s)\,dx\,ds \tag{4.5}$$

and

$$E'(t)=-\int_0^\pi u_t\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\,z(x,1,\sigma,t)\,d\sigma\,dx-\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\,z_\rho\,z(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx$$
$$-\gamma_1\int_0^\pi u_t^2\,dx-\tau\int_0^\pi\varphi_t^2\,dx-\frac{1}{2}\int_0^\infty\kappa(s)\,\frac{\partial}{\partial s}\int_0^\pi\eta_x^2(x,s)\,dx\,ds \tag{4.6}$$

On the other hand we have $z(x,0,\sigma,t) = u_t(x,t)$, then

$$
\begin{aligned}
&-\int_0^{\pi}\int_0^{1}\int_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\ z_\rho\, z(x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx \\
&= -\frac{1}{2}\int_0^{\pi}\int_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\ z^2(x,1,\sigma,t)\, d\sigma\, dx + \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\ d\sigma\right)\int_0^{\pi} u_t^2\, dx
\end{aligned}
\tag{4.7}
$$

using integration by part and bringing in mind $(H7)$ we find

$$
\begin{aligned}
&-\frac{1}{2}\int_0^{\infty}\kappa(s)\,\frac{\partial}{\partial s}\int_0^{\pi}\eta_x^2(x,s)\, dx\, ds \\
&= -\frac{1}{2}\lim_{s\to b}\left[\kappa(s)\int_0^{\pi}\eta_x^2(x,s)\, dx\right]_{b=0}^{b=\infty} + \frac{1}{2}\int_0^{\infty}\kappa'(s)\int_0^{\pi}\eta_x^2(x,s)\, dx\, ds \\
&= \frac{1}{2}\int_0^{\infty}\kappa'(s)\int_0^{\pi}\eta_x^2(x,s)\, dx\, ds
\end{aligned}
\tag{4.8}
$$

Then, using Young's inequality on the first term in (4.6) we have

$$
\begin{aligned}
-\int_0^{\pi} u_t\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\ z(x,1,\sigma,t)\, d\sigma\, dx \le{}& \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\ d\sigma\right)\int_0^{\pi} u_t^2\, dx \\
&+\frac{1}{2}\int_0^{\pi}\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\ z^2(x,1,\sigma,t)\, d\sigma\, dx
\end{aligned}
\tag{4.9}
$$

Inserting (4.7) , (4.8) and (4.9) in (4.6), we get (4.2) ■

Remark 4.2. *The energy function $E(t)$ defined in* (4.1) *is nonnegative. In fact,*

$$
\begin{aligned}
\mu\, u_x^2 + \xi\,\varphi^2 + 2b\, u_x\,\varphi = {}& \frac{1}{2}\left[\mu\left(u_x + \frac{b}{\mu}\varphi\right)^2 + \xi\left(\varphi + \frac{b}{\xi}u_x\right)^2\right] \\
&+ \frac{1}{2}\left[\left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \left(\xi - \frac{b^2}{\mu}\right)\varphi^2\right]
\end{aligned}
$$

from (1.3) *we deduce that*

$$
\mu\, u_x^2 + \xi\,\varphi^2 + 2b\, u_x\,\varphi \ge \frac{1}{2}\left[\left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \left(\xi - \frac{b^2}{\mu}\right)\varphi^2\right]
$$

consequently

$$
\begin{aligned}
E(t)\ >\ & \frac{1}{2}\int_0^{\pi}\left(\rho_1\, u_t^2 + J\,\varphi_t^2 + c\,\theta^2 + \frac{1}{2}\left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \alpha\,\varphi_x^2 + \frac{1}{2}\left(\xi - \frac{b^2}{\mu}\right)\varphi^2\right)dx \\
&+ \frac{1}{2}\int_0^{\pi}\int_0^{1}\int_{\tau_1}^{\tau_2}\sigma\ |\gamma_2(\sigma)|\ z^2(x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx + \frac{1}{2}\int_0^{\infty}\kappa(s)\int_0^{\pi}\eta_x^2(x,s)\, dx\, ds
\end{aligned}
$$

then $E(t)$ is nonnegative.

Remark 4.3. *From $(H2)$ we conclude that the energy functional $E(t)$ is decreasing and bounded above by $E(0)$*

Lemma 4.4. *Let $(u,\varphi,\theta,z,\eta)$ be a solution of* (3.3) - (3.4). *Then, the functional*

$$I_1(t)=\rho_1\int_0^{\pi}u_t u dx+\frac{\gamma_1}{2}\int_0^{\pi}u^2dx\ ,\ t\geq 0,$$

satisfies

$$\begin{aligned}I_1{}'(t)\leq&-\frac{\mu}{2}\int_0^{\pi}u_x^2\,dx+\rho_1\int_0^{\pi}u_t^2\,dx+c_0\int_0^{\pi}\left(\varphi^2+\theta^2\right)dx\\&+c_0\int_0^{\pi}\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\ \ z^2(x,1,\sigma,t)\ d\sigma\,dx\end{aligned}\tag{4.10}$$

Proof. By differentiating $I_1(t)$, using $(3.3)_1$ and integrating by parts together with the boundary conditions, we obtain

$$\begin{aligned}I_1{}'(t)=&-\mu\int_0^{\pi}u_x^2\,dx-b\int_0^{\pi}\varphi\,u_x\,dx+\beta\int_0^{\pi}\theta\,u_x\,dx+\rho_1\int_0^{\pi}u_t^2\,dx\\&-\int_0^{\pi}u\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\ \ z(x,1,\sigma,t)\ d\sigma\,dx\end{aligned}\tag{4.11}$$

Young's , Poincaré and Cauchy Schwarz inequalities lead to

$$-b\int_0^{\pi}\varphi\,u_x\,dx\leq\frac{\mu}{6}\int_0^{\pi}u_x^2\,dx+\frac{3\,b^2}{2\,\mu}\int_0^{\pi}\varphi^2dx\tag{4.12}$$

$$\beta\int_0^{\pi}\varphi\,u_x\,dx\leq\frac{\mu}{6}\int_0^{\pi}u_x^2\,dx+\frac{3\,\beta^2}{2\,\mu}\int_0^{\pi}\theta^2dx\tag{4.13}$$

and

$$\begin{aligned}&-\int_0^{\pi}u\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\ \ z(x,1,\sigma,t)\ d\sigma\,dx\\&\leq\frac{\mu}{6}\int_0^{\pi}u_x^2\,dx+\frac{3}{2\,\mu}\left(\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\ d\sigma\right)\int_0^{\pi}\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\ \ z^2(x,1,\sigma,t)\ d\sigma\,dx\end{aligned}\tag{4.14}$$

Substituting (4.12) , (4.13) and (4.14) in (4.11), we get (4.10). ■

Lemma 4.5. *Let $(u,\varphi,\theta,z,\eta)$ be a solution of* (3.3) - (3.4). *Then, the functional*

$$I_2(t)=J\int_0^{\pi}\varphi\,\varphi_t\,dx+\frac{\tau}{2}\int_0^{\pi}\varphi^2dx+\frac{b\,\rho_1}{\mu}\int_0^{\pi}\varphi\int_0^{x}u_t(y)\,dy\,dx\ ,\ t\geq 0,$$

satisfies

$$I_2{}'(t) \le -\alpha \int_0^\pi \varphi_x^2 dx - \frac{\chi}{4}\int_0^\pi \varphi^2 dx + c_0 \int_0^\pi \left(u_t^2 + \varphi_t^2 + \theta^2\right)\, dx \tag{4.15}$$

$$+ c_0 \int_0^\pi \int_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\, z^2(x,1,\sigma,t)\, d\sigma\, dx \tag{4.16}$$

where $\chi = \xi - \dfrac{b^2}{\mu}$

Proof. By differentiating $I_2(t)$, using $(3.3)_2$ and integrating by parts together with the boundary conditions, we obtain

$$\begin{aligned} I_2{}'(t) &= -\alpha \int_0^\pi \varphi_x^2 dx - \chi \int_0^\pi \varphi^2 dx + J\int_0^\pi \varphi_t^2 dx + \left(\delta - \frac{b\,\beta}{\mu}\right)\int_0^\pi \varphi\,\theta\, dx \\ &+ \frac{b\,\rho_1}{\mu}\int_0^\pi \varphi_t \int_0^x u_t(y)\, dy\, dx - \frac{b\,\gamma_1}{\mu}\int_0^\pi \varphi \int_0^x u_t(y)\, dy\, dx \\ &- \frac{b}{\mu}\int_0^\pi \varphi \int_0^x \int_{\tau_1}^{\tau_2} \gamma_2(\sigma)\, z(y,1,\sigma,t)\, d\sigma\, dy\, dx \end{aligned} \tag{4.17}$$

Using Young's and Cauchy Schwarz inequalities, we get

$$\delta - \frac{b\,\beta}{\mu}\int_0^\pi \varphi\,\theta\, dx \le \frac{\chi}{4}\int_0^\pi \varphi^2 dx + \frac{1}{\chi}\left(\delta - \frac{b\,\beta}{\mu}\right)^2 \int_0^\pi \theta^2 dx \tag{4.18}$$

$$\frac{b\,\rho_1}{\mu}\int_0^\pi \varphi_t \int_0^x u_t(y)\, dy\, dx \le \frac{b\,\rho_1}{2\mu}\int_0^\pi \varphi_t^2 dx + \frac{b\,\rho_1\,\pi^2}{2\mu}\int_0^\pi u_t^2 dx \tag{4.19}$$

$$-\frac{b\,\gamma_1}{\mu}\int_0^\pi \varphi \int_0^x u_t(y)\, dy\, dx \le \frac{\chi}{4}\int_0^\pi \varphi^2 dx + \frac{1}{\chi}\left(\frac{b\,\gamma_1\,\pi}{\mu}\right)^2 \int_0^\pi u_t^2 dx \tag{4.20}$$

and

$$\begin{aligned} &- \frac{b}{\mu}\int_0^\pi \varphi \int_0^x \int_{\tau_1}^{\tau_2} \gamma_2(\sigma)\, z(y,1,\sigma,t)\, d\sigma\, dy\, dx \\ &\le \frac{\chi}{4}\int_0^\pi \varphi^2 dx + \frac{1}{\chi}\left(\frac{b\,\pi}{\mu}\right)^2 \left(\int_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\, d\sigma\right) \int_0^\pi \int_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\, z^2(x,1,\sigma,t)\, d\sigma\, dx \end{aligned} \tag{4.21}$$

Inserting (4.18)-(4.21) in (4.17), we obtain (4.16). ■

Lemma 4.6. *Let* $(u,\varphi,\theta,z,\eta)$ *be a solution of* (3.3) - (3.4). *Then, the functional*

$$I_3(t) = -c\,\rho_1 \int_0^\pi \theta \int_0^x u_t(y)\, dy\, dx \ ,\ \ t \ge 0,$$

satisfies, for any $\varepsilon_1 > 0$, the following estimate

$$\begin{aligned} I_3{}'(t) \leq & -\frac{\rho_1\,|\beta|}{2}\int_0^{\pi} u_t^2\,dx + \varepsilon_1 \int_0^{\pi} \left(u_x^2+\varphi^2\right)\,dx + c_0\left(1+\frac{1}{\varepsilon_1}\right)\int_0^{\pi}\theta^2\,dx \\ & + \frac{\rho_1\,\kappa_0}{|\beta|}\int_0^{\infty}\kappa(s)\int_0^{\pi}\eta_x^2\,dx\,ds + \pi^2\varepsilon_1\left(\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\,d\sigma\right)\int_0^{\pi}\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\;z^2(x,1,\sigma,t)\;d\sigma\,dx \end{aligned} \tag{4.22}$$

Proof. Differentiating the functional $I_3(t)$ using $(3.3)_1, (3.3)_3$ and integrating by parts together with the boundary conditions,, we obtain

$$\begin{aligned} I_3{}'(t) = & -\rho_1\,\beta\int_0^{\pi} u_t^2\,dx - \mu\,c\int_0^{\pi}\theta\,u_x\,dx - b\,c\int_0^{\pi}\theta\,\varphi\,dx + \beta\,c\int_0^{\pi}\theta^2\,dx - c\,\gamma_1\int_0^{\pi}u_t\,\theta\,dx \\ & + c\int_0^{\pi}\theta\int_0^{x}\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\;z(y,1,\sigma,t)\;d\sigma\,dy\,dx + \rho_1\int_0^{\pi}u_t\int_0^{\infty}\kappa(s)\,\eta_x(x,s)\,ds\,dx \end{aligned} \tag{4.23}$$

Using Young's and Cauchy Schwarz inequalities,

$$-\mu\,c\int_0^{\pi}\theta\,u_x\,dx \leq \varepsilon_1\int_0^{\pi}u_x^2\,dx + \frac{\mu^2c^2}{4\varepsilon_1}\int_0^{\pi}\theta^2\,dx \tag{4.24}$$

$$-b\,c\int_0^{\pi}\theta\,\varphi\,dx \leq \varepsilon_1\int_0^{\pi}\varphi^2\,dx + \frac{b^2c^2}{4\varepsilon_1}\int_0^{\pi}\theta^2\,dx \tag{4.25}$$

$$\begin{aligned} & \int_0^{\pi}\theta\int_0^{x}\int_{\tau_1}^{\tau_2}\gamma_2(\sigma)\;z(y,1,\sigma,t)\;d\sigma\,dy\,dx \\ & \leq \frac{c^2}{4\varepsilon_1}\int_0^{\pi}\theta^2\,dx + \pi^2\varepsilon_1\left(\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\,d\sigma\right)\int_0^{\pi}\int_{\tau_1}^{\tau_2}|\gamma_2(\sigma)|\;z^2(x,1,\sigma,t)\;d\sigma\,dx \end{aligned} \tag{4.26}$$

$$\rho_1\int_0^{\pi}u_t\int_0^{\infty}\kappa(s)\,\eta_x(x,s)\,ds\,dx \leq \frac{\rho_1\,|\beta|}{4}\int_0^{\pi}u_t^2\,dx + \frac{\rho_1\,\kappa_0}{|\beta|}\int_0^{\infty}\kappa(s)\int_0^{\pi}\eta_x^2\,dx\,ds \tag{4.27}$$

$$-c\,\gamma_1\int_0^{\pi}u_t\,\theta\,dx \leq \frac{|\beta|\,\rho_1}{4}\int_0^{\pi}u_t^2dx + \frac{(c\,\gamma_1)^2}{|\beta|\,\rho_1}\int_0^{\pi}\theta^2dx \tag{4.28}$$

Substituting (4.24)-(4.28) in (4.23), we get (4.22). ■

Lemma 4.7. *Let $(u, \varphi, \theta, z, \eta)$ be a solution of* (3.3) - (3.4). *Then, the functional*

$$I_4(t) = -\frac{c}{\kappa_0}\int_0^{\pi}\theta\int_0^{\infty}\kappa(s)\;\eta(x,s)\,ds\,dx\;,\;\; t \geq 0,$$

satisfies, for any $\varepsilon_2, \varepsilon_3 > 0$, the following estimate

$$I_4{}'(t) \leq -\frac{c}{2}\int_0^\pi \theta^2\,dx + \varepsilon_2\int_0^\pi \varphi_t^2\,dx + \varepsilon_3\int_0^\pi u_t^2\,dx - \frac{c\,k(0)}{2\kappa_0^2}\int_0^\infty \kappa'(s)\int_0^\pi \eta_x^2\,dx\,ds$$
$$+\,c_0\left(1+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}\right)\int_0^\infty \kappa(s)\int_0^\pi \eta_x^2\,dx\,ds \tag{4.29}$$

Proof. By differentiating $I_4(t)$, using $(3.3)_3$, $(3.3)_5$ and integrating by parts together with the boundary conditions,, we obtain

$$I_4{}'(t) = -c\int_0^\pi \theta^2\,dx + \frac{c}{\kappa_0}\int_0^\pi \theta\int_0^\infty \kappa(s)\ \eta_s(x,s)\,ds\,dx + \frac{1}{\kappa_0}\int_0^\pi\left(\int_0^\infty \kappa(s)\ \eta_x(x,s)\,ds\right)^2 dx$$
$$-\,\frac{\beta}{\kappa_0}\int_0^\pi u_t\int_0^\infty \kappa(s)\ \eta_x(x,s)\,ds\,dx + \frac{\delta}{\kappa_0}\int_0^\pi \varphi_t\int_0^\infty \kappa(s)\ \eta(x,s)\,ds\,dx \tag{4.30}$$

Young's , Poincaré and Cauchy Schwarz inequalities lead to

$$\frac{\delta}{\kappa_0}\int_0^\pi \varphi_t\int_0^\infty \kappa(s)\ \eta(x,s)\,ds\,dx \leq \varepsilon_2\int_0^\pi \varphi_t^2\,dx + \frac{\delta^2}{4\varepsilon_2\,\kappa_0}\int_0^\infty \kappa(s)\int_0^\pi \eta_x^2\,dx\,ds \tag{4.31}$$

$$-\frac{\beta}{\kappa_0}\int_0^\pi u_t\int_0^\infty \kappa(s)\ \eta_x(x,s)\,ds\,dx \leq \varepsilon_3\int_0^\pi u_t^2\,dx + \frac{\beta^2}{4\varepsilon_3\,\kappa_0}\int_0^\infty \kappa(s)\int_0^\pi \eta_x^2\,dx\,ds \tag{4.32}$$

$$\frac{1}{\kappa_0}\int_0^\pi\left(\int_0^\infty \kappa(s)\ \eta_x(x,s)\,ds\right)^2 dx \leq \int_0^\infty \kappa(s)\int_0^\pi \eta_x^2\,dx\,ds \tag{4.33}$$

and

$$\frac{c}{\kappa_0}\int_0^\pi \theta\int_0^\infty \kappa(s)\ \eta_s(x,s)\,ds\,dx \leq \frac{c}{2}\int_0^\pi \theta^2\,dx - \frac{c\,k(0)}{2\kappa_0^2}\int_0^\infty \kappa'(s)\int_0^\pi \eta_x^2\,dx\,ds \tag{4.34}$$

Estimate (4.29) follows by substituting (4.31)-(4.34) into (4.30). ■

Lemma 4.8. *Let $(u,\varphi,\theta,z,\eta)$ be a solution of* (3.3) - (3.4). *Then, the functional*

$$I_5(t) = \int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} \sigma\,e^{-\sigma\,\rho}\,|\gamma_2(\sigma)|\,z^2(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx \qquad t\geq 0$$

satisfies the estimate

$$I_5{}'(t) \leq -m_1\int_0^\pi\int_{\tau_1}^{\tau_2} |\gamma_2(\sigma)|\,z^2(x,1,\sigma,t)\,d\sigma\,dx + \gamma_1\int_0^1 u_t^2\,dx$$
$$-\,m_1\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2} \sigma\ |\gamma_2(\sigma)|\,z^2(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx \qquad t\geq 0 \tag{4.35}$$

Proof. By differentiating $I_5(t)$, using $(3.3)_4$, integrating by parts and using the fact that $z(x,0,\sigma,t) = u_t(x,t)$ gives, we obtain

$$I_5{}'(t) = -\int_0^\pi \int_{\tau_1}^{\tau_2} e^{-\sigma} |\gamma(\sigma)| z^2(x,1,\sigma,t)\, d\sigma\, dx + \left(\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, d\sigma\right) \int_0^\pi u_t^2\, dx$$
$$- \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} \sigma\, e^{-\sigma\rho}\, |\gamma(\sigma)| z^2(x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx$$

using the fact that $e^{-\sigma} \le e^{-\sigma\rho} \le 1$ we get for all $\rho \in [0,1]$

$$I_5{}'(t) \le -\int_0^\pi \int_{\tau_1}^{\tau_2} e^{-\sigma} |\gamma(\sigma)| z^2(x,1,\sigma,t)\, d\sigma\, dx + \left(\int_{\tau_1}^{\tau_2} |\gamma(\sigma)|\, d\sigma\right) \int_0^\pi u_t^2\, dx$$
$$- \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} \sigma\, e^{-\sigma}\, |\gamma(\sigma)| z^2(x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx$$

Since $-e^{-\sigma}$ is an increasing function, we have $-e^{-\sigma} \le -e^{-\tau_2}$ for all $\sigma \in [\tau_1, \tau_2]$. Finally , setting $m_1 = e^{-\tau_2}$ and bringing in mind (2.6) we get (4.35) ■

Now, we define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = N\, E(t) + I_1(t) + N_1 I_2(t) + \frac{2}{|\beta|\ \rho_1} I_3(t) + N_2 I_4(t) + N_3 I_5(t) \tag{4.36}$$

where N, N_1, N_2, N_3 are positive constants.

Lemma 4.9. *Let $(u, \varphi, \theta, z, \eta)$ be a solution of* (3.3) - (3.4). *Then, there exist two positive constants λ_1 and λ_2 such that the Lyapunov functional* (4.36) *satisfies*

$$\lambda_1 E(t) \le \mathcal{L}(t) \le \lambda_2 E(t)\,,\ \forall t \ge 0, \tag{4.37}$$

and

$$\mathcal{L}'(t) \le -\varsigma_1 E(t) + \varsigma_2 \int_0^\infty \kappa(s) \|\eta_x\|^2 ds\ \ ;\ \ \varsigma_1\ ,\ \varsigma_2 > 0. \tag{4.38}$$

Proof. From (4.36), we have

$$|\mathcal{L}(t) - NE(t)| \le \rho_1 \int_0^\pi |u_t u|\, dx + \frac{\gamma_1}{2} \int_0^\pi u^2 dx + N_1 J \int_0^\pi |\varphi_t \varphi|\, dx + N_1 \frac{\tau}{2} \int_0^\pi \varphi^2 dx$$
$$N_1 \frac{b\,\rho_1}{\mu} \int_0^\pi \varphi \int_0^x u_t(y)\, dy\, dx + \frac{2c}{|\beta|} \int_0^\pi \left|\theta \int_0^x u_t(y)\, dy\right| dx$$
$$+ \frac{N_2\, c}{\kappa_0} \int_0^\pi \left|\theta \int_0^\infty \kappa(s)\, \eta(x,s)\, ds\right| dx$$
$$+ N_3 \int_0^\pi \int_0^1 \int_{\tau_1}^{\tau_2} \sigma e^{-\sigma\rho} |\gamma(\sigma)| z^2(x,\rho,\sigma,t)\, d\sigma\, d\rho\, dx$$

By using the Young's, Poincaré and Cauchy-Schwarz inequalities, we obtain

$$|\mathcal{L}(t)-NE\,(t)|\leq\varsigma E\,(t)\,,\;\varsigma>0,$$

which yields

$$(N-\varsigma)\,E\,(t)\leq\mathcal{L}\,(t)\leq(N+\varsigma)\,E\,(t)\,,$$

by choosing N (depending on N_1, N_2, and N_3) sufficiently large we obtain (4.37).

Now, By differentiating $\mathcal{L}\,(t)$, exploiting (4.2), (4.10), (4.16), (4.22), (4.29), (4.35) and setting $\varepsilon_1=\frac{\mu\,\rho_1\,|\beta|}{8}$, $\varepsilon_2=\frac{1}{N_2}$, $\varepsilon_3=\frac{\rho_1}{N_2}$, we get

$$\begin{aligned}
\mathcal{L}'\,(t)\leq&-\left[\left(\left(\gamma_1-\int_{\tau_1}^{\tau_2}|\gamma_2\,(\sigma)|\,d\sigma\right)N+1\right)-2\rho_1-c_0N_1-\gamma_1N_3\right]\int_0^\pi u_t^2dx\\
&-(N\,\tau-c_0N_1-1)\int_0^\pi\varphi_t^2dx-\left(\frac{N_1\chi}{4}-c_0-\frac{\mu}{4}\right)\int_0^\pi\varphi^2dx-\frac{\mu}{4}\int_0^\pi u_x^2dx\\
&-\left(\frac{c\,N_2}{2}-c_0-c_0N_1-\frac{2c_0}{\rho_1\,|\beta|}\left(1+\frac{8}{\mu\,\rho_1\,|\beta|}\right)\right)\int_0^\pi\theta^2dx-\alpha\,N_1\int_0^\pi\varphi_x^2dx\\
&-m_1N_3\int_0^\pi\int_0^1\int_{\tau_1}^{\tau_2}\sigma\,|\gamma_2\,(\sigma)|\,z^2\,(x,\rho,\sigma,t)\,d\sigma\,d\rho\,dx\\
&-\left(m_1N_3-c_0-c_0N_1-\frac{\mu\,\pi^2}{4}\int_{\tau_1}^{\tau_2}|\gamma_2\,(\sigma)|\,d\sigma\right)\int_0^\pi\int_{\tau_1}^{\tau_2}|\gamma_2\,(\sigma)|\,z^2\,(x,1,\sigma,t)\,d\sigma\,dx\\
&+\left(\frac{N}{2}-\frac{c\,k\,(0)}{2\kappa_0^2}N_2\right)\int_0^\infty\kappa'\,(s)\,\|\eta_x\|^2ds\\
&+\left(\frac{2\kappa_0}{\beta^2}+c_0N_2\left(1+N_2+\frac{N_2}{\rho_1}\right)\right)\int_0^\infty\kappa\,(s)\,\|\eta_x\|^2ds
\end{aligned}\tag{4.39}$$

Now, we select our parameters appropriately as follows:
First, we choose N_1 large enough so that

$$\alpha_1=\frac{N_1\chi}{4}-c_0-\frac{\mu}{4}>0.$$

Next, we select N_2 large enough so that

$$\alpha_2=\frac{c\,N_2}{2}-c_0-c_0N_1-\frac{2c_0}{\rho_1\,|\beta|}\left(1+\frac{8}{\mu\,\rho_1\,|\beta|}\right)>0.$$

We take N_3 large such that

$$m_1N_3-c_0-c_0N_1-\frac{\mu\,\pi^2}{4}\int_{\tau_1}^{\tau_2}|\gamma_2\,(\sigma)|\,d\sigma>0$$

Finally, we choose N large enough so that (4.37) remains valid, further

$$\alpha_3=N\,\tau-c_0N_1-1>0\quad,\quad\frac{N}{2}-\frac{c\,k\,(0)}{2\kappa_0^2}N_2>0$$

and

$$\alpha_4 = \left(\left(\gamma_1 - \int_{\tau_1}^{\tau_2} |\gamma_2(\sigma)| \, d\sigma \right) N + 1 \right) - 2\rho_1 - c_0 N_1 - \gamma_1 N_3 > 0.$$

Let $\alpha_5 = \frac{\mu}{4}$, $\alpha_6 = \alpha\, N_1$, $\alpha_7 = m_1\, N_3$, $\alpha_8 = \frac{2\kappa_0}{\beta^2} + c_0 N_2 \left(1 + N_2 + \frac{N_2}{\rho_1}\right)$

Ultimately, (4.39) turns out to be

$$\mathcal{L}'(t) \le -\omega \left[\int_0^{\pi} \left(u_t^2 + \varphi_t^2 + \theta^2 + u_x^2 + \varphi_x^2 + \varphi^2 \right) dx \right]$$
$$- \omega \int_0^{\pi} \int_0^{1} \int_{\tau_1}^{\tau_2} \sigma \, |\gamma_2(\sigma)| \, z^2(x, \rho, \sigma, t) \, d\sigma \, d\rho \, dx + \alpha_8 \int_0^{\infty} \kappa(s) \, \|\eta_x\|^2 ds$$

Meanwhile, by revisiting the energy functional (4.1) and utilizing Young's inequality we find (4.38) ■

Now, we can state and prove the following stability result

Theorem 4.10. *Assume that* (1.3) *holds and* κ *satisfies* $(H1)$ - $(H7)$. *Then system* (3.3)-(3.4) *is exponentially stable. In other words there exist two positive constants* υ_1 *and* υ_2 *such that*

$$E(t) \le \upsilon_2 \, e^{-\upsilon_1 t} \quad , \quad \forall t \ge 0 \tag{4.40}$$

Proof. Multiplying (4.1) by r, using $(H6)$, we end up with

$$\mathcal{Y}'(t) \le -r\,\varsigma_1 E(t) \quad , \quad \forall t \ge 0 \tag{4.41}$$

where$\mathcal{Y}(t) = r\mathcal{L}(t) + 2\varsigma_2 E(t)$.Using(4.37),it's readily follows,for some $a_0, b_0 > 0$

$$a_0 E(t) \le Y(t) \le b_0 E(t) \quad , \quad \forall t \ge 0 \tag{4.42}$$

Consequently, inequality (4.41) becomes

$$Y'(t) \le -\,\upsilon_1 Y(t) \quad , \quad \forall t \ge 0 \tag{4.43}$$

where $\upsilon_1 = \dfrac{r\,\varsigma_1}{b_0}$. A simple integration of (4.43) over $(0, t)$ induces

$$Y(t) \le Y(0) \, e^{-\,\upsilon_1 t} \quad , \quad \forall t \ge 0 \tag{4.44}$$

Accordingly, by merging (4.42) and (4.44), we get (4.40). which leads to the conclusion of our stability result. ■

5. Acknowledgement

The author is thankful to the referee for his valuable suggestions which improved the presentation of the paper.

References

[1] T.A. Apalara, A general decay for a weakly nonlinearly damped porous system, *J. Dyn. Control Syst.*, **25(3)**(2019), 311–322.

[2] T.A. Apalara and O.B. Almutairi, Well-Posedness and Exponential Stability of Swelling Porous with Gurtin Pipkin Thermoelasticity, *Mathematics*, **10(23)**(2022), 4498.

[3] P.S. CASAS AND R. QUINTANILLA, Exponential decay in one-dimensional porous-thermo-elasticity, *Mech. Res. Commun*, **32(6)**(2005), 652–658.

[4] F. DELL'ORO, Asymptotic stability of thermoelastic systems of Bresse type, *J. Differ. Equ*, **258(11)**(2015), 3902–3927.

[5] F. DELL'ORO, On the stability of Bresse and Timoshenko systems with hyperbolic heat conduction, *J. Differ. Equ*, **281(25)**(2021), 148–198.

[6] F. DELL'ORO AND V. PATA, On the stability of Timoshenko systems with Gurtin Pipkin thermal law, *J. Differ. Equ*, **257(2)**(2014), 523–548.

[7] A. FAREH, Exponential stability of a porous thermoelastic system with Gurtinâ Pipkin thermal law, *RACSAM*, **116(6)**(2022).

[8] A. FAREH, Exponential stability of a Timoshenko type thermoelastic system with Gurtin-Pipkin thermal law and frictional damping, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat*, **71**(2022), 95–115.

[9] A. FAREH AND S.A. MESSAOUDI, Energy decay for a porous thermoelastic system with thermoelasticity of second sound and with a non-necessary positive definite energy, *Appl. Math. Comput*, **293**(2017), 493–507.

[10] L.H. FATORI AND J.E. MUÑOZ RIVERA, Energy decay for hyperbolic thermoelastic systems of memory type, *Q.Appl. Math*, **59(3)**(2001), 441–458.

[11] L. H. FATORI, J.E. MUÑOZ RIVERA AND R. NUNES MONTEIRO, Energy decay to Timoshenko's system with thermoelasticity of type III, *Asympt. Anal*, **86**(2014), 227–247.

[12] H.D. FERNÁNDEZ SARE AND R. RACKE, On the stability of damped Timoshenko systems: Cattaneo versus Fourier's law, *Arch. Ration. Mech. Anal*, **194**(2009), 221–251.

[13] F. FOUGHALI, S. ZITOUNI, H. E. KHOCHEMANE AND A. DJEBABLA, Well-posedness and exponential decay for a porous-thermoelastic system with second sound and a distributed delay term, *Mathematics in Engineering, Science and Aerospace*, **11(4)**(2020), 1003–1020.

[14] A.E. GREEN AND P.M. NAGHDI, A re-examination of the basic postulates of thermomechanics, *Proc. R. Soc. Lond. A*, **432**(1991), 171–194.

[15] A.E. GREEN AND P.M. NAGHDI, Thermoelasticity without energy-dissipation, *J. Elast*, **31(3)**(1993), 189–208.

[16] M.E. GURTIN AND A.C. PIPKIN, A general theory of heat conduction with finite wave speeds, *Arch. Ration. Mech. Anal*, **31**(1968), 113–126.

[17] Z.J. HAN AND G.Q. XU, Exponential decay result in non-uniform porous-thermo-elasticity model of LordShulman type, *Discrete Contin. Dyn. Syst. B*, **17**(2012), 57–77.

[18] D. HANNI, B. FENG AND K. ZENNIR, Stability of Timoshenko system coupled with thermal law of Gurtin-Pipkin affecting on shear force, *Appl. Anal*, **101**(2022), 5171–5192.

[19] M. KAFINI, General energy decay in a Timoshenko-type system of thermoelasticity of type III with a viscoelastic damping, *J. Math. Anal. Appl*, **375(2)**(2011), 523–537.

[20] H. E. KHOCHEMANE, General stability result for a porous thermoelastic system with infinite history and microtemperatures effects, *Math Meth Appl Sci*, **45(3)**(2021), 1–20.

[21] M.C. Leseduarte, A. Magaña and R. Quintanilla, On the time decay of solutions in porous-thermo-elasticity of type II, *Discrete Contin. Dyn. Syst. B*, **13(2)**(2010), 375–391.

[22] H.W. Lord and Y. Shulman, A generalized dynamical theory of thermoelasticity, *J. Mech. Phys. Solids*, **15**(1967), 299–309.

[23] S. Loucif, R. Guefaifia, S. Zitouni and H. E. Khochemane, Global well-posedness and exponential decay of fully dynamic and electrostatic or quasi-static piezoelectric beams subject to a neutral delay, *Z. Angew. Math. Phys*, **74(3)**(2023).

[24] A. Magaña, and R. Quintanilla, On the time decay of solutions in one-dimensional theories of porous materials, *Int. J. Solids Struct*, **43**(2006), 3414–3427.

[25] S.A. Messaoudi and A. Fareh, Energy decay in a Timoshenko type system of thermoelasticity type III with different wave-propogation speeds, *Arab. J. Math*, **2**(2013), 199–207.

[26] S.A. Messaoudi and A. Fareh, Exponential decay for linear damped porous thermoelastic systems with second sound, *Discrete Contin. Dyn. Syst. B*, **20(2)**(2015), 599–612.

[27] S.A. Messaoudi and B. Said Houari, Energy decay in a Timoshenko-type system of thermoelasticity type III, *J. Math. Anal. Appl*, **348**(2008), 298–307.

[28] S.A. Messaoudi and B. Said-Houari, Energy decay in a Timoshenko-type system with history in thermoelasticity of type III, *Adv. Differ. Equ*, **14(3-4)**(2009), 375–400.

[29] A. Miranville and R. Quintanilla, Exponential decay in one-dimensional type III thermoelasticity with voids, *Appl. Math. Lett*, **94**(2019), 30–37.

[30] J.E. Muñoz Rivera and R. Quintanilla, On the time polynomial decay in elastic solids with voids, *J. Math. Anal. Appl*, **338**(2008), 1296–1309.

[31] J.E. Muñoz Rivera and R. Racke, Mildly dissipative nonlinear Timoshenko systems's global existence and exponential stability, *J. Math. Anal. Appl*, **276**(2002), 248–278.

[32] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM J. Contr. Optim*, **45(5)**(2006), 1561–1585.

[33] V. Pata and E. Vuk, On the exponential stability of linear thermoelasticity, *Contin. Mech. Thermodyn*, **12**(2000), 121–130.

[34] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, *Springer New York,* (1983).

[35] M.L. Santos, D.S. Almeida Júnior and J.E. Muñoz Rivera, The stability number of the Timoshenko system with second sound, *J. Differ. Equ*, **253**(2012), 2715–2733.

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **12(04)**(2024), 437–448.
http://doi.org/10.26637/mjm1204/006

Fractional Hermite-Hadamard type inequalities for co-ordinated convex functions

Badreddine Meftah[*1]

[1] *Laboratory of Analysis and Control of Differential Equations "ACED", Faculty MISM, Department of Mathematics, University of 8 May 1945 Guelma, P.O. Box 401, 24000 Guelma, Algeria.*

***R**eceived 12 April 2024; **A**ccepted 27 August 2024*

Abstract. In this paper, we first construct a new integral equality. Using this equality, we establish Hermite-Hadamard type fractional integral inequalities involving two variables via convexity.

AMS Subject Classifications: 26D51, 26D15, 26A20.

Keywords: Co-ordinated convex function, Hermite-Hadamard inequality, Hölder inequality.

Contents

1. Introduction

The Hermite-Hadamard integral inequality, which may be expressed as follows: for every convex function $\mathcal{S}$ on the finite interval $[k,l]$, we have

$$\mathcal{S}\left(\tfrac{k+l}{2}\right) \leq \tfrac{1}{l-k}\int\limits_{k}^{l}\mathcal{S}(x)dx \leq \tfrac{\mathcal{S}(k)+\mathcal{S}(l)}{2} \tag{1.1}$$

is one of the most well-known mathematical inequalities for convex functions. (1.1) holds in the opposite way if the function $\mathcal{S}$ is concave (see [15]).

Dragomir determined the bidimentionnal analogue of (1.1) provided by in [3].

$$\begin{aligned}\mathcal{S}\left(\tfrac{k+l}{2},\tfrac{u+v}{2}\right) \leq& \tfrac{1}{2}\left(\tfrac{1}{l-k}\int\limits_{k}^{l}\mathcal{S}(x,\tfrac{u+v}{2})dx+\tfrac{1}{v-u}\int\limits_{u}^{v}\mathcal{S}(\tfrac{k+l}{2},y)dy\right)\\ \leq& \tfrac{1}{(l-k)(v-u)}\int\limits_{k}^{l}\int\limits_{u}^{v}\mathcal{S}(x,y)dydx\end{aligned}$$

---

[*] **Corresponding author.** Email address: **badrimeftah@yahoo.fr** (B. Meftah)

$$\leq \frac{1}{4}\left(\frac{1}{l-k}\int_k^l \mathcal{S}(x,u)dx+\frac{1}{l-k}\int_k^l \mathcal{S}(x,v)dx\right.$$
$$\left.+\ \frac{1}{v-u}\int_u^v \mathcal{S}(k,y)dy+\frac{1}{v-u}\int_u^v \mathcal{S}(l,y)dy\right)$$
$$\leq \frac{\mathcal{S}(k,u)+\mathcal{S}(k,v)+\mathcal{S}(l,u)+\mathcal{S}(l,v)}{4}. \tag{1.2}$$

Numerous scholars have been drawn to the inequalities (1.2), and numerous generalizations, improvements, expansions, and modifications of (1.1) have been documented in the literature (see $[1,2,4,6,7,14,16-20]$ and the references therein).

The following results was provided by Sarikaya [16].

Theorem 1.1. *Let $\mathcal{S}:\Delta\to\mathbb{R}$ partially differentiable map on $\Delta=[k,l]\times[u,v]\subset\mathbb{R}^2$. If $\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}\right|$ is a co-ordinated convex function on Δ, then we have*

$$\left|\frac{\mathcal{S}(k,u)+\mathcal{S}(k,v)+\mathcal{S}(l,u)+\mathcal{S}(l,v)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(l-k)^\alpha(v-u)^\beta}\left(J^{\alpha,\beta}_{k^+,u^+}\mathcal{S}(l,v)+J^{\alpha,\beta}_{k^+,v^-}\mathcal{S}(l,u)\right.\right.$$
$$\left.\left.+\ J^{\alpha,\beta}_{l^-,u^+}\mathcal{S}(k,v)+J^{\alpha,\beta}_{l^-,v^-}\mathcal{S}(k,u)\right)-\mathfrak{A}\right|$$
$$\leq\frac{(l-k)(v-u)}{4(\alpha+1)(\beta+1)}\left(\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(k,u)\right|+\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(k,v)\right|+\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(l,u)\right|+\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(l,v)\right|\right),$$

where

$$\mathfrak{A}=\frac{\Gamma(\beta+1)}{4(v-u)^\beta}\left(J^{\beta}_{u^+}\mathcal{S}(k,v)+J^{\beta}_{u^+}\mathcal{S}(l,v)+J^{\beta}_{v^-}\mathcal{S}(k,u)+J^{\beta}_{v^-}\mathcal{S}(l,u)\right)$$
$$+\ \frac{\Gamma(\alpha+1)}{4(l-k)^\alpha}\left(J^{\alpha}_{k^+}\mathcal{S}(l,c)+J^{\alpha}_{k^+}\mathcal{S}(l,v)+J^{\alpha}_{l^-}\mathcal{S}(k,c)+J^{\alpha}_{l^-}\mathcal{S}(k,v)\right). \tag{1.3}$$

Theorem 1.2. *Under the assumptions of Theorem 1.1. If $\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}\right|^q$ is a co-ordinated convex function on Δ, then we have*

$$\left|\frac{\mathcal{S}(k,u)+\mathcal{S}(k,v)+\mathcal{S}(l,u)+\mathcal{S}(l,v)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(l-k)^\alpha(v-u)^\beta}\left(J^{\alpha,\beta}_{k^+,u^+}\mathcal{S}(l,v)+J^{\alpha,\beta}_{k^+,v^-}\mathcal{S}(l,u)\right.\right.$$
$$\left.\left.+\ J^{\alpha,\beta}_{l^-,u^+}\mathcal{S}(k,v)+J^{\alpha,\beta}_{l^-,v^-}\mathcal{S}(k,u)\right)-\mathfrak{A}\right|$$
$$\leq\frac{(l-k)(v-u)}{((\alpha p+1)(\beta p+1))^{\frac{1}{p}}}\left(\frac{\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(k,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(k,v)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(l,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial s\partial t}(l,v)\right|^q}{4}\right)^{\frac{1}{q}},$$

where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\mathfrak{A}$ is as in (1.3).

The aim of this work is to establish some integral inequalities of the Hermite-Hadamard type via convexity on co-ordinates by using fractional integral operators. The obtained results are based on a new integral equality.

2. Preliminaries

Here, we revisit few definitions. We also assume throughout that $\Delta\subset\mathbb{R}^2$ with $\Delta:=[k,l]\times[u,v]$ where $k<l$ and $u<v$.

Definition 2.1. *[6] Convexity on the co-ordinates on Δ is the property of a function $\mathcal{S}:\Delta\to\mathbb{R}$ that holds when the inequality*

$$\begin{aligned}\mathcal{S}\left(gx+(1-g)\,\xi,\lambda y+(1-\lambda)\,j\right)\leq& g\lambda\mathcal{S}(x,y)+g\left(1-\lambda\right)\mathcal{S}(x,j)\\&+(1-g)\,\lambda\mathcal{S}(\xi,y)+(1-g)\left(1-\lambda\right)\mathcal{S}(\xi,j)\end{aligned}$$

remains true for any $(x,y),(x,j),(\xi,y),(\xi,j)\in\Delta$ and $g,\lambda\in[0,1]$.

Definition 2.2. *[5] The Riemann-Liouville integrals $J_{k^+}^{\alpha}\mathcal{S}$ and $J_{l^-}^{\alpha}\mathcal{S}$ of order α are defined by:*

$$\begin{aligned}J_{k^+}^{\alpha}\mathcal{S}(\xi)=&\frac{1}{\Gamma(\alpha)}\int\limits_{k}^{\xi}(\xi-t)^{\alpha-1}\mathcal{S}(t)dt,\quad \xi>k,\\J_{l^-}^{\alpha}\mathcal{S}(\xi)=&\frac{1}{\Gamma(\alpha)}\int\limits_{\xi}^{l}(t-\xi)^{\alpha-1}\mathcal{S}(t)dt,\quad l>\xi,\end{aligned}$$

respectively, where $\alpha>0,k\geq0,\mathcal{S}\in L^1[k,l]$ and $\Gamma(\alpha)=\int\limits_0^{\infty}e^{-t}t^{\alpha-1}dt$, is the gamma function and $J_{k^+}^{0}\mathcal{S}(\xi)=J_{l^-}^{0}\mathcal{S}(\xi)=\mathcal{S}(\xi)$.

Definition 2.3. *[5] The Riemann-Liouville integrals $J_{k^+,u^+}^{\alpha,\beta}$, $J_{k^+,v^-}^{\alpha,\beta}$, $J_{l^-,u^+}^{\alpha,\beta}$, and $J_{l^-,v^-}^{\alpha,\beta}$ of order $\alpha,\beta>0$ with $k,u\geq0$, $k<l$ and $u<v$ are defined by*

$$J_{k^+,u^+}^{\alpha,\beta}\mathcal{S}\left(l,v\right)=\tfrac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_{k}^{l}\int\limits_{u}^{v}(l-\xi)^{\alpha-1}(v-y)^{\beta-1}\mathcal{S}\left(\xi,y\right)dyd\xi,\tag{2.1}$$

$$J_{k^+,v^-}^{\alpha,\beta}\mathcal{S}\left(l,u\right)=\tfrac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_{k}^{l}\int\limits_{u}^{v}(l-\xi)^{\alpha-1}(y-u)^{\beta-1}\mathcal{S}\left(\xi,y\right)dyd\xi\tag{2.2}$$

$$J_{l^-,u^+}^{\alpha,\beta}\mathcal{S}\left(k,v\right)=\tfrac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_{k}^{l}\int\limits_{u}^{v}(\xi-k)^{\alpha-1}(v-y)^{\beta-1}\mathcal{S}\left(\xi,y\right)dyd\xi,\tag{2.3}$$

$$J_{l^-,v^-}^{\alpha,\beta}\mathcal{S}\left(k,u\right)=\tfrac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_{k}^{l}\int\limits_{u}^{v}(\xi-k)^{\alpha-1}(y-u)^{\beta-1}\mathcal{S}\left(\xi,y\right)dyd\xi,\tag{2.4}$$

where $\mathcal{S}\in L^1(\Delta),\Gamma$ is the gamma function, and

$$J_{k^+,u^+}^{0,0}\mathcal{S}\left(l,v\right)=J_{k^+,v^-}^{0,0}\mathcal{S}\left(l,u\right)=J_{l^-,u^+}^{0,0}\mathcal{S}\left(k,v\right)=J_{l^-,v^-}^{0,0}\mathcal{S}\left(k,u\right)=\mathcal{S}\left(\xi,y\right).$$

Definition 2.4. *[16] The Riemann-Liouville integrals $J_{l^-}^{\alpha}\mathcal{S}\left(k,u\right)$, $J_{k^+}^{\alpha}\mathcal{S}\left(l,u\right)$, $J_{v^-}^{\beta}\mathcal{S}\left(k,u\right)$ and $J_{u^+}^{\alpha}\mathcal{S}\left(k,v\right)$ of order $\alpha,\beta>0$ with $k,u\geq0$, $k<l$ and $u<v$, are defined by*

$$J_{l^-}^{\alpha}\mathcal{S}\left(k,u\right)=\tfrac{1}{\Gamma(\alpha)}\int\limits_{k}^{l}(\xi-k)^{\alpha-1}\mathcal{S}\left(\xi,u\right)d\xi,\tag{2.5}$$

$$J_{k^+}^{\alpha}\mathcal{S}\left(l,u\right)=\tfrac{1}{\Gamma(\alpha)}\int\limits_{k}^{l}(l-\xi)^{\alpha-1}\mathcal{S}\left(\xi,u\right)d\xi,\tag{2.6}$$

$$J_{v^-}^{\beta}\mathcal{S}(k,u)=\tfrac{1}{\Gamma(\beta)}\int\limits_{c}^{v}(y-u)^{\beta-1}\mathcal{S}(k,y)\,dy, \tag{2.7}$$

$$J_{u^+}^{\alpha}\mathcal{S}(k,v)=\tfrac{1}{\Gamma(\beta)}\int\limits_{u}^{v}(v-y)^{\beta-1}\mathcal{S}(k,y)\,dy, \tag{2.8}$$

where $\mathcal{S}\in L^1(\Delta)$ and Γ represents the gamma function.

3. Main results

Lemma 3.1. *Assume that $\mathcal{S}:\Delta\to\mathbb{R}$ be a partially differentiable map. If $\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\in L(\Delta)$, then we have*

$$\begin{aligned}
&\mathcal{S}\left(\tfrac{k+l}{2},\tfrac{u+v}{2}\right)-\tfrac{\mathcal{S}\left(k,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},u\right)+\mathcal{S}\left(l,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},v\right)}{2}+\mathfrak{A}-\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(l-k)^{\alpha}(v-u)^{\beta}}\\
&\times\left(J_{k^+,u^+}^{\alpha,\beta}\mathcal{S}(l,v)+J_{l^-,u^+}^{\alpha,\beta}\mathcal{S}(k,v)+J_{k^+,v^-}^{\alpha,\beta}\mathcal{S}(l,u)+J_{l^-,v^-}^{\alpha,\beta}\mathcal{S}(k,u)\right)\\
=&\tfrac{(l-k)(v-u)}{4}\left(\int\limits_0^1\int\limits_0^1\mathcal{K}\mathcal{H}\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v)\,dF\,d\chi_2\right.\\
&-\int\limits_0^1\int\limits_0^1\left((1-\chi_1)^{\alpha}-\chi_1^{\alpha}\right)\left((1-\chi_2)^{\beta}-\chi_2^{\beta}\right)\\
&\times\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v)\,d\chi_1d\lambda,
\end{aligned}\tag{3.1}$$

where

$$\mathcal{K}=\left\{\begin{cases}1 & \text{if } 0\le\chi_1<\frac{1}{2},\\ -1 & \text{if } \frac{1}{2}\le\chi_1<1,\end{cases}\right. \tag{3.2}$$

$$\mathcal{H}=\left\{\begin{cases}1 & \text{if } 0\le\chi_2<\frac{1}{2},\\ -1 & \text{if } \frac{1}{2}\le\chi_2<1,\end{cases}\right. \tag{3.3}$$

$$\begin{aligned}
\mathfrak{A}=&\tfrac{\Gamma(\beta+1)}{4(v-u)^{\beta}}\left(J_{u^+}^{\beta}\mathcal{S}(k,v)+J_{u^+}^{\beta}\mathcal{S}(l,v)+J_{v^-}^{\beta}\mathcal{S}(k,u)+J_{v^-}^{\beta}\mathcal{S}(l,u)\right)\\
&+\tfrac{\Gamma(\alpha+1)}{4(l-k)^{\alpha}}\left(J_{k^+}^{\alpha}\mathcal{S}(l,u)+J_{k^+}^{\alpha}\mathcal{S}(l,v)+J_{l^-}^{\alpha}\mathcal{S}(k,u)+J_{l^-}^{\alpha}\mathcal{S}(k,v)\right).
\end{aligned}\tag{3.4}$$

Proof. Let

$$I=\tfrac{(l-k)(v-u)}{4}(I_1-I_2), \tag{3.5}$$

where

$$I_1=\int\limits_0^1\int\limits_0^1\mathcal{K}\mathcal{H}\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v)\,d\chi_1d\chi_2,$$

$$I_2=\int\limits_0^1\int\limits_0^1\left((1-\chi_1)^{\alpha}-\chi_1^{\alpha}\right)\left((1-\chi_2)^{\beta}-\chi_2^{\beta}\right)\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v)\,d\chi_1d\chi_2.$$

Clearly, we have

$$I_1=\int\limits_0^{\frac{1}{2}}\int\limits_0^{\frac{1}{2}}\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1k+(1-\chi_1)l,\chi_2u+(1-\chi_2)v)\,d\chi_1d\chi_2$$

$$
\begin{aligned}
&-\int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1k+\left(1-\chi_1\right)l,\chi_2u+\left(1-\chi_2\right)v\right)d\chi_1d\chi_2\\
&-\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1k+\left(1-\chi_1\right)l,\chi_2u+\left(1-\chi_2\right)v\right)d\chi_1d\chi_2\\
&+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1k+\left(1-\chi_1\right)l,\chi_2u+\left(1-\chi_2\right)v\right)d\chi_1d\chi_2\\
=&\tfrac{1}{(l-k)(v-u)}\left(\left(\mathcal{S}\left(\tfrac{k+l}{2},\tfrac{u+v}{2}\right)-\mathcal{S}\left(l,\tfrac{u+v}{2}\right)-\mathcal{S}\left(\tfrac{k+l}{2},v\right)+\mathcal{S}\left(l,v\right)\right)\right.\\
&-\mathcal{S}\left(k,\tfrac{u+v}{2}\right)+\mathcal{S}\left(\tfrac{k+l}{2},\tfrac{u+v}{2}\right)+\mathcal{S}\left(k,v\right)-\mathcal{S}\left(\tfrac{k+l}{2},v\right)\\
&-\mathcal{S}\left(\tfrac{k+l}{2},u\right)+\mathcal{S}\left(l,u\right)+\mathcal{S}\left(\tfrac{k+l}{2},\tfrac{u+v}{2}\right)-\mathcal{S}\left(l,\tfrac{u+v}{2}\right)\\
&\left.+\mathcal{S}\left(k,u\right)-\mathcal{S}\left(\tfrac{k+l}{2},u\right)-\mathcal{S}\left(k,\tfrac{u+v}{2}\right)+\mathcal{S}\left(\tfrac{k+l}{2},\tfrac{u+v}{2}\right)\right)\\
=&\tfrac{4}{(l-k)(v-u)}\left(\left(\mathcal{S}\left(\tfrac{k+l}{2},\tfrac{u+v}{2}\right)\right)+\tfrac{\mathcal{S}(l,v)+\mathcal{S}(k,v)+\mathcal{S}(l,u)+\mathcal{S}(k,u)}{4}\right.\\
&\left.-\tfrac{\mathcal{S}\left(k,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},u\right)+\mathcal{S}\left(l,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},v\right)}{2}\right).
\end{aligned}
\tag{3.6}
$$

Using the integration by parts, I_2 gives

$$
\begin{aligned}
I_2=&\int_{0}^{1}\left(\left(1-\chi_2\right)^{\beta}-\chi_2^{\beta}\right) \qquad (3.7)\\
&\times\left(\int_{0}^{1}\left(\left(1-\chi_1\right)^{\alpha}-\chi_1^{\alpha}\right)\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1k+\left(1-\chi_1\right)l,\chi_2u+\left(1-\chi_2\right)v\right)d\chi_1\right)d\chi_2\\
=&\tfrac{1}{(l-k)(v-u)}\left(\mathcal{S}\left(k,u\right)+\mathcal{S}\left(k,v\right)+\mathcal{S}\left(l,u\right)+\mathcal{S}\left(l,v\right)\right)\\
&-\tfrac{\beta}{(l-k)(v-u)}\left(\int_{0}^{1}\left(1-\chi_2\right)^{\beta-1}\mathcal{S}\left(k,\chi_2u+\left(1-\chi_2\right)v\right)d\chi_2\right.\\
&+\int_{0}^{1}\chi_2^{\beta-1}\mathcal{S}\left(k,\chi_2u+\left(1-\chi_2\right)v\right)d\chi_2+\int_{0}^{1}\chi_2^{\beta-1}\mathcal{S}\left(l,\chi_2u+\left(1-\chi_2\right)v\right)d\chi_2\\
&\left.+\int_{0}^{1}\left(1-\chi_2\right)^{\beta-1}\mathcal{S}\left(l,\chi_2u+\left(1-\chi_2\right)v\right)d\chi_2\right)\\
&-\tfrac{\alpha}{(l-k)(v-u)}\left(\int_{0}^{1}\left(1-\chi_1\right)^{\alpha-1}\mathcal{S}\left(\chi_1k+\left(1-\chi_1\right)l,u\right)d\chi_1\right.\\
&+\int_{0}^{1}\chi_1^{\alpha-1}\mathcal{S}\left(\chi_1k+\left(1-\chi_1\right)l,u\right)d\chi_1+\int_{0}^{1}\chi_1^{\alpha-1}\mathcal{S}\left(\chi_1k+\left(1-\chi_1\right)l,v\right)d\chi_1\\
&\left.+\int_{0}^{1}\left(1-\chi_1\right)^{\alpha-1}\mathcal{S}\left(\chi_1k+\left(1-\chi_1\right)l,v\right)d\chi_1\right)
\end{aligned}
$$

$$+\frac{\alpha\beta}{(l-k)(v-u)}\left(\int_0^1\int_0^1 \chi_1^{\alpha-1}\chi_2^{\beta-1}\mathcal{S}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)d\chi_2 d\chi_1\right.$$
$$+\int_0^1\int_0^1 (1-\chi_1)^{\alpha-1}\chi_2^{\beta-1}\mathcal{S}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)d\chi_2 d\chi_1$$
$$+\int_0^1\int_0^1 \chi_1^{\alpha-1}(1-\chi_2)^{\beta-1}\mathcal{S}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)d\chi_2 d\chi_1$$
$$\left.+\int_0^1\int_0^1 (1-\chi_1)^{\alpha-1}(1-\chi_2)^{\beta-1}\mathcal{S}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)d\chi_2 d\chi_1\right).$$

Combining (3.5)-(3.7) and making a changes $\xi=\chi_1 k+(1-\chi_1)l$ and $y=\chi_2 u+(1-\chi_2)v$, we get

$$\begin{aligned}I=&\mathcal{S}\left(\tfrac{k+l}{2},\tfrac{u+b}{2}\right)-\tfrac{\mathcal{S}\left(k,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},u\right)+\mathcal{S}\left(l,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},v\right)}{2}\\
&+\tfrac{\beta}{4(v-u)^\beta}\left(\int_u^v (y-u)^{\beta-1}\mathcal{S}(k,y)\,dy+\int_u^v (y-u)^{\beta-1}\mathcal{S}(l,y)\,dy\right.\\
&\left.+\int_u^v (v-y)^{\beta-1}\mathcal{S}(k,y)\,dy+\int_u^v (v-y)^{\beta-1}\mathcal{S}(l,y)\,dy\right)\\
&+\tfrac{\alpha}{4(l-k)^\alpha}\left(\int_k^l (\xi-k)^{\alpha-1}\mathcal{S}(\xi,u)\,d\xi+\int_k^l (\xi-k)^{\alpha-1}\mathcal{S}(\xi,v)\,d\xi\right.\\
&\left.+\int_k^l (l-\xi)^{\alpha-1}\mathcal{S}(\xi,u)\,dx+\int_k^l (l-\xi)^{\alpha-1}\mathcal{S}(\xi,v)\,d\xi\right)\\
&-\tfrac{\alpha\beta}{4(l-k)^\alpha(v-u)^\beta}\left(\int_k^l\int_u^v (l-\xi)^{\alpha-1}(v-y)^{\beta-1}\mathcal{S}(\xi,y)\,dyd\xi\right.\\
&+\int_k^l\int_u^v (\xi-k)^{\alpha-1}(v-y)^{\beta-1}\mathcal{S}(\xi,y)\,dyd\xi\\
&+\int_k^l\int_u^v (l-\xi)^{\alpha-1}(y-u)^{\beta-1}\mathcal{S}(\xi,y)\,dyd\xi\\
&\left.+\int_k^l\int_u^v (\xi-k)^{\alpha-1}(y-u)^{\beta-1}\mathcal{S}(\xi,y)\,dyd\xi\right).\end{aligned}\tag{3.9}$$

(3.8)

The proof is thus finished. ■

In what follows, we note

$$\begin{aligned}&\Lambda(k,l,u,v,\mathcal{S})\\
=&\mathcal{S}\left(\tfrac{k+l}{2},\tfrac{u+v}{2}\right)-\tfrac{\mathcal{S}\left(k,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},u\right)+\mathcal{S}\left(l,\frac{u+v}{2}\right)+\mathcal{S}\left(\frac{k+l}{2},v\right)}{2}+\mathfrak{A}-\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(l-k)^\alpha(v-u)^\beta}\end{aligned}$$

$$\times\left(J^{\alpha,\beta}_{k^+,u^+}\mathcal{S}\left(l,v\right)+J^{\alpha,\beta}_{l^-,u^+}\mathcal{S}\left(k,v\right)+J^{\alpha,\beta}_{k^+,v^-}\mathcal{S}\left(l,u\right)+J^{\alpha,\beta}_{l^-,v^-}\mathcal{S}\left(k,u\right)\right),$$

where $\mathfrak{A}$ is given by (3.4).

Theorem 3.2. *For a partial differentiable map $\mathcal{S}:\Delta\to\mathbb{R}$ whose $\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\right|$ is co-ordinated convex, we have*

$$\begin{aligned}\left|\Lambda\left(k,l,u,v,\mathcal{S}\right)\right|\le&\tfrac{(l-k)(v-u)}{4}\left(\tfrac{1}{4}+\tfrac{1}{(\alpha+1)(\beta+1)}\right)\\&\times\left(\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|+\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|+\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|+\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|\right).\end{aligned}\tag{3.10}$$

Proof. Using the absolute value on both sides of (3.1), we get

$$\begin{aligned}&\left|\Lambda\left(k,l,u,v,\mathcal{S}\right)\right|\\\le&\tfrac{(l-k)(v-u)}{4}\left(\int_0^1\int_0^1\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1k+\left(1-\chi_1\right)l,\chi_2u+\left(1-\chi_2\right)v\right)\right|d\chi_1d\chi_2\right.\\&+\int_0^1\int_0^1\left(1-\chi_1\right)^{\alpha}\left(1-\chi_2\right)^{\beta}\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1k+\left(1-\chi_1\right)l,\chi_2u+\left(1-\chi_2\right)v\right)\right|d\chi_1d\chi_2\\&+\int_0^1\int_0^1\chi_1^{\alpha}\left(1-\chi_2\right)^{\beta}\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1k+\left(1-\chi_1\right)l,\chi_2u+\left(1-\chi_2\right)v\right)\right|d\chi_1d\chi_2\\&+\int_0^1\int_0^1\left(1-\chi_1\right)^{\alpha}\chi_2^{\beta}\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1k+\left(1-\chi_1\right)l,\chi_2u+\left(1-\chi_2\right)v\right)\right|d\chi_1d\chi_2\\&+\int_0^1\int_0^1\chi_1^{\alpha}\chi_2^{\beta}\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1k+\left(1-\chi_1\right)l,\chi_2u+\left(1-\chi_2\right)v\right)\right|d\chi_1d\chi_2.\end{aligned}\tag{3.11}$$

Since $\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\right|$ is co-ordinated convex, (3.11) gives

$$\begin{aligned}&\left|\Lambda\left(k,l,u,v,\mathcal{S}\right)\right|\\\le&\tfrac{(l-k)(v-u)}{4}\left(\int_0^1\int_0^1\left(\chi_1\chi_2\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|+\chi_1\left(1-\chi_2\right)\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|\right.\right.\\&+\left(1-\chi_1\right)\chi_2\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|+\left(1-\chi_1\right)\left(1-\chi_2\right)\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|\right)d\chi_1d\chi_2\\&+\int_0^1\int_0^1\left(1-\chi_1\right)^{\alpha}\left(1-\chi_2\right)^{\beta}\left(\chi_1\chi_2\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|+\chi_1\left(1-\chi_2\right)\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|\right.\\&+\left(1-\chi_1\right)\chi_2\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|+\left(1-\chi_1\right)\left(1-\chi_2\right)\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|\right)d\chi_1d\chi_2\\&+\int_0^1\int_0^1\chi_1^{\alpha}\left(1-\chi_2\right)^{\beta}\left(\chi_1\chi_2\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|+\chi_1\left(1-\chi_2\right)\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|\right.\\&+\left(1-\chi_1\right)\chi_2\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|+\left(1-\chi_1\right)\left(1-\chi_2\right)\left|\tfrac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|\right)d\chi_1d\chi_2\end{aligned}$$

$$+\int_0^1\int_0^1 (1-\chi_1)^\alpha \chi_2^\beta \left(\chi_1\chi_2 \left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right| + \chi_1(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|\right.$$
$$\left.+ (1-\chi_1)\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right| + (1-\chi_1)(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|\right) d\chi_1 d\chi_2$$
$$+\int_0^1\int_0^1 \chi_1^\alpha \chi_2^\beta \left(\chi_1\chi_2 \left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right| + \chi_1(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|\right.$$
$$\left.\left.+ (1-\chi_1)\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right| + (1-\chi_1)(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|\right) d\chi_1 d\chi_2\right)$$
$$=\frac{(l-k)(v-u)}{4}\left(\frac{1}{4}+\frac{1}{(\alpha+1)(\beta+1)}\right)$$
$$\times\left(\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right| + \left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right| + \left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right| + \left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|\right),$$

which is the desired outcome. ■

Theorem 3.3. *Suppose that all the assumptions of Theorem 3.2 hold. If* $\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}\right|^q$ *is co-ordinated convex function, then we have*

$$|\Lambda(k,l,u,v,\mathcal{S})|$$
$$\leq\frac{(l-k)(v-u)}{4^{1+\frac{1}{p}}}\left(\left(\frac{\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+9\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q}{64}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q+9\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q}{64}\right)^{\frac{1}{q}}$$
$$+\left(\frac{3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+9\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q}{64}\right)^{\frac{1}{q}}$$
$$+\left(\frac{9\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q}{64}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{4^{1+\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\right)\left(\frac{\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q}{4}\right)^{\frac{1}{q}}\right),$$

where $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $\mathfrak{A}$ *is given by* (3.4).

Proof. Using the absolute value on both sides of (3.1) and then applying Hölder's inequality, it yields

$$|\Lambda(k,l,u,v,\mathcal{S})|$$
$$\leq\frac{(l-k)(v-u)}{4}\left(\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} d\chi_1 d\chi_2\right)^{\frac{1}{p}}\right.$$
$$\times\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$
$$+\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 d\chi_1 d\chi_2\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}d\chi_1 d\chi_2\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}\left|\frac{\partial^2 \mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}d\chi_1 d\chi_2\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left|\frac{\partial^2 \mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\left(\int_0^1\int_0^1 d\chi_1 d\chi_2\right)^{\frac{1}{p}}+\left(\int_0^1\int_0^1 (1-\chi_1)^{\alpha p}(1-\chi_2)^{\beta p}d\chi_1 d\chi_2\right)^{\frac{1}{p}}\right.$$

$$+\left(\int_0^1\int_0^1 \chi_1^{\alpha p}(1-\chi_2)^{\beta p}d\chi_1 d\chi_2\right)^{\frac{1}{p}}+\left(\int_0^1\int_0^1 (1-\chi_1)^{\alpha p}\chi_2^{\beta p}d\chi_1 d\chi_2\right)^{\frac{1}{p}}$$

$$\left.+\left(\int_0^1\int_0^1 \chi_1^{\alpha p}\chi_2^{\beta p}d\chi_1 d\chi_2\right)^{\frac{1}{p}}\right)$$

$$\left.\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 \mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}\right)$$

$$=\frac{(l-k)(v-u)}{4^{1+\frac{1}{p}}}\left(\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\frac{\partial^2 \mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}\left|\frac{\partial^2 \mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}\left|\frac{\partial^2 \mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left|\frac{\partial^2 \mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\frac{4^{1+\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\right)$$

$$\left.\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 \mathcal{S}}{\partial\chi_1\partial\chi_2}\left(\chi_1 k+(1-\chi_1)l,\chi_2 u+(1-\chi_2)v\right)\right|^q d\chi_1 d\chi_2\right)^{\frac{1}{q}}\right). \tag{3.12}$$

Now, using the convexity of $\left|\frac{\partial^2 \mathcal{S}}{\partial\chi_1\partial\chi_2}\right|^q$, (3.12) gives

$$|\Lambda(k,l,u,v,\mathcal{S})|$$

$$\leq \frac{(l-k)(v-u)}{4^{1+\frac{1}{p}}}\left(\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(\chi_1\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+\chi_1(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q\right.\right.\right.$$

$$\left.\left.+(1-\chi_1)\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+(1-\chi_1)(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q\right)d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left(\chi_1\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+\chi_1(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q\right.\right.$$

$$\left.\left.+(1-\chi_1)\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+(1-\chi_1)(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q\right)d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left(\chi_1\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+\chi_1(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q\right.\right.$$

$$\left.\left.+(1-\chi_1)\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+(1-\chi_1)(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q\right)d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1\left(\chi_1\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+\chi_1(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q\right.\right.$$

$$\left.\left.+(1-\chi_1)\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+(1-\chi_1)(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q\right)d\chi_1 d\chi_2\right)^{\frac{1}{q}}$$

$$+\left(\frac{4^{1+\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\right)\left(\int_0^1\int_0^1\left(\chi_1\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+\chi_1(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q\right.\right.$$

$$\left.\left.\left.+(1-\chi_1)\chi_2\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+(1-\chi_1)(1-\chi_2)\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q\right)d\chi_1 d\chi_2\right)^{\frac{1}{q}}\right)$$

$$=\frac{(l-k)(v-u)}{4^{1+\frac{1}{p}}}\left(\left(\frac{\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+9\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q}{64}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q+9\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q}{64}\right)^{\frac{1}{q}}$$

$$+\left(\frac{3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+9\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q}{64}\right)^{\frac{1}{q}}$$

$$+\left(\frac{9\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q+3\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q}{64}\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{4^{1+\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\right)\left(\frac{\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(k,v)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,u)\right|^q+\left|\frac{\partial^2\mathcal{S}}{\partial\chi_1\partial\chi_2}(l,v)\right|^q}{4}\right)^{\frac{1}{q}}\right).$$

The proof is over. ■

References

[1] A. Alomari and M. Darus, The Hadamard's inequality for s-convex function of 2-variables on the co-ordinates, *Int. J. Math. Anal. (Ruse)*, 2 (2008), no. 13-16, 629–638.

[2] A. Alomari and M. Darus, On the Hadamard's inequality for $\log$-convex functions on the coordinates *J. Inequal. Appl.,* 2009, Art. ID 283147 13 pp.

[3] S. S. Dragomir, On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwanese J. Math.,* 5 (2001), no. 4, 775–788.

[4] K.-C. Hsu, Refinements of Hermite-Hadamard type inequalities for differentiable co-ordinated convex functions and applications. *Taiwanese J. Math.,* 19 (2015), no. 1, 133–157.

[5] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[6] M. A. Latif and M. Alomari, Hadamard-type inequalities for product two convex functions on the co-ordinates, *Int. Math. Forum*, 4 (2009), no. 45-48, 2327–2338.

[7] M. A. Latif and S. S. Dragomir, On some new inequalities for differentiable co-ordinated convex functions, *J. Inequal. Appl.,* 2012, 2012:28, 13 pp.

[8] B. Meftah, Some new Ostrwoski's inequalities for functions whose nth derivatives are r-convex, *Int. J. Anal.,* 2016, Art. ID 6749213, 7 pp.

[9] B. Meftah, Ostrowski inequalities for functions whose first derivatives are logarithmically preinvex, *Chin. J. Math. (N.Y.)*, 2016, Art. ID 5292603, 10 pp.

[10] B. Meftah, Ostrowski inequality for functions whose first derivatives are s-preinvex in the second sense, *Khayyam J. Math.,* 3 (2017), no. 1, 61-80.

[11] B. Meftah, Some new Ostrowski's inequalities for n-times differentiable mappings which are quasi-convex, *Facta Univ. Ser. Math. Inform*, 32 (2017), no. 3, 319-327.

[12] B. Meftah, New Ostrowski's inequalities, *Rev. Colombiana Mat.,* 51 (2017), no. 1, 57-69.

[13] B. Meftah, Fractional Hermite-Hadamard type integral inequalities for functions whose modulus of derivatives are co-ordinated $\log$-preinvex, *Punjab Univ. J. Math.*, 51 (2019), no. 2, 21-37.

[14] M. E. Özdemir, H. Kavurmaci, A. O. Akdemir and M. Avci, Inequalities for convex and s-convex functions on $\Delta = [a,b] \times [c,d]$, *J. Inequal. Appl.,* 2012, 2012:20, 19 pp.

[15] J. Pečarić, F. Proschan, Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.

[16] M. Z. Sarikaya, On the Hermite-Hadamard-type inequalities for co-ordinated convex function via fractional integrals, *Integral Transforms Spec. Funct.*, 25 (2014), no. 2, 134–147.

[17] M. Z. Sarikaya, E.Set, M. E. Özdemir and S. S. Dragomir, New some Hadamard's type inequalities for co-ordinated convex functions, *Tamsui Oxf. J. Inf. Math. Sci.,* 28 (2012), no. 2, 137–152.

[18] K.-L. Tseng and K.-C. Hsu, New fractional integral inequalities for differentiable convex functions and their applications; translated from Ukraïn. Mat. Zh. 69 (2017), no. 3, 407–425 *Ukrainian Math. J.,* 69 (2017), no. 3, 478–499.

[19] B.-Y. Xi, J. Hua and F. Qi, Hermite-Hadamard type inequalities for extended s-convex functions on the co-ordinates in a rectangle, *J. Appl. Anal.,* 20 (2014), no. 1, 29–39.

[20] H. Yaldiz, M. Z. Sarikaya and Z. Dahmani, On the Hermite-Hadamard-Fejer-type inequalities for co-ordinated convex functions via fractional integrals, *Int. J. Optim. Control. Theor. Appl.,* 7 (2017), no. 2, 205–215.

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **12(04)**(2024), 449–456.
http://doi.org/10.26637/mjm1204/007

On β-γ-connectedness and $\beta_{(\gamma,\delta)}$-continuous functions

Sanjay Tahiliani[*1]

[1] *Lecturer, N.K. Bagrodia Public School, Sector 9, Rohini, Delhi 110085, India.*

***R**eceived 16 May 2024; **A**ccepted 25 September 2024*

Abstract. The purpose of this work is to present the idea of β-γ-separated sets, examine their characteristics in topological spaces and then define the notation for β-γ-connected and β-γ-disconnectedness. In addition, the study of topological qualities that involves for β-γ-connected spaces via β-γ-separated sets. An analysis is conducted on the properties of β-γ-connected spaces and how they behave under $\beta_{(\gamma,\delta)}$-continuous functions. We also provide the ideas of β-γ-components in a space X and β-γ-locally connected spaces.

AMS Subject Classifications: 54C08.

Keywords: operation, β-γ-open, β-γ-closed, β-γ-closure, β-γ-connected and $\beta_{(\gamma,\delta)}$-continuous functions.

Contents

1. Introduction

One of the most significant, practical and basic notations in general topology and other high level mathematical discipline now a days is connectedness. The notation of connectedness is fruitful in computing, topology, algebraic topology and advanced calculus. Many researchers across the globe have investigated properties of connectedness ([2], [3], [4], [5], [6]) and obtained new and interesting results.

The idea of β-open set in topological spaces was first proposed by M.E. Abd El-Monsef, S.N. El-Deeb and R.A. Mahmoud in 1983. Their proof was that the set of all β-open sets in (X,τ) is finer topology on X then τ. The researchers worked on two related topologies that were tested on the same underlying structure to determine if they share the same topological properties. The basic properties of β-connectedness were obtained by Jafari and Noiri [7] in 2003. Several other forms of connectedness can be introduced and studied using it. Tahiliani [8] discussed and studied the characterisations of β-γ-open sets in topological spaces in 2011. This work presents and investigates an additional kind of connectivity that is defined on β-open sets in (X,τ) via operations. Their behavior under is $\beta_{(\gamma,\delta)}$-continuous,as well as their attributes are discussed in this study.

The procedures γ and δ are defined on the set of all β-open sets of topological spaces (X,τ) and (Y,σ) correspondingly during the conversion.For any subset A of X, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ stands for the closure and interior of A, respectively, for any subset A of X.

[*]**Corresponding author.** Email addresses: **sanjaytahiliani@yahoo.com** (Sanjay Tahiliani)

2. Preliminaries

Here we lay down the groundwork by defining key terms and showing key findings:

The condition that $A \subseteq \mathrm{Cl}(\mathrm{Int}(Cl(A)))$ merely indicates that subset A of topological space X is β-open [1]. A β-open sets counterpart is a β-closed set, and $\beta O(X)$ [1] is the collection of all β-open sets. $\beta\,\mathrm{Cl}(A)$ [2] is the symbol for intersection of all β-closed sets that include A, while $\beta\,\mathrm{Int}(A))$ [2] is the symbol for union of all β-open sets that contain A.

The condition $V \in V^{\gamma}$ satisfied for each $V \in \beta O(X)$ in an operation $\gamma : \beta O(X) \to P(X)$.The function $V^{id} = V$ for each set $V \in \beta O(X)$ is called the identity operation on $\beta O(X)$.

As γ and δ are always defined on the family of β-open sets in space, We always mean them as operations. From [8], we retrieve the following definitions and findings:

Definition 2.1. *(i): If there exists a β-open set U of X that contains x and $U^{\gamma} \subseteq A$, then for any point $x \in A$, a subset A of X is termed as of β-γ-open set. The β-γ-closed is counterpart of β-γ-open set. The set symbolized by $\beta O(X)\gamma$ includes all β-γ-open sets of (X, τ).*

(ii): $\beta_{\gamma}\,\mathrm{Cl}(A)$ notation represents β-γ-closure of A, which is the intersection of all β-γ-closed sets set containing A. The $\beta\gamma\,\mathrm{Int}(A)$ notation represents β-γ-interior of A, which is the union of all β-γ-open set included in A. The β-γ-boundary of a set A is represented by $\beta\gamma Bd(A)$ and is defined by $(\beta_{\gamma}\,\mathrm{Cl}(A) - \beta_{\gamma}\,\mathrm{Int}(A))$.

(iii): If, for every element x in X and each β-δ-open set V that contains $f(x)$, there exists a β-γ-open set U such that $x \in U$ and $f(U) \subseteq V$, then we say that $f : (X, \tau) \to (Y, \sigma)$ is $\beta_{(\gamma,\delta)}$-continuous.

(iv): For any β-γ-closed set A of (X, τ), the set $f(A)$ is β-δ-closed in (Y, σ) we say that mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be $\beta_{(\gamma,\delta)}$-closed.

(v): For any β-γ-open set A of (X, τ), the set $f(A)$ is β-δ-open in (Y, σ) we say that mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be $\beta_{(\gamma,\delta)}$-open.

Theorem 2.2. *Suppose X be a subset of a topological space and A is a subset of it. Then*

(i) $x \in \beta\gamma\,\mathrm{Cl}(A)$ *if and only if every β_{γ}-open set U containing x has non empty intersection with A.*

(ii) $\beta_{\gamma}\,\mathrm{Cl}(X - A) = X - \beta_{\gamma}\,\mathrm{Int}(A)$.

3. β-γ-connected spaces

Definition 3.1. *(i): If $(\beta\,\mathrm{Cl}(A) \cap B) \cup (A \cap (\beta\,\mathrm{Cl}(B)) = \emptyset$, then the subsets A and B of a topological space (X, τ) are said to be β-separated.*

(ii): The term "β-γ-separated" is used to describe a pair of subsets A and B of a topological space (X, τ), where

$$(\beta_{\gamma}\,\mathrm{Cl}(A) \cap B) \cup (A \cap (\beta_{\gamma}\,\mathrm{Cl}(B)) = \emptyset.$$

Remark 3.2. *Each two β-γ-separated sets are always disjoint, since $A \cap B \subseteq A \cap \beta_{\gamma}\,\mathrm{Cl}(B) = \emptyset$. The converse may not hold in general.*

Example 3.3. *The set $X = \{a, b, c\}$, and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ are defined as follows: $A^{\gamma} = A$ if $b \in A$, $A^{\gamma} = \mathrm{Cl}(A)$ if $b \notin A$, then $\{a, b\}$ and $\{c\}$ are disjoint subsets of X which are not β-γ-separated.*

Given that $\beta\,\mathrm{Cl}(A) \subseteq \beta_{\gamma}\,\mathrm{Cl}(A)$, for all subsets A of X, it follows that every β-γ-separated set is β-separated. The preceding example, however suggests that reverse may not be true. Both $\{a\}$ and $\{b, c\}$ are β-separated in this case,but they are not β-γ-separated.

Theorem 3.4. *The following claims hold if A and B are two non empty subsets of space X*

(1) *If A and B are β-γ-separated and $A_1 \subseteq A$ and $B_1 \subseteq B$, then A_1 and B_1 are also β-γ-separated.*

(2) *If A and B are disjoint and are both β-γ-closed (both β-γ-open), then A and B are β-γ-separated.*

(3) *If A and B are both β-γ-closed (both β-γ-open) then $H = A \cap (X - B)$ and $G = B \cap (X - A)$ are β-γ-separated.*

Proof. 1. Since $A_1 \subseteq A$ implies $\beta_\gamma \operatorname{Cl}(A_1) \subseteq \beta_\gamma \operatorname{Cl}(A)$ for every pair of A and A_1, $\beta_\gamma \operatorname{Cl}(A) \cap B = \emptyset$ and $\beta_\gamma \operatorname{Cl}(B) \cap A = \emptyset$ implies $\beta_\gamma \operatorname{Cl}(A_1) \cap B_1 = \emptyset$ and $\beta_\gamma \operatorname{Cl}(B_1) \cap A_1 = \emptyset$. Hence A_1 and B_1 are β-γ-separated.

2. The equations $A = \beta_\gamma \operatorname{Cl}(A)$ and $B = \beta_\gamma \operatorname{Cl}(B)$ hold if A and B are both β-γ-closed. Hence because $A \cap B = \emptyset$, it follows that $\beta_\gamma \operatorname{Cl}(A) \cap B = \emptyset$ and $\beta_\gamma \operatorname{Cl}(B) \cap A = \emptyset$, A and B are β-γ-separated. In other words, the complement of disjoint β-γ-open sets A and B are also β-γ-closed sets. Specifically X-A and X-B are β-γ-separated. If A and B are disjoint and are both, then their complements are disjoint and β-γ-closed. Furthermore, $A \subseteq \beta_\gamma \operatorname{Cl}(A) \subseteq \beta_\gamma \operatorname{Cl}(X - B) = X - B$ and $B \subseteq \beta_\gamma \operatorname{Cl}(B) \subseteq X - A$. Hence by given part (1), A and B are β-γ-separated.

3. Since A and B are β-γ-open, it follows that $X - A$ and $X - B$ are β-γ-closed. Also, $H \subseteq X - B$ means that $\beta_\gamma \operatorname{Cl}(H) \subseteq \beta_\gamma \operatorname{Cl}(X - B)$. Then because $\beta_\gamma \operatorname{Cl}(H) \cap B = \emptyset$ and it follows that $\beta_\gamma \operatorname{Cl}(H) \cap G = \emptyset$. Similarly if $H \cap \beta_\gamma \operatorname{Cl}(G) = \emptyset$. i.e. H and G are β-γ-separated. $(X - A)$ and $(X - B)$ are β-γ-open if and only if A and B are β-γ-closed. Consequently, H and G are β-γ-separated. ■

Theorem 3.5. *If there is a set U and set V in $\beta O(X)\gamma$ such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \emptyset$ and $B \cap U = \emptyset$, then the subsets A and B of a space X are β-γ-separated and conversely.*

Proof. We have $\beta_\gamma \operatorname{Cl}(A) \cap B = \emptyset$ and $\beta_\gamma \operatorname{Cl}(B) \cap A = \emptyset$ as A and B are β-γ-separated sets. Therefore the sets $V = X - \beta_\gamma \operatorname{Cl}(A)$ and $U = X - \beta_\gamma \operatorname{Cl}(B)$ are β-γ-open, such that $A \subseteq U$, $B \subseteq V$ with $A \cap V = \emptyset$ and $B \cap U = \emptyset$. On the other hand, if U and V exists in $\beta O(X)\gamma$ such that $A \subseteq U$, $B \subseteq V$, $A \cap V = \emptyset$ and $B \cap U = \emptyset$, then $X - V$ and $X - U$ are β-γ-closed and $\beta_\gamma \operatorname{Cl}(A) \subseteq X - V \subseteq X - B$ and $\beta_\gamma \operatorname{Cl}(B) \subseteq X - U \subseteq X - A$ respectively. Hence $\beta_\gamma \operatorname{Cl}(A) \cap B = \emptyset$ and $\beta_\gamma \operatorname{Cl}(B) \cap A = \emptyset$ were determined. ■

Theorem 3.6. *In any topological space (X, τ), the following statements are equivalent:*

(1) *$\emptyset$ and X are the only sets which are both β-γ-open and β-γ-closed in X.*

(2) *X is not the union of two disjoint non empty β-γ-open sets.*

(3) *X is not the union of two disjoint non empty β-γ-closed sets.*

(4) *X is not the union of non empty β-γ-separated sets.*

Proof. (1)⇒(2): It is assumed that (2) is not true. Given that A and B are disjoint, non empty and are β-γ-open so let $X = A \cup B$. So $X - A = B$ is a nonempty set which is proper β-γ-open. It follows that (1) is not true, since A is non empty proper β-γ-open and β-γ-closed in X.

(2)⇒(3): Clear.

(3)⇒(4): If (4) is false, then $X = A \cup B$, where A and B are nonempty and β-γ-separated sets. Then $\beta\gamma \operatorname{Cl}(B) \cap A = \emptyset$ implies $\beta_\gamma \operatorname{Cl}(B) \subseteq B$ and hence B is β-γ-closed. Similarly A is also β-γ-closed. i.e. (3) is false.

(4)⇒(1). Assuming that (1) is not true, assume that there is a non empty proper subset A of X, that is both β-γ-open and β-γ-closed. If A and B are β-γ-separated and $X = A \cup B$, then (4) is not true since. $B = X - A$ is non empty, β-γ-open and β-γ-closed. ■

Definition 3.7. *The condition that a subset C of a space X is β-γ-disconnected is that $C = A \cup B$, where A and B are non empty β-γ-separated or that C is β-γ-connected if there exists no non empty β-γ-separated sets A and B of X such that $C = A \cup B$.*
A pair of sets A and B is referred to as a β-γ-disconnection of C if C is β-γ-disconnected.

In Example 3.3, X is β-γ-disconnected, since $\{c\}$ and $\{a, b\}$ are β-γ-separated sets and hence there union is X.

Example 3.8. (i) *Assume X is a set comprising $\{a, b, c\}$ and $\tau = \{\theta, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Let γ be an operation on $\beta O(X)$ such that $A^\gamma = A$ if $c \in A$ and $A^\gamma = \mathrm{Cl}(A)$ if $c \notin A$. Then X is β-γ-disconnected.*

(ii) *Assume X is a set comprising $\{a, b, c\}$ and $\tau = \{\theta, X, \{a\}, \{c\}, \{a, c\}\}$. Let γ be an operation on $\beta O(X)$ such that $A^\gamma = A$ if $b \in A$, $A^\gamma = X$, if $b \notin A$. So X is β-γ-connected, because there is no non empty pair A, B of non empty β-γ-separated subsets of X such that $X = A \cup B$.*

Remark 3.9. (1) *Every indiscrete space is β-γ-connected.*

(2) *Every discrete space with more than one point is β_{id}-disconnected.*

(3) *A space X is β-γ-connected if any of the conditions (1) to (4) in Theorem 3.6 holds.*

(4) *A space X is β-γ-disconnected if $X = A \cup B$, satisfies any one of the following statements:*

(i) *A and B are disjoint, non-empty and β-γ-open sets.*

(ii) *A and B are disjoint, non-empty and β-γ-closed sets.*

(iii) *A and B are disjoint, non-empty and β-γ-separated sets.*

Theorem 3.10. *If there is non empty proper subset A of X which is both β-γ-open and β-γ-closed in X, then we say that space X is β-γ-disconnected.*

Proof. Follows from above remarks. ■

Theorem 3.11. *Every non empty proper subset of X must have a non-empty β-γ-boundary for a space X to be β-γ-connected.*

Proof. Let A be nonempty proper subset of X with $\beta_\gamma \mathrm{Bd}(A) = \emptyset$. Then $\beta_\gamma \mathrm{Cl}(A) = \beta_\gamma \mathrm{Int}(A) \cup \beta_\gamma \mathrm{Bd}(A)$ implies $\beta_\gamma \mathrm{Cl}(A) = \beta_\gamma \mathrm{Int}(A)$. Because A is both β-γ-open and β-γ-closed and $\beta_\gamma \mathrm{Int}(A) \subseteq A$ is nonempty proper subset of X, by Theorem 3.10, X is β-γ-disconnected, which is a contradictory. Due to this, A has a non-empty β-γ-boundary. On the other hand, let X be β-γ-disconnected. Next, by Theorem 3.10, X contain a valid subset A that is non empty proper subset and is both β-γ-open and β-γ-closed. i.e. $\beta_\gamma \mathrm{Cl}(A) = A$, $\beta_\gamma \mathrm{Cl}(X - A) = X - A$ and $\beta_\gamma \mathrm{Cl}(A) \cap \beta_\gamma \mathrm{Cl}(X - A) = \emptyset$. So A has empty β-γ-boundary, which is again a contradiction. Hence X is β-γ-connected. ■

Lemma 3.12. *Suppose M and N are β-γ-separated subsets of X. If $C \subseteq M \cup N$ and C is β-γ-connected, then $C \subseteq M$ or $C \subseteq N$.*

Proof. Since $C \cap M \subseteq M$ and $C \cap N \subseteq N$ then $C \cap M$ and $C \cap N$ are β-γ-separated sets. Also $C = C \cap (M \cup N) = (C \cap M) \cup (C \cap N)$. Since C is β-γ-connected, so $(C \cap M)$ and $(C \cap N)$ cannot form a β-γ-disconnection of C. Therefore, either $C \cap M = \emptyset$, so $C \subseteq N$ or $C \cap M = \emptyset$ so $C \subseteq M$. ■

Theorem 3.13. *Suppose C and C_i $(i \in I)$ are β-γ-connected but not β-γ-separated subsets of X, then $S = C \cup C_i$ is β-γ-connected for each i.*

Proof. Where M and N are β-γ-separated, then $C \cup C_i$ is equal to $S = M \cup N$ if S is β-γ-disconnected. Either $C \subseteq M$ or $C \subseteq N$ and $C_i \subseteq M$ or $C_i \subseteq N$ are required by Lemma 3.12. Assume $C \subseteq M$ without sacrificing generality. A contradiction would occur if for some i, $C_i \subseteq N$, and C and C_i would be β-γ-separated. Therefore every $C_i \subseteq M$. Therefore $N = \emptyset$. i.e. M and N are not β-γ-disconnected in S. ■

Corollary 3.14. *Assume that, C_i is β-γ-connected subset of X for every $i \in I$, and if C_i, share a point, then $C_i \cup \{C_i : i \in I\}$ is β-γ-connected.*

Proof. With $I = \emptyset$, the set $\cup C_i = \emptyset$ is obviously β-γ-connected for all i in I. In Theorem 3.13, choose $i_0 \in I$ and C_{i0} be the central set C. If I is not equal to $\emptyset$. It is not true that $C_i \cap C_{i0}$ equal to $\emptyset$ for every $i \in I$. So C_i and C_{i0} are not β-γ-separated. The β-γ-connectedness of $\cup\{C_i : i \in I\}$ is shown by Theorem 3.13. ■

Corollary 3.15. *Suppose that for all $x, y \in X$, there exists a β-γ-connected set $C_{xy} \subseteq X$ with $x, y \in C_{xy}$. Then X is β-γ-connected.*

Proof. Obviously $X = \emptyset$ is β-γ-connected. By hypothesis, there exists a β-γ-connected set C_{ay} that contains both a and y for any $y \in X$ where $X \neq \emptyset$, and let $a \in X$ be a fixed element. The β-γ-connection of $X = \cup\{C_{ay} : y \in X\}$ is established by Corollary 3.14. ■

Corollary 3.16. *Let C be a β-γ-connected subset of X and $A \subseteq X$. If $C \subseteq A \subseteq \beta_\gamma \operatorname{Cl}(C)$, then A is also β-γ-connected.*

Proof. If $a \in \beta_\gamma \operatorname{Cl}(C)$ is true for all $a \in A$, then $\{a\} \cap \beta_\gamma \operatorname{Cl}(C)$ is not equal to $\emptyset$. C and $\{a\}$ are not β-γ-separated. Thus, $A = C \cup \cup\{\{a\} : a \in A\}$ is β-γ-connected by Theorem 3.13. ■

Remark 3.17. *In particular, the β-γ-closure of a β-γ-connected set is β-γ-connected.*

Corollary 3.18. *If for every β-δ-open set V of Y, $f^{-1}(V)$ is β-γ-open in X, then function $f : X \to Y$ is $\beta_{(\gamma,\delta)}$-continuous.*

Proof. Assume that V be β-δ-open in Y. Then $Y - V$ is a set in Y that is β-δ-closed. Following the reasoning in ([8, Theorem 16(ii)]), the set $f^{-1}(Y - V)$ is β-γ-closed set in X. The reason for this is because $f^{-1}(V)$ is β-γ-open set in X, since $f^{-1}(Y - V) = X - f^{-1}(V)$.

On the other side, consider $x \in X$ and V as a β-δ-open subset of Y that contains $f(x)$. Then $x \in f^{-1}(V)$. Given x and $f(f^{-1}(V)) \subseteq V$. It may be inferred that $f^{-1}(V)$ is β-γ-open in X. Hence f is $\beta_{(\gamma,\delta)}$-continuous. ■

Theorem 3.19. *If $f : (X, \tau) \to (Y, \sigma)$ is onto $\beta_{(\gamma,\delta)}$-continuous function and X is β-γ-connected, then Y is β-δ-connected.*

Proof. Y is β-δ-disconnected if and only if A and B give a β-δ-disconnection of Y. A and B are both β-δ-open sets according to Remark 3.9. Both $f^{-1}(A)$ and $f^{-1}(B)$ are both non empty β-γ-open set in X because f is $\beta_{(\gamma,\delta)}$-continuous, according to Corollary 3.18. Now, for function f, $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X$. Remark 3.9 states that $f^{-1}(A)$ and $f^{-1}(B)$ are two β-γ-disconnections of X. Then Y is β-δ-disconnected is contradicted by this. ■

Theorem 3.20. *Let $f : (X, \tau) \to (Y, \sigma)$ be an injective function. Then the following are equivalent:*

(i) *f is $\beta_{(\gamma,\delta)}$-continuous.*

(ii) *$f^{-1}(V) \subseteq \beta_\gamma \operatorname{Int}(f^{-1}(V))$ for every subset β-γ-open set V of Y.*

(iii) *$f(\beta_\gamma \operatorname{Cl}(A)) \subseteq \beta_\delta \operatorname{Cl}(f(A))$ for every subset A of X.*

(iv) *$\beta_\gamma \operatorname{Cl}(f^{-1}(B)) \subseteq f^{-1}(\beta_\delta \operatorname{Cl}(B))$ for every subset B of Y.*

(v) $f^{-1}(\beta_\delta \operatorname{Int}(B)) \subseteq \beta_\gamma \operatorname{Int}(f^{-1}(B))$ *for every subset B of Y.*

Proof. (i)⇒(ii): Let $x \in f^{-1}(V)$, where V is a β-δ-open subset of Y. Then $f(x) \in V$. Since f is $\beta_{(\gamma,\delta)}$-continuous, there exists β-γ-open set U of X containing x such that $f(U) \subseteq V$ and so $U \subseteq f^{-1}(V)$, this implies that $x \in \beta_\gamma \operatorname{Int}(f^{-1}(V))$. Thus $f^{-1}(V) \subseteq \beta_\gamma \operatorname{Int}(f^{-1}(V))$ for every β-δ-open subset V of Y.

(ii)⇒(iii): Let A be any subset of X and $f(x) \notin \beta_\delta \operatorname{Cl}(f(A))$, then by Theorem 2.2(i), there exists a β-γ-open set V of Y containing $f(x)$ such that $V \cap f(A) = \emptyset$ and hence $f^{-1}(V) \cap A = \emptyset$. Also $f(x) \in V$ implies $x \in f^{-1}(V)$, which implies $x \in \beta_\gamma \operatorname{Int}(f^{-1}(V))$. Hence, there exists a β-γ-open set U of X containing x such that $U \subseteq f^{-1}(V)$. Then $U \cap A = \emptyset$ and so $x \notin \beta_\gamma \operatorname{Cl}(A)$ and hence $f(x) \notin (\beta_\gamma \operatorname{Cl}(A))$. Thus $f(\beta_\gamma \operatorname{Cl}(A)) \subseteq \beta_\delta \operatorname{Cl}(f(A))$.

(iii)⇒(iv): Let B be any subset of Y. Since $f(f^{-1}(B)) \subseteq B$, so we have $\beta_\delta \operatorname{Cl}(f(f^{-1}(B)) \subseteq \beta_\delta \operatorname{Cl}(B)$. Also $f^{-1}(B) \subseteq X$. Then by (iii), we have $f(\beta_\gamma \operatorname{Cl}(f^{-1}(B)) \subseteq (\beta_\delta \operatorname{Cl} f(f^{-1}(B)) \subseteq \beta_\delta \operatorname{Cl}(B)$. Thus $\beta_\gamma \operatorname{Cl}(f^{-1}(B)) \subseteq f^{-1}(\beta_\delta \operatorname{Cl}(B))$.

(iv)⇒(v): Let B be any subset of Y and $x \in f^{-1}(\beta_\delta \operatorname{Int}(B))$. Then by Theorem 2.2(ii), $x \notin X - f^{-1}(\beta_\gamma \operatorname{Int}(B)) = f^{-1}(\beta_\gamma \operatorname{Cl}(Y - B)$. By (iv), $x \notin (\beta_\gamma \operatorname{Cl}(f^{-1}(Y - B)) = X - (\beta_\gamma \operatorname{Int}(f^{-1}(B))$ and hence $x \in \beta_\gamma \operatorname{Int} f^{-1}(B)$. Thus $f^{-1}(\beta_\delta \operatorname{Int}(B)) \subseteq \beta_\gamma \operatorname{Int}(f^{-1}(B))$.

(v)⇒(i): Let $x \in X$ and V be any β-δ-open set of Y containing $f(x)$. Since $V \cap (Y - V) = \emptyset$, we have $f(x) \notin \beta_\gamma \operatorname{Cl}(Y - V) = Y - \beta_\gamma \operatorname{Int}(V))$ and hence $f(x) \notin \beta_\gamma \operatorname{Cl}(Y - B) = Y - \beta_\gamma \operatorname{Int}(V)$ and so $f(x) \in \beta_\gamma \operatorname{Int}(f^{-1}(V))$, which implies that $x \in f^{-1}(\beta_\delta \operatorname{Int}(v))$. By (v), we obtain that $x \in \beta_\gamma(\operatorname{Int} f^{-1}(V))$. This means that there exists a β-γ-open set U of X containing x such that $U \subseteq f^{-1}(V)$ and so $f(U) \subseteq V$. This shows that f is $\beta_{(\gamma,\delta)}$-continuous. ∎

Corollary 3.21. *Let $f : X \to Y$ be a $\beta_{(\gamma,\delta)}$-continuous and injective function. If K is β-γ-connected in X, then $f(K)$ is β-δ-connected in Y.*

Proof. Suppose that $f(K)$ is β-δ-disconnected in Y. Then there exists two β-δ-separated sets P and Q of Y such that $f(K) = P \cup Q$. Let $A = K \cap f^{-1}(P)$ and $B = K \cap f^{-1}(Q)$. Since $f(K) \cap P$ is not empty, so is $K \cap f^{-1}(P)$. Hence A and B are non empty. Now $A \cup B = (K \cap f^{-1}(P)) \cup (K \cap f^{-1}(Q)) = K \cap (f^{-1}(P) \cup f^{-1}(Q)) = K \cap (f^{-1}(P \cup Q)) = K \cap (f^{-1}(f(K)) = K$. Since f is $\beta_{(\gamma,\delta)}$-continuous, then by Theorem 3.20, $\beta_\gamma \operatorname{Cl}(f^{-1}(Q)) \subseteq f^{-1}(\beta_\delta \operatorname{Cl}(Q))$ and this together with $B \subseteq f^{-1}(Q)$, implies $\beta_\delta \operatorname{Cl}(B) \subseteq f^{-1}(\beta_\gamma \operatorname{Cl}(Q))$. Since $P \cap \beta_\gamma \operatorname{Cl}(Q) = \emptyset$, $A \cap \beta_\gamma \operatorname{Cl}(B) \subseteq A \cap f^{-1}(\beta_\gamma \operatorname{Cl}(Q)) \subseteq f^{-1}(P) \cap f^{-1}(\beta_\gamma \operatorname{Cl}(Q)) = \emptyset$. i.e. $A \cap \beta_\gamma \operatorname{Cl}(B) = \emptyset$. Similarly $B \cap \beta_\gamma \operatorname{Cl}(A) = \emptyset$. Thus A and B are β-γ-separated, therefore K is a β-γ-disconnected, a contradiction. Hence $f(K)$ is β-δ-connected. ∎

Theorem 3.22. *A space X is β-γ-disconnected if and only if there exists an $\beta_{(\gamma,\mathrm{id})}$-continuous function from X onto discrete space $\{0,1\}$.*

Proof. Suppose that X is β-γ-disconnected. Then, there exists disjoint β-γ-open sets G_1 and G_2 of X such that $X = G_1 \cup G_2$. Define a function $f : X \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in G_1, \\ 1 & \text{if } x \in G_2. \end{cases}$$

Now,the only β_{id}-open sets in $\{0,1\}$ are $\emptyset, \{0\}, \{1\}, \{0,1\}$. So, $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{0\}) = G_1$, $f^{-1}(\{1\}) = G_2$ and $f^{-1}(\{0,1\}) = X$, which are β-γ-open sets in X. Thus by Corollary 3.18, f is $\beta_{(\gamma,\mathrm{id})}$-continuous function from X onto discrete space $\{0,1\}$. Conversely, let the hypothesis holds and if possible suppose that X is β-γ-connected. Therefore by Theorem 3.19, $\{0,1\}$ is β_{id}-connected which is a contradiction by Remark 3.9. So X must be β-γ-disconnected. ∎

Theorem 3.23. *A space X is β-γ-connected if and only if every $\beta_{(\gamma,\mathrm{id})}$-continuous function from space X to the discrete space $\{0,1\}$ is constant.*

Proof. Consider X be β-γ-connected and consider any $\beta_{(\gamma,\mathrm{id})}$-continuous function $f : X \to \{0,1\}$. Since the space $\{0,1\}$ is discrete, we may say that $\{y\}$ is both β_{id}-open and β_{id}-closed in space $\{0,1\}$. If we let $y \in f(X) \subseteq \{0,1\}$, then $\{y\} \subseteq \{0,1\}$. For any y in Y, $f^{-1}(\{y\})$ is both β-γ-open and β-γ-closed in X according to Corollary 3.18 and ([8, Theorem 16(ii)]) since f is $\beta_{(\gamma,\mathrm{id})}$-continuous function. We may deduce that $f(x) = y$ for every $x \in X$ because $y \in f(X)$, so x is a function of $f^{-1}(\{y\})$. Therefore $f^{-1}(\{y\})$ does not include empty set. If $f^{-1}(\{y\})$ is not equal to X, then $f^{-1}(\{y\})$ is a non empty subset of X which is both β-γ-open and β-γ-closed in X. So there is a contradiction as, X is β-γ-connected. By Theorem 3.10. Therefore if $f^{-1}(\{y\}) = X$, then $f(X) = \{y\}$. This indicates that f is constant since for each $x \in X$, $f(x) = y$. ∎

Definition 3.24. *A set C is called maximal β-γ-connected set if it is β-γ-connected and if D is β-γ-connected such that $C \subseteq D \subseteq X$, then $C = D$. A maximal β-γ-connected subset C of a space X is called a β-γ-component of X, if X itself β-γ-connected, then X is only β-γ-component of X.*

Theorem 3.25. *For β-γ-component of X containing x, for each $x \in X$, there is exactly one β-γ-component of X containing x.*

Proof. For any $x \in X$, let $C_x = \cup\{A : x \in A \subseteq X$ and A is β-γ-connected$\}$. Then $\{x\} \in C_x$, since C_x is union of β-γ-connected sets each containing x, is β-γ-connected by Corollary 3.14. If $C_x \subseteq D$ and D is β-γ-connected, then D was one of the sets A in the collection whose union defined C_x. So $D \subseteq C_x$ and therefore $C_x = D$. Therefore C_x is a β-γ-component of X containing x. ∎

Corollary 3.26. *Two β-γ-components either are disjoint or coincide.*

Proof. Let C_x and C_y be two β-γ-components and C_x not equal to C_y. If they are not disjoint, let $p \in C_x \cap C_y$. Then by Corollary 3.14, $C_x \cup C_y$ would be a β-γ-connected set strictly larger then C_x. Therefore $C_x \cap C_y = \emptyset$. ∎

Theorem 3.27. *Each β-γ-connected subset of X is contained in exactly one β-γ-component of X.*

Proof. Let A be a β-γ-connected subset of X which is not in exactly one β-γ-component of X. Suppose that C_1 and C_2 are β-γ-component of X such that, $A \subseteq C_1$ and $A \subseteq C_2$. Since C_1 and C_2 are not disjoint and by Corollary 3.14, $C_1 \cup C_2$ is another β-γ-connected subset which contain C_1 and C_2, a contradiction to the fact that C_1 and C_2, are β-γ-components. This proves that A is contained in exactly one β-γ-component of X. ∎

Theorem 3.28. *A β-γ-component is a non empty β-γ-connected subset of X that is both β-γ-open and β-γ-closed.*

Proof. Assume that A be a β-γ-connected subset of X which is both β-γ-open and β-γ-closed. A is included in precisely one β-γ-component C of X, according to Theorem 3.27. It is contradictory because if A is proper subset of C, then equation $C = (C \cap A) \cup (C \cap (X - A))$ results in a β-γ-disconnection of C. Thus, $A = C$. ∎

Theorem 3.29. *Every β-γ-component of X is β-γ-closed.*

Proof. Assume that C be a β-γ-component of X. according to Remark 3.17, $\beta_\gamma\,\mathrm{Cl}(C)$ is a β-γ-connected which appropriately includes the β-γ-component C of X. C is therefore β-γ-closed as $C = \beta_\gamma\,\mathrm{Cl}(C)$. ∎

Definition 3.30. *For every point $x \in X$ and every β-γ-open set U containing x, there exists a β-γ-open β-γ-connected set V such that $x \in V \subseteq U$, we say that X is said to be β-γ-locally connected at x.*

Theorem 3.31. *Let $f : X \to Y$ be a $\beta_{(\gamma,\delta)}$-continuous, $\beta_{(\gamma,\delta)}$-open and bijective. If X is β-γ-locally connected, then Y is β-δ-locally connected.*

Proof. By $y \in Y$, find an element $x \in X$ such that y is equal to $f(x)$. Let U be a β-δ-open set of Y that contains y. According to Corollary 3.18, $f^{-1}(U)$ is β-γ-open in X containing x, because f is $\beta_{(\gamma,\delta)}$-continuous. There is a β-γ-open β-γ-connected set V that contains x such that $x \in V \subseteq f^{-1}(U)$ because X is β-γ-locally connected. This means that $f(x) \in f(V) \subseteq f(f^{-1}(U)) = U$ or $y \in f(V) \subseteq U$. The reason for $f(V)$ is also β-δ-open because f is $\beta_{(\gamma,\delta)}$-open. In addition according to Corollary 3.21, $f(V)$ is β-δ-connected. This establishes that Y is β-δ-locally connected. ■

4. Concluding Remarks and Acknowledgements

Our research in this study focused on β-γ-connected and β-γ-locally connected spaces and we also presented the concept of β-γ-separated sets. There is much scope of further work based on operational approach and variants of open sets. The authors would like to express their profound gratitude to the referees who helped us to enhance the paper quality and the findings.

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, β-open sets and β-continuous mappings, *Bull. Fac. Sci. Assuit. Univ.*, **12** (1983), 77–90.

[2] M. E. Abd El-Monsef, R. A. Mahmoud and E. R. Lashin, β-closure and β-interior, *J. Fac. Ed. Ain Shams. Univ.*, **10** (1986), 235–245.

[3] A. v. Arhangelskii and R. Wiegandt, Connectedness and disconnectedness in topology, *Top. App.*, **5** (1975).

[4] J. A. Guthrie, D. P. Reynolds and H. E. Stone, Connected expansions of topologies, *Bull. Austral. Math. Soc.*, **9** (1973), 259–265.

[5] J. A. Guthrie and H. E. Stone, Spaces whose connected expansion preserve connected subsets, *Fund. Math.*, **80**(1) (1973), 91–100.

[6] J. A. Guthrie, H. E. Stone and M. L. Wage, Maximal connected expansion of the reals, *Proc. Amer. Math. Soc.*, **69**(1) (1978), 159–165.

[7] S. Jafari and T. Noiri, Properties of β-connected spaces, *Acta. Math. Hungar.*, **101**(3) (2003), 227–235.

[8] S. Tahiliani, Operational approach on β-open sets and applications, *Math. Commun.*, **16**(2011), 577–591.

MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **12(04)**(2024), 457–463.
http://doi.org/10.26637/mjm1204/001

On generalized pseudo conformally symmetric manifolds

Akshoy Patra *[1]

[1] *Government College of Enggineering and Textile Technology, Berhampore, 742101, Dist. Murshidabad, West Bengal, India.*

***R*eceived 02 June 2024; *A*ccepted 04 August 2024**

Abstract. In this paper, a type of Riemannian manifold, namely generalized pseudo conformally symmetric manifold is studied. Several geometric properties of such spaces are studied. By imposing different restrictions on the conformal curvature tensor, we have obtained several properties. If the conformal curvature tensor is harmonic, then the form of the scalar curvature is obtained. Also, the relations among the 1-forms under various conditions are obtained.

AMS Subject Classifications: 53C20, 53C21, 53C44.

Keywords: Pseudo symmetry, Second Bianchi Identity, Conformal curvature tensor, Harmonic curvature tensor.

Contents

1. Introduction

The geometry of a space mainly depends on the curvature of the space. One of the most important geometric properties of a space is symmetry. Cartan began the study of local symmetry of Riemannian spaces and studied elaborately ([2], [3]). According to him, a Riemannian manifold is said to be locally symmetric if $\nabla R = 0$. During the last sixty years, the notion of locally symmetric manifolds has been generalized by many authors in a weaker sense. They have weakened in different directions with several defining conditions by giving some curvature restrictions. Various weaker symmetries are studied as generalizations or extensions of Cartan's notion, such as recurrent manifolds by Walker [18], semi-symmetric manifolds by Szabó ([17]), pseudosymmetric manifolds in the sense of Deszcz [7], pseudosymmetric manifolds in the sense of Chaki [4], generalized pseudosymmetric manifolds by Chaki [6], weakly symmetric manifolds by Selberg [11] and weakly symmetric manifolds by Támassy and Binh [16].
According to Chaki, a Riemannian manifold is said to be pseudo symmetric if

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= 2\alpha(X)R(Y,Z,U,V)\\ &+ \alpha(Y)R(X,Z,U,V) + \alpha(Z)R(Y,X,U,V)\\ &+ \alpha(U)R(Y,Z,X,V) + \alpha(V)R(Y,Z,U,X)\end{aligned} \tag{1.1}$$

where and α is a 1-form, $X, Y, Z, U, V \in \chi(M)$.

*Corresponding author. Email address: **akshoyp@gmail.com** (Akshoy Patra)

Pseudo symmetric manifolds are studied by many authors ([5] , [6], [12]). Ishii [8], introduced the notion of conharmonic transformation under which a harmonic function transforms into a harmonic function. $\overline{C}$, the conharmonic curvature tensor of type (0,4) on an (M^n, g) is defined as follows

$$\overline{C} = R - \frac{1}{n-2} g \wedge S, \tag{1.2}$$

which remains invariant under conharmonic transformation where R and S are the Riemannian curvature and Ricci curvature tensor respectively. $g \wedge S$ is the Kulkarni-Nomizu product [14].

In [13], Shaikh and Hui showed that the conharmonic curvature tensor satisfies the symmetric and skew-symmetric properties of the Riemannian curvature tensor as well as cyclic ones. They also studied it ellaborately [15]. The conharmonic curvature tensor has many applications in the theory of general relativity.
The conformal curvature tensor of type(0,4) is defined by

$$C_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{jk}r_{il} - g_{ik}r_{jl} + g_{il}r_{jk} - g_{jl}r_{ik}) + \frac{s}{(n-1)(n-2)}(g_{il}g_{jk} - g_{ik}g_{jl}) \tag{1.3}$$

It should be noted that the conformal curvature tensor C_{ijkl} remains invariant under conformal transformation.

In 2021, Ali, Khan and Vasiulla [1] introduced generalized pseudo symmetric manifold and studied various properties. Also in 2017, Kim introduced pseudo semiconformally symmetric manifolds [10] and studied various properties. According to him, a Riemannian manifold (M^n, g) is said to be pseudo semiconformally symmetric if

$$P_{ijkl;m} = 2A_m P_{ijkl} + A_i P_{mjkl} + A_j P_{imkl} + A_k P_{ijml} + A_l P_{ijkm}, \tag{1.4}$$

where P is the semiconformal curvature tensor [9]and A is a non zero 1-form.
Motivating by the above studies in this paper, I would like to introduce generalized pseudo conformally symmetric manifold, which is defined by

$$\begin{aligned}(\nabla_X C)(Y, Z, U, V) &= 2\alpha(X)C(Y, Z, U, V) \\ &+ \beta(Y)C(X, Z, U, V) + \gamma(Z)C(Y, X, U, V) \\ &+ \delta(U)C(Y, Z, X, V) + \eta(V)C(Y, Z, U, X)\end{aligned} \tag{1.5}$$

where and α, β, γ, δ, η are 1-forms.
In terms of local coordinates

$$C_{ijkl;m} = 2\alpha_m C_{ijkl} + \beta_i C_{mjkl} + \gamma_j C_{imkl} + \delta_k C_{ijml} + \eta_l C_{ijkm} \tag{1.6}$$

2. Generalized pseudo conformally symmetric manifolds

Definition 2.1. *The conformal curvature tensor is said to be harmonic if the divergence of the curvature tensor C^i_{jkl} of type (1,3) vanishes, i.e.,*

$$C^h_{jkl;h} = 0. \tag{2.1}$$

By virtue of second Bianchhi Identity we have

$$R^h_{jkl;h} = r_{jk;l} - r_{jl;k}. \tag{2.2}$$

And then

$$r^k_{l;k} = \frac{1}{2}s;l. \tag{2.3}$$

We have

$$C^h_{jkl;h} = [(r_{jk;l} - r_{jl;k}) - \frac{1}{2(n-2)}(g_{jk}s_{;l} - g_{jl}s_{;k})]. \tag{2.4}$$

Multiplying (2.4) by g^{jk} and using the condition (**??**) we get

$$0 = \frac{s_{;l}}{n-2}. \tag{2.5}$$

Which on siplification gives $s_{;l} = 0$, that is the scalar curvature is constant. Hence we have the following:

Theorem 2.2. *If the confrmal curvature tensor of a generalized pseudo confrmally symmetric Riemannian manifold is harmonic, then the scalar curvature of the space is constant.*

Let the confrmal curvature tensor of a generalized pseudo confrmally symmetric Riemannian manifold is harmonic. Then we have

$$0 = 2\alpha_m C^m_{jkl} + \beta^m C_{mjkl} + \gamma_j C^m_{mkl} + \delta_k C^m_{mjl} + \eta_l C^m_{jkm}. \tag{2.6}$$

Multiplying the above equation by g^{jk} we have

$$\frac{s}{n-2}[2\alpha_l + \beta_l - \delta_l + n\eta_l] = 0. \tag{2.7}$$

If the scalar curvature does not vanishes, then we have

$$2\alpha_l + \beta_l - \delta_l + n\eta_l = 0. \tag{2.8}$$

Thus we can state the following:

Theorem 2.3. *Let the confrmal curvature tensor of a generalized pseudo confrmally symmetric Riemannian manifold is harmonic. If $2\alpha + \beta - \delta + n\eta \neq 0$, then the scalar curvature of the space vanishes.*

If the space is pseudo confrmally symmetric Riemannian manifold then $\alpha = \beta = \gamma = \delta = \eta = A$ then we have:

Corollary 2.4. *If the confrmal curvature tensor of a pseudo confrmally symmetric Riemannian manifold is harmonic, then the scalar curvature of the space vanishes.*

Definition 2.5. *A Riemannian manifold (M^n, g) is said to be recurrent if its curvature tensor R_{ijkl} of type (0,4) satisfies the condition*

$$R_{ijkl;m} = B_m R_{ijkl} \tag{2.9}$$

where the 1-form B is non zero.

Multiplying the equation (2.9) by g^{il} and then multiplying by g^{jk} we get

$$r_{jk;m} = B_m r_{jk} \tag{2.10}$$

and then

$$s_{;m} = B_m s \tag{2.11}$$

Using (2.10), (2.11) and (2.1) we get

$$g^{il}g^{jk}C_{ijkl;m} = -\frac{n}{n-2}B_m s. \tag{2.12}$$

From (1.6)

$$g^{il}g^{jk}C_{ijkl;m} = -\frac{s}{n-2}[2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m]. \tag{2.13}$$

If $[a+(n-2)b] \neq 0$, then from the above two equations we get

$$B_m = \frac{2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m}{n} \tag{2.14}$$

Thus we can state that:

Theorem 2.6. *If a generalized pseudo conformally symmetric Riemannian manifold is recurrent, then the 1-forms B, α, β, γ, δ, η satisfy the relation $B = \frac{2n\alpha+\beta+\gamma+\delta+\eta}{n}$.*

If a generalized pseudo conformally symmetric Riemannian manifold is pseudo semiconformally symmetric then $\alpha = \beta = \gamma = \delta = \eta$. Then we can state that :

Corollary 2.7. *If a pseudo conformally symmetric Riemannian manifold is recurrent, then the 1-forms B and α are related by $B = \frac{2(n+2)}{n}\alpha$.*

From the Ricci identity and a parallel vector field V, it follows that

$$0 = V^t_{;jk} - V^t_{;kj} = V^m R^t_{mjk}. \tag{2.15}$$

Taking covariant derevative of the above equation we get

$$V^m R^t_{mjk;l} = 0. \tag{2.16}$$

Multiplying by g_{ti} we get

$$V^m R_{imjk;l} = 0 \tag{2.17}$$

Using second Bianchi identity we obtain

$$V^m R_{jkli;m} = 0. \tag{2.18}$$

Multiplying by g^{ji} and then multiplying by g^{kl} we have

$$V^m r_{kli;m} = 0 \tag{2.19}$$

$$V^m s_{;m} = 0. \tag{2.20}$$

Using the above equations it follows that

$$V^m C_{ijkl;m} = 0. \tag{2.21}$$

Or,

$$[2\alpha_m C_{ijkl} + \beta_i C_{mjkl} + \gamma_j C_{imkl} + \delta_k C_{ijml} + \eta_l C_{ijkm}]V^m = 0. \tag{2.22}$$

Or,

$$[\frac{2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m}{n}]V^m = 0. \tag{2.23}$$

Which leads the following:

Theorem 2.8. *If a generalized pseudo conformally symmetric manifold (M^n, g) admits a parallel vector field V, then either s=0 or* $[\frac{2n\alpha_m+\beta_m+\gamma_m+\delta_m+\eta_m}{n}]V^m = 0$.

Let a generalized pseudo conformally symmetric manifold is pseudo semiconformally symmetric then $\alpha = \beta = \gamma = \delta = \eta$. Then we can state that:

Corollary 2.9. *If a pseudo conformally symmetric manifold (M^n, g) admits a parallel vector field V and $[a + (n-2)b] \neq 0$, then either s=0 or $\alpha_m V^m = 0$.*

and then Multiplying (1.6) by g^{il} and then multiplying the relation thus obtained by g^{jk}, we obtain

$$-(\frac{s_{;m}}{n-2})n = -(\frac{s}{n-2})[2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m]. \tag{2.24}$$

Since $[a + (n-2)] \neq 0$, we have

$$s_{;m} = \frac{[2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m]}{n}s. \tag{2.25}$$

Taking covariant derivative of (2.25), we get

$$\begin{aligned} s_{;mt} = &\frac{[2n\alpha_{m;t} + \beta_{m;t} + \gamma_{m;t} + \delta_{m;t} + \eta_{m;t}]}{n}s \\ &+ \frac{[2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m)s_{;t}]}{n}. \end{aligned} \tag{2.26}$$

Or,

$$\begin{aligned} s_{;mt} = &\frac{[2n\alpha_{m;t} + \beta_{m;t} + \gamma_{m;t} + \delta_{m;t} + \eta_{m;t}]}{n}s \\ &+ \frac{[(2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m)(2n\alpha_t + \beta_t + \gamma_t + \delta_t + \eta_t)]s}{n}. \end{aligned} \tag{2.27}$$

Therefore from the above relation we can write

$$\begin{aligned} 0 = s_{;mt} - s_{;tm} = &\frac{s}{n}[2n(\alpha_{m;t} - \alpha_{t;m}) + (\beta_{m;t} - \beta_{t;m}) \\ &+ (\gamma_{m;t} - \gamma_{t;m}) + (\delta_{m;t} - \delta_{t;m}) + (\eta_{m;t} - \eta_{t;m})]. \end{aligned} \tag{2.28}$$

Thus we can state that:

Theorem 2.10. *Let the scalar curvature of a generalized pseudo conformally symmetric manifold does not vanish. Then if four 1-forms are closed then all the 1-forms are closed.*

If the manifold pseudo conformally symmetric manifold then, $\alpha = \beta = \gamma = \delta = \eta = A$. Then we have from (2.28)

$$0 = s_{;mt} - s_{;tm} = \frac{2n+4}{n}s[A_{m;t} - A_{t;m}]. \tag{2.29}$$

If $s \neq 0$ then

$$A_{m;t} - A_{t;m} = 0. \tag{2.30}$$

Thus we have the following

Corollary 2.11. *If the scalar curvature of a generalized pseudo conformally symmetric manifold does not vanish, then the 1-form A is closed.*

3. Conclusions

I have studied a new space namely generalized pseudo conformally symmetric manifold. Some geometric properties of such spaces are obtained. we have studied the harmonic nature of conformal curvature tensor. In future, different properties of these spaces can be obtained by imposing differnt restriction on the Ricci tensor of such spaces.

Acknowledgements

I would like to thank my guide Prof. Shyamal Kumar Hui, Professor of the University of Burdwan , West Bengal, India. He always encourages me in research.

References

[1] M. ALI, Q. KHAN, M. VASIULLA, On generalized pseudo symmetric manifolds,*Bull. Cal. Math. Soc.*, **113(4)** (2021), 353-362.

[2] É. CARTAN, Sur une classe remarquable d'espaces de Riemann, I,*Bull. de la Soc. Math. de France*, **54**, (1926), 214–216.

[3] É. CARTAN, Sur une classe remarquable d'espaces de Riemann, II, *Bull. de la Soc. Math. de France*, **55**, (1927), 114–134.

[4] M. C. CHAKI , On pseudosymmetric manifolds,*An. Sti. Ale Univ., "AL. I. CUZA" Din Ias*, **33** (1987), 53–58.

[5] M. C. CHAKI , On pseudosymmetric Ricci symmetric manifolds, *Bulg. J. Phys.*, **15** (1988), 526–531.

[6] M. C. CHAKI , On generalized pseudo-symmetric manifolds, *Publ. Math. Debrecen*, **45** (1994), 305–312.

[7] R. DESZCZ, On pseudosymmetric spaces,*Bull. Soc. Math. Belg. Sér. A*, **44** No. 1, (1992), 1–34.

[8] Y. ISHII, On conharmonic transformations, *Tensor, N. S.*, **11**, (1957), 73–80.

[9] J. KIM, A type of conformal curvature tensor,*Far East J. Math. Sci.*, **99** No. 1, (2016), 61-74.

[10] J. KIM, On pseudo semi-conformal symmetric manifolds,*Bull. Korean. Math. Soc.*, **54** ,(2017), 177-186.

[11] A. SELBERG, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, *J. Indian Math. Soc.*, **20** ,(1956), 47–87.

[12] A. A. SHAIKH, R. DESZCZ, M. HOTLOŚ, J. JELOWICKI, H. KUNDU, On pseudosymmetric manifolds,*Publ. Math. Debrecen*, **86(3-4)**, (2015), 433–456.

[13] A. A. SHAIKH, S. K. HUI, On weakly conharmonically symmetric manifolds,*Tensor, N. S.*, **70**, No. 2, (2008), 119–134.

[14] A. A. SHAIKH, H. KUNDU, On equivalency of various geometric structures, *J. Geom.*, **105**, (2014), 139–165.

[15] A. A. SHAIKH, I. ROY, S. K. HUI, On totally umbilical hypersurfaces of weakly conharmonically symmetric spaces, *Global J. Science Frontier Research*, **10(4)**, (2010), 28–31.

[16] L. TAMÁSSY, T. Q. BINH, On weakly symmetric and weakly projective symmetric Riemannian manifolds,*Coll. Math. Soc., J. Bolyai*, **56**, (1989), 663–670.

[17] Z. I. SZABÓ, Stucture theorems on Riemannian spaces satisfying $R(X,Y).R = 0$, I, The local version, *J. Diff. Geom.*, **17**, (1982), 531–582.

[18] A. G. WALKER, On Ruse's space of recurrent curvature ,*Proc. of London Math. Soc.*, **52** , (1950), 36-54.