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Stochastic delayed fractional-order differential equations driven by fractional Brownian motion

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Abstract. In this paper, we presents results on existence and uniqueness of mild solutions to stochastic differential equations with time delay driven by fractional Brownian motion (fBM) with Hurst index $(1/2, 1)$ in a Hilbert space with non-Lipschitzian coefficients.

AMS Subject Classifications: 60G22, 45N05, 34G20, 60H15, 60G15, 35R12.

Keywords: Fractional calculus, Mild solution, Semigroup of bounded linear operator, Fractional Brownian Motion, Stochastic differential equation with time delay, Young integral, Wiener integral.

Contents

1. Introduction and Background

Fractional differential equations have been widely applied in many fields of science and engineering, such as physics ([1]-[3]), chemical ([4]-[6]), etc. For example the nonlinear oscillation of earthquake can be modeled with fractional derivatives [7] and the fluid dynamic traffic model with fractional derivatives ([8]) can eliminate the deficiency arising from the assumption of continuum traffic flow. Actually, the concepts of fractional derivatives are not only generalization of the ordinary derivatives, but also it has been found that they can efficiently and properly describe the behavior of many real-life phenomena more accurately than integer order derivatives.

Stochastic differential equations (SDEs) are playing an increasingly important role in applications to finance, physics, and biology. A stochastic differential equation (SDE) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process. SDEs are used to model various phenomena such as unstable stock prices or physical systems subject to thermal fluctuations. Typically, SDEs contain a variable which represents random white noise calculated as the derivative of Brownian motion or the Wiener process. Stochastic differential equations are considered by many authors (see for example, ([9])) where the stochastic disturbances are described by stochastic integrals with respect to

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semimartingale (Brownian motion processes). However, the Brownian motion process is not suitable to represent a noise process if long-range dependence is modeled. It is then desirable to replace the Brownian motion process by fractional Brownian motion (fBM).

Fractional Brownian motion appears naturally in the modeling of many situations, for example , when describing the level of water in a river as a function of time, Financial turbulence. The existence of the fBM follows from the general existence theorem of centered Gaussian processes with given covariance functions ([10]). The fBM is divided into three very different families corresponding to $0 < H < 1/2$, $H = 1/2$ and $1/2 < H < 1$, respectively. The fBM (B^H) is not a semimartingale, as a result , the usual Itô calculus is not available for use. When $H > 1/2$, it happens that the regularity of the sample paths of B^H is enough and allows for using Young integral. In the case that $H < 1/2$ a powerful approach (Rough path theory) may be used.

In Ferrante and Rovira ([11]), the existence and uniqueness of solutions and the smoothness of the density for delayed SDEs driven by fBM is proved when $H > 1/2$, but under strong hypotheses, using only techniques of the classical stochastic calculus, and preventing, for instance, the presence of a hereditary drift in the equations. Neuenkirch et al. ([12]), using rough path theory, the authors prove existence and uniqueness of solutions to fractional equations with delays when $H > 1/3$. Recently, T. Caraballo et al. ([13]) prove the existence of solutions to stochastic delay evolution equations with a fBM.

Inspired by the above discussions, in this paper we study the following fractional stochastic differential equations (FSDEs) described in the form:

$$\begin{aligned} {}^{C}D_t^{\alpha}u(t) &= [Au(t) + f(t, u(\tau(t)))] + \sigma(t)\frac{dB_Q^H}{dt}, \quad 0 \le t \le T \\ u(t) &= \phi(t), \qquad -r \le t \le 0 \end{aligned} \tag{1.1}$$

where A is the infinitesimal generator of an analytic semigroup, $\{S(t)\}_{t\ge 0}$, of bounded linear operators in a separable Hilbert space $\mathfrak{X}$; B_Q^H is a fBM on a Hilbert space $\mathcal{Y}$, f and σ are given functions, $\tau : [0, \infty) \to [0, \infty)$ is a suitable delay function and $\phi : [-r, 0] \times \Omega \to \mathfrak{X}$ is the initial value.

The outline of this paper is structured as follows: section 2 contains some notations and preliminary facts. In section 3, the existence and uniqueness of solutions for equation (1.1) are established. The last section contains an example to illustrate our main results.

In the next part we give a brief review and preliminaries needed to establish our results.

Definition 1.1. *The Reimann-Liouville fractional derivative of f is defined as*

$$ {}^{R}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds$$

where $t > 0$, $n - 1 < \alpha < n$, $\Gamma(\cdot)$ stands for the gamma function and $n = [\alpha] + 1$ with $[\alpha]$ denotes the integer part of α (see e.g., [14]).

The Reimann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^{α} proposed by M. Caputo in his work on the theory of viscoelasticity.

Definition 1.2. *The Caputo-type derivative of order α for a function f can be written as*

$$ {}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds$$

where $t > 0$, $n - 1 < \alpha < n$. (see e.g., [14]).

Remark 1.1. *1. The relationship between the Riemann-Liouville derivative and the Caputo-type derivative can be written as*

$$^{C}D_t^\alpha f(t) = {^{R}D_t^\alpha} f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)$$

2. The Caputo-type derivative of a constant is equal to zero.

$$\mathcal{I}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \qquad t > 0. \tag{1.2}$$

Let $(\Omega, \Im, \mathbb{P})$ be a complete probability space and let $\{\beta^H(t), t \in [0,T]\}$ the one-dimensional fractional Brownian motion with Hurst index $H \in (1/2, 1)$. This means by definition that β^H is a centered Gaussian process with covariance function:

$$R_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - \mid t-s \mid^{2H})$$

It is known that β^H has the following Wiener integral representation (see, for example, [10]):

$$\beta^H(t) = \int_0^t \mathbb{K}_H(t,s) dB(s)$$

where $B = \{B(t) : t \in [0,T]\}$ is a standard Brownian motion process and $\mathbb{K}_H(t,s)$ is an explicit square integrable kernel given by

$$\mathbb{K}_H(t,s) = \mathbb{C}_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad t > s$$

where

$$\mathbb{C}_H = \sqrt{\frac{H(2H-1)}{\int_0^t (1-x)^{1-2H} x^{H-\frac{3}{2}} dx}} = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$$

and $\beta(\cdot,\cdot)$ denotes the Beta function. Let $\mathcal{H}$ be the closure of the set of indicator functions $\{\mathbb{I}_{[0,t]}, t \in [0,T]\}$ with respect to the scalar product

$$\langle \mathbb{I}_{[0,t]}, \mathbb{I}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s)$$

We recall that for $\varphi, \psi \in \mathcal{H}$ their scalar product in $\mathcal{H}$ is given by ([15]):

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T \varphi(s)\psi(t) \mid t-s \mid^{2H-2} dsdt$$

Let the operator $\mathbb{K}_H^* : \mathcal{H} \to \mathcal{L}^2([0,T])$ defined by ([15]):

$$(\mathbb{K}_H^* \varphi)(s) = \int_s^T \varphi(\tau) \frac{\partial \mathbb{K}_H}{\partial \tau}(\tau, s) d\tau$$

and for any $\varphi \in \mathcal{H}$, we have

$$\beta^H(\varphi) = \int_0^T \mathbb{K}_H^*(\varphi)(t) dB(t)$$

It is known that the elements of $\mathcal{H}$ may be not functions but distributions of negative order. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $\mathcal{H}^*$ generated by the measurable functions ψ such that

$$\parallel \psi \parallel_{\mathcal{H}^*}^2 := H(2H-1) \int_0^T \int_0^T \mid \psi(\tau) \mid\mid \psi(s) \mid\mid \tau - s \mid^{2H-2} d\tau ds$$

It is clear that, the space $(\mathcal{H}^*; \| \psi \|^2_{\mathcal{H}^*})$ is a Banach space and we have, ([10]):

$$\mathcal{L}^2([0,T]) \subseteq \mathcal{L}^{\frac{1}{H}}([0,T]) \subseteq \mathcal{H}^* \subseteq \mathcal{H}$$

and for any $\varphi \in \mathcal{L}^2([0,T])$, we have

$$\|\psi\|^2_{\mathcal{H}^*} \leq 2HT^{2H-1} \int_0^T \mid \psi(s) \mid^2 ds$$

Let $\mathfrak{L}(\mathcal{Y}, \mathfrak{X})$ be the space of bounded linear operator from $\mathcal{Y}$ to $\mathfrak{X}$ and let $Q \in \mathfrak{L}(\mathcal{Y}, \mathcal{Y})$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $TrQ = \sum_{n=1}^{\infty} \lambda_n < \infty$, $\lambda_n \geq 0$ are nonnegative real numbers and e_n is a complete orthonormal basis in $\mathcal{Y}$. Let $\mathcal{B}^H_Q = \{\mathcal{B}^H_Q(t)\}$ be $\mathcal{Y}$-valued fBM on $(\Omega, \Im, \mathbb{P})$ with covariance Q defined as:

$$\mathcal{B}^H_Q(t) = \sum_{n=1}^{\infty} \beta^H_n(t) e_n \sqrt{\lambda_n}$$

It is clear that the process $\mathcal{B}^H_Q$ is Gaussian, it starts from zero, has zero mean and covariance

$$\mathbb{E}[\langle \mathcal{B}^H_Q(t), x\rangle \langle \mathcal{B}^H_Q(s), y\rangle] = R(t,s)\langle Q(x), y\rangle, \quad x, y \in \mathcal{Y}, \quad t, s \in [0,T]$$

In order to define Wiener integrals with respect to the Q-fBM, we introduce the space $\mathbb{L}^2(\mathcal{Y}, \mathfrak{X})$ of all Q-Hilbert-Schmidt operators $\Psi : \mathcal{Y} \to \mathfrak{X}$. We recall that $\Psi \in \mathbb{L}^2(\mathcal{Y}, \mathfrak{X})$ is called a a Q-Hilbert-Schmidt operator if

$$\| \Psi \|^2_{\mathbb{L}^2} := \sum_{n=1}^{\infty} \left\| \Psi e_n \sqrt{\lambda_n} \right\|^2 < \infty$$

We note that the space $\mathbb{L}^2$ equipped with the inner product

$$\langle \varphi, \psi \rangle_{\mathbb{L}^2} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle$$

is a separable Hilbert space ([13]). Now, the Wiener integral of $\varphi \in \mathbb{L}^2(\mathcal{Y}, \mathfrak{X})$ with respect to $\mathcal{B}^H_Q$ is defined by:

$$\int_0^t \varphi(s) dB^H_Q(s) := \sum_{n=1}^{\infty} \int_0^t \varphi(s) \sqrt{\lambda_n} e_n d\beta^H_n(s) = \sum_{n=1}^{\infty} \int_0^t \mathbb{K}^*_H(\varphi e_n)(s) \sqrt{\lambda_n} dB_n(s)$$

Lemma 1.1. *If* $\Phi : [0,T] \to \mathbb{L}^2(\mathcal{Y}, \mathfrak{X})$ *satisfies* $\int_0^T \| \Phi(s) \|^2_{\mathbb{L}^2} ds < \infty$. *Then the above sum in the previous equation is well-defined as a* $\mathfrak{X}$*-valued random variable and we have:*

$$\mathbb{E} \left\| \int_0^t \Phi(s) d\mathcal{B}^H_Q \right\|^2 \leq 2HT^{2H-1} \int_0^t \|\Phi(s)\|^2_{\mathbb{L}^2} ds.$$

We recall that for any strongly continuous semigroup $\{S(t); t \geq 0\}$ on $\mathfrak{X}$, we define the generator

$$Au = \lim_{t \to 0^+} \frac{S(t)u - u}{t}.$$

Throughout this paper, let A is the infinitesimal generator of a strongly continuous semigroup $\{S(t); t \geq 0\}$ of operators on a Hilbert space $\mathfrak{X}$. Clearly,

$$M = \sup_{t \in [0,T]} \| S(t) \| < \infty.$$

We suppose that $\| S(t) \| \leq C_1$

Lemma 1.2. *([16],[18]) Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and non-decreasing function and let also g, h and λ be non-negative functions on $\mathbb{R}_+$ such that:*

$$g(t) \le h(t) + \int_0^t \lambda(s)\xi(g(s))ds, \quad t \ge 0$$

, then

$$g(t) \le \rho^{-1}\left\{\rho(h^*(t)) + \int_0^t \lambda(s)ds\right\},$$

where $\rho(x) = \int_{t_0}^t \frac{dx}{\xi(x)}$ is well-defined for $t_0 > 0$ and $h^*(t) = \sup_{s \le t} h(s)$. In particular, we have the Gronwall-Bellman Lemma: If

$$g(t) \le h(t) + \int_0^t \lambda(s)g(s)ds, \quad t \ge 0$$

, then

$$g(t) \le h^*(t)e^{\int_0^t \lambda(s)ds}.$$

Definition 1.3. *A $\mathfrak{X}$-valued process $\{u(t), t \in [-r, T]\}$ is called a mild solution of equation (1.1) if:*

1. *$u(t) \in \mathbf{C}([-r, T], \mathcal{L}^2(\Omega, \mathfrak{X}))$,*
2. *$u(t) = \phi(t), \qquad -r \le t \le 0$,*
3. *For any $t \in [0, T]$, we have*

$$u(t) = J(t)\phi(0) + \int_0^t J^*(t-s)f(s, u(\tau(s)))ds + \int_0^t J^*(t-s)\sigma(s)d\mathcal{B}_Q^H(s), \quad a.s.$$

where

$$J(t) = \int_0^\infty M_\alpha(\theta)S(t^\alpha\theta)d\theta,$$
$$J^*(t) = \alpha\int_0^\infty \theta t^{\alpha-1}M_\alpha(\theta)S(t^\alpha\theta)d\theta$$

and $M_\alpha(\theta) \ge 0$ is a probability function on $(0, \infty)$, that is

$$M_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\omega_\alpha(\theta^{-\frac{1}{\alpha}}),$$
$$\omega_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin n\pi\alpha.$$

and $\int_0^\infty M_\alpha(\theta)d\theta = 1$

Lemma 1.3. *([19]) The operators J and J^* have the following properties:*

1. *For any fixed $t \ge 0$, $J(t)$ and $J^*(t)$ are linear and bounded, i.e., for any $x \in \mathfrak{X}$*

$$\| J(t)x \| \le C_2 \| x \|, \quad \| J^*(t)x \| \le \frac{C_2T^\alpha}{\Gamma(\alpha+1)} \| x \|$$

2. *$\{J(t), t \ge 0\}$ and $\{J^*(t), t \ge 0\}$ are strongly continuous.*
3. *For every $t > 0$, $J(t)$ and $J^*(t)$ are compact operators if $S(t)$ is compact.*

2. Main Results

To prove the existence and the uniqueness of mild solutions of equation (1.1), the following weaker conditions (instead of the Lipschitz and linear growth conditions, see, e.g., ([16],[17])) are listed:

($\mathcal{H}1$) $f : [0,T] \times \mathfrak{X} \to \mathfrak{X}$ and $\sigma : [0,T] \to \mathbb{L}^2(\mathcal{Y},\mathfrak{X})$ are satisfying the following conditions: there exists a function $\zeta : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that:

1. For all t, $\zeta(t,\cdot)$ is continuous non-decreasing , concave, and for each fixed $x \in \mathbb{R}_+$, $\int_0^T \zeta(s,x)ds < \infty$.
2. For any $t \in [0,T]$ and $x \in \mathfrak{X}$
$$\|f(t,x)\|^2 \leq \zeta(t, \| x \|^2), \quad \int_0^T \|f(t,x)\|^2_{\mathbb{L}^2} < \infty.$$
3. For any constant $q > 0$, $x_0 \geq 0$, the integral equation
$$x(t) = x_0 + q\int_0^t \zeta(s,x(s))ds$$
has a global solution on $[0,T]$.

($\mathcal{H}2$) There exists a function $\eta : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that:

1. For all t, $\eta(t,\cdot)$ is continuous non-decreasing , concave, with $\eta(t,0) = 0$, and for each fixed $x \in \mathbb{R}_+$, $\int_0^T \eta(s,x)ds < \infty$.
2. For any $t \in [0,T]$ and $x, y \in \mathfrak{X}$
$$\|f(t,x) - f(t,y)\|^2 \leq \eta(t, \|x-y\|^2).$$
3. For any constant $C_3 > 0$, if a non-negative function $h(t), t \in [0,T]$ satisfies $h(0) = 0$ and
$$h(t) \leq C_3\int_0^t \eta(s,h(s))ds$$
, then $h(t) = 0, \forall t \in [0,T]$.

($\mathcal{H}3$) τ is a continuous function satisfying the condition:
$$-r \leq \tau(t) \leq t, \qquad \forall t \geq 0$$

($\mathcal{H}4$) we assume that $\phi \in \mathbf{C}([-r,T], \mathcal{L}^2(\Omega,\mathfrak{X}))$

Lemma 2.1. *Let $b \in \mathcal{L}^2([0,T],\mathfrak{X})$, $\tilde{\sigma} \in \mathcal{L}^2([0,T],\mathbb{L}^2)$ and consider the following equation:*

$$\begin{aligned} {}^C D_t^\alpha u(t) &= [Au(t) + b(t)] + \tilde{\sigma}(t)\frac{dB_Q^H}{dt}, \quad 0 \leq t \leq T \\ u(t) &= \phi(t), \qquad -r \leq t \leq 0 \end{aligned} \tag{2.1}$$

, then equation (2.1) has a unique mild solution on $[-r,T]$

Proof. Let $\mathcal{C}_T := \mathbf{C}([-r,T],\mathcal{L}^2(\Omega,\mathfrak{X}))$ be a Banach space of all continuous functions from $[-r,T]$ into $\mathcal{L}^2(\Omega,\mathfrak{X})$, endowed with the norm

$$\|u\|^2_{\mathcal{C}_T} = \sup_{-r\leq t\leq T} \|u(t,\omega)\|^2, \quad \omega\in\Omega. \tag{2.2}$$

Let us consider,

$$\mathcal{G}_T := \{u\in\mathcal{C}_T : u(s)=\phi(s), s\in[-r,0]\}.$$

It is clear that, $\mathcal{G}_T$ is a closed subset of $\mathcal{C}_T$ provided with the norm (2.2).
Let $\mathcal{F}$ be the function defined on $\mathcal{G}_T$ by:

$$\begin{aligned}
\mathcal{F}_x(t) &= \phi(t), \qquad t\in[-r,0],\\
\mathcal{F}_x(t) &= J(t)\phi(0) + \int_0^t J^*(t-s)b(s)ds + \int_0^t J^*(t-s)\tilde{\sigma}(s)d\mathcal{B}^H_Q(s), \quad t\in[0,T]\\
&= \sum_{k=1}^{3} I_k.
\end{aligned}$$

In the next step, we are going to prove that each function $t\mapsto I_k, k=1,2,3$ is contionouous on $[0,T]$ in the $\mathcal{L}^2(\Omega,\mathfrak{X})$ sense.

1. The contionuity of I_1 followes directly from the continuity of $t\mapsto J(t)z$ (see, Lemma (1.3)), and by using some simple computations we can show the continuity of I_2.

2. For the term I_3, by using (Lemma 1.1, Lemma 1.3), we have

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^{t+z} J^*(t+z-s)\tilde{\sigma}(s)d\mathcal{B}^H_Q(s) - \int_0^t J^*(t-s)\tilde{\sigma}(s)d\mathcal{B}^H_Q(s)\right\|\\
&\quad\leq \left\|\int_0^t (J^*(t+z-s)-J^*(t-s))\tilde{\sigma}(s)d\mathcal{B}^H_Q(s)\right\|\\
&\quad\quad+\left\|\int_t^{t+z} J^*(t+z-s)\tilde{\sigma}(s)d\mathcal{B}^H_Q(s)\right\|\\
&\leq I_{31}(z)+I_{32}(z).
\end{aligned}$$

 It is clear that, $I_{31}, I_{32}\to 0$ as $z\to 0$, and then

$$\lim_{z\to 0}\mathbb{E}\|\mathcal{F}_x(t+z)-\mathcal{F}_x(t)\|^2 = 0.$$

Hence, we conclude that the function $\mathcal{F}_x(t)$ is continuous on $[0,T]$ in the $\mathcal{L}^2(\Omega,\mathfrak{X})$ sense. By using classical computations we can show that

$$\sup_{-r\leq t\leq T}\mathbb{E}\|\mathcal{F}_x(t)\|^2<\infty.$$

Hence, we conclude that $\mathcal{F}$ is well defined. It is clear that $\mathcal{F}$ is a contraction mapping in $\mathcal{G}_{T_1}$ with some $T_1\leq T$ and therefore has a unique fixed point, which is a mild solution of equation (2.1) on $[0,T_1]$. This procedure cen be repeated in order to extend the solution to the entire interval $[-r,T]$ in finitly many steps.

■

By using a Picard type iteration with the help of Lemma (2.1), we can construct a successive approximation sequence as: Let u_0 be the solution of equation (2.1) with $b \equiv 0$ and $\tilde{\sigma} \equiv 0$. For $n \geq 0$, let u_{n+1} ne the solution of equation (2.1) with $b(t) \equiv f(t, u(\tau(t)))$ and $\tilde{\sigma}(t) \equiv \sigma(t)$. Therefore,

$$\begin{aligned} u_{n+1}(t) &= \phi(t), \qquad t \in [-r, 0], \\ u_{n+1}(t) &= J(t)\phi(0) + \int_0^t J^*(t-s) f(t, u_n(\tau(t)))ds \\ &\quad + \int_0^t J^*(t-s)\sigma(s) d\mathcal{B}_Q^H(s), \quad t \in [0, T] \end{aligned} \tag{2.3}$$

Lemma 2.2. *Let $(\mathcal{H}1 - \mathcal{H}4)$ hold. The sequence $\{u_n, n \geq 0\}$ is well-defined and there exist positive constants C_4, C_5, C_6 such that $\forall n, m \in \mathbb{N}$ and $t \in [0, T]$, we have:*

$$\sup_{-r \leq s \leq t} \mathbb{E} \|u_{m+1}(s) - u_{n+1}(s)\|^2 \leq C_4 \int_0^t \eta(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|u_m(\theta) - u_n(\theta)\|^2) ds \tag{2.4}$$

and

$$\sup_{-r \leq s \leq t} \mathbb{E} \|u_{n+1}(s)\|^2 \leq C_5 + C_6 \int_0^t \zeta(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|u_n(\theta)\|^2) ds \tag{2.5}$$

Proof. For inequality (2.4), we have

$$\|u_{m+1}(t) - u_{n+1}(t)\|^2 = \left\| \int_0^t J^*(t-s)[f(t, u_m(\tau(t))) - f(t, u_n(\tau(t)))]ds \right\|^2.$$

By using condition $(\mathcal{H}2)$, we get

$$\sup_{-r \leq s \leq t} \mathbb{E} \left\| \int_0^t J^*(t-s)[f(t, u_m(\tau(t))) - f(t, u_n(\tau(t)))]ds \right\|^2 \leq C_4 \int_0^t \eta(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|u_m(\theta) - u_n(\theta)\|^2) ds.$$

and hence the result. For inequality (2.5), we have

$$\|u_{n+1}(t)\|^2 = \left\| J(t)\phi(0) + \int_0^t J^*(t-s)[f(t, u_n(\tau(t)))]ds + \int_0^t J^*(t-s)\sigma(s) d\mathcal{B}_Q^H(s) \right\|^2.$$

By using the identity $\| x + y \|^2 \leq 2 \| x \|^2 + 2 \| y \|^2$, Lemma (1.3), condition $\mathcal{H}2$ and Lemma (1.1), we get

$$\sup_{-r \leq s \leq t} \|u_{n+1}(s)\|^2 \leq C_5 + C_6 \int_0^t \zeta(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|u_n(\theta)\|^2.$$

■

Lemma 2.3. *Suppose that $(\mathcal{H}1 - \mathcal{H}4)$ are satisfied, then there exists an $x(t)$ such that*

$$x(t) = x_0 + q \int_0^t \zeta(s, x(s)) ds \tag{2.6}$$

for $x_0 \geq 0$, $q > 0$ and the sequence $\{u_n, n \geq 0\}$ satisfies, for all $n \in \mathbb{N}, t \in [0, T]$

$$\sup_{-r \leq s \leq t} \mathbb{E} \|u_n(s)\|^2 \leq x(t). \tag{2.7}$$

Proof. Let $x : [0, T] \to \mathbb{R}$ be a global solution of the integral equation (2.6) with an initial condition $x_0 = \max(C_6, \sup_{-r \leq t \leq T} \mathbb{E} \| u_0(t) \|^2)$, then by using mathematical induction we can prove that inequality (2.7). ■

Theorem 2.1. *Let $(\mathcal{H}1-\mathcal{H}4)$ be satisfied. Then for all $T>0$, the equation (1.1) has a unique mild solution on* $[-r,T]$.

Proof. Existence: For $t\in[0,T]$, by using Lemma (2.2), Lemma (2.3) and Fatou's theorem, we get:

$$\limsup_{m,n\to\infty}\{\sup_{-r\leq s\leq t}\mathbb{E}\left\|u_{m+1}(s)-u_{n+1}(s)\right\|^2\}\leq C_4\int_0^t\eta(s,\limsup_{m,n\to\infty}\sup_{-r\leq\theta\leq s}\mathbb{E}\left\|u_m(\theta)-u_n(\theta)\right\|^2)ds.$$

By using condition $(\mathcal{H}2)$, we have

$$\lim_{m,n\to\infty}\sup_{-r\leq s\leq T}\mathbb{E}\left\|u_m(s)-u_n(s)\right\|^2)=0.$$

Hence, the sequence $\{u_n\}_{n\geq 0}$ is a cauchy sequence in $\mathcal{C}_T$ and from the completeness of $\mathcal{C}_T$ we guarantees the existence of a process $u\in\mathcal{C}_T$ such that

$$\lim_{n\to\infty}\sup_{-r\leq s\leq T}\mathbb{E}\left\|u_n(s)-u(s)\right\|^2)=0,$$

and if $n\to\infty$ in equation (2.3), then we can see that u is a mild solution to equation (1.1) on $[-r,T]$.
Uniqueness: Let u,v be two mild solutions of equation (1.1), then

$$\sup_{-r\leq s\leq t}\mathbb{E}\left\|u(s)-v(s)\right\|^2\leq C_4\int_0^t\eta(s,\sup_{-r\leq\theta\leq s}\mathbb{E}\left\|u(\theta)-v(\theta)\right\|^2)ds$$

and by using condition $(\mathcal{H}2)$, we get $\sup_{-r\leq s\leq t}\mathbb{E}\parallel u(s)-v(s)\parallel^2=0$, which implies that $u\equiv v$. ∎

3. Applications.

In this section, we give an example to illustrate our main results.

Example 3.1.

$$\begin{aligned}{}^{C}D_t^{1/2}[u(t,\zeta)]&=\frac{\partial^2}{\partial\zeta^2}u(t,\zeta)+\frac{e^{-2t}u(\frac{1}{3}(1+\cos t))}{70(1+u^2(\frac{1}{3}(1+\cos t))}+e^{-\pi^2 t}\frac{dB_Q^H}{dt},\quad t\in(0,T],\zeta\in[0,\pi]\\ u(t,0)=u(t,\pi)&=0,t\in(0,T],\\ u(t,\zeta)&=\phi(t,\zeta),\qquad -r\leq t\leq 0.\end{aligned}\tag{3.1}$$

where $A:D(A)\subset\mathfrak{X}\to\mathfrak{X}$, which is defined by $A\varpi=\varpi''$ with $D(A)=\{u\in\mathfrak{X}:u''\in\mathfrak{X},u(0)=u(\pi)=0\}$, u,u' are absolutely continuous and then A can be written as $Au=\sum_{n=1}^{\infty}n^2\langle u,u_n\rangle u_n$ where $u_n(s)=\sqrt{\frac{2}{\pi}}\sin(nu)$ is the orthonormal set of eigenvectors of A. Also A is the the infinitesimal generator of an analytic semigroup, $\{S(t)\}_{t\geq 0}$ in $\mathfrak{X}$ and there exists M, such that $\|S(t)\|\leq M$. From (3.1), we know that the delay term $\frac{1}{3}(1+\cos t)$ and

$$\begin{aligned}f(t,u)&=\frac{e^{-2t}u}{70(1+u^2)},\\ \sigma(t)&=e^{-\pi^2 t}\end{aligned}$$

and with the above choices (3.1) can be formulated in the abstract form of (1.1) and it is easy to verify the conditions of Theorem (2.1) all hold, and then (3.1) must have a mild solution on $[0,1]$.

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Results using primitive function module

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Abstract. A real or complex valued function defined on the set of all positive integers is called an arithmetic function and an arithmetic function is said to be completely multiplicative function if f is not identically zero and $f(mn) = f(m)f(n)$ for all m, n. The objective of this paper is to present a result of completely multiplicative function of two variables using primitive function module.

AMS Subject Classifications: 11A07, 11A25.

Keywords: Arithmetic function, Multiplicative function, Primitive function module.

Contents

1. Introduction

A real or complex valued function defined on the set of all positive integers is called an arithmetic function. An arithmetic function f is said to be multiplicative function in one argument if f is not identically zero and $\mathrm{f(mn)} = \mathrm{f(m)f(n)}$ whenever $(\mathrm{m}, \mathrm{n}) = 1$. The function $\mathrm{f(m, n)}$ of two variables defined for pairs of positive integers m and n is said to be multiplicative in both the arguments m and n if $\mathrm{f}(1, 1) = 1$ and $\mathrm{f}\left(\mathrm{m}_1\ \mathrm{m}_2, \mathrm{n}_1\mathrm{n}_2\right) = \mathrm{f}\left(\mathrm{m}_1,\ \mathrm{m}_2\right)\mathrm{f}\left(\mathrm{m}_2, \mathrm{n}_2\right)$ where $\left(\mathrm{m}_1\mathrm{n}_1,\ \mathrm{m}_2\mathrm{n}_2\right) = 1$. Many identities have been established by various researchers discussed in [3, 7, 9].

Definition 1.1. *An arithmetic function is said to be completely multiplicative function if f is not identically zero and $f(mn) = f(m)f(n)$ for all m, n.*

Definition 1.2. *Strongly Multiplicative function: A multiplicative arithmetic function* f *is said to be strongly multiplicative function if for every prime* P, *we have*

$$f(p) = f\left(p^2\right) = f\left(p^3\right) = \cdots \ldots \ldots \ldots \ldots$$

Definition 1.3. *An arithmetic function $f(n, r)$ is said to be primitive function module r if $f(n, r) = f(\gamma(n, r), r)$ for all $\gamma(n, r) = \gamma((n, r))$.*

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Definition 1.4. *An arithmetic function $f(n,r)$ is said to be completely primitive function module r if $f(n,r)=f(n',r')$ for all n,n^1 and all positive r,r' Such that*

$$\frac{\gamma(r)}{\gamma(n,r)}=\frac{\gamma(r')}{\gamma(n',r')}$$

Let $g(r)$ and $h(r)$ be two arithmetic functions.
Define

$$f(n,r)=\sum_{d\mid(n,r)} h(d)g\left(\frac{r}{d}\right)\mu\left(\frac{r}{d}\right) \tag{1.1}$$

and

$$F(r)=f(0,r)=\sum_{d\mid r} h(d)g\left(\frac{r}{d}\right)\mu\left(\frac{r}{d}\right). \tag{1.2}$$

2. Preliminaries

We use the following lemma proved by E.Cohen ([4], P404).

Lemma 2.1. *Suppose $f(n,r)$ is completely primitive module r. Then*

$$f(n,r)=\sum_{\substack{d\mid\gamma(r)\\(d,n)=r_1}} G(d)\Leftrightarrow G(r_1)=\sum_{d\mid r_1} f\left(\frac{r_1}{d},d\right)\mu\left(\frac{r_1}{d}\right)$$

for any square free r_1.

Lemma 2.2. *Let $h(v)$ is completely multiplicative function. Then $F(v)=h\left(\frac{v}{\gamma(v)}F(\gamma(v))\right)$.*

Proof. From (1.2), We have

$$\begin{aligned}
F(v) &= \sum_{d\mid n} h(d)g\left(\frac{v}{d}\right)\mu\left(\frac{v}{d}\right)\\
&= \sum_{d\delta=v} h\left(\frac{v}{\delta}\right)g(\delta)\mu(\delta)\\
&= \sum_{d\delta=v} h\left(\frac{v}{\gamma(v)}\cdot\frac{\gamma(n)}{\delta}\right)g(\delta)\mu(\delta)\\
&= h\left(\frac{v}{\gamma(v)}\right)\sum_{d\delta=v} h\left(\frac{\gamma(v)}{\delta}\right)g(\delta)\mu(\delta)\\
&= h\left(\frac{v}{\gamma(v)}\right)F(\gamma(v)).
\end{aligned}$$

Because of factor $\mu(\delta)$, since $\mu(\delta)=0$ for square number. Hence ranging d over divisor of v or over divisor of $\gamma(v)$ is same. This completes the lemma.

■

Also we need the following result:

Lemma 2.3. *Let $g(r)$ be multiplicative, $h(r)$ is completely multiplicative and for all primes $P, h(p)\neq 0, h(p)\neq g(P)$. Then $F(r)\neq 0$ for all r.*

Proof. Since $F(1)=1$ We may assume that $r>1$. Note that $F(r)=(h*\mu g)(r)$.

$F(r)$ is multiplicative, since $h(r), g(r)$ and $\mu(r)$ are multiplicative functions. To prove $F(P^{\alpha})\neq 0$ for all primes P and $\alpha>1$.

Consider

$$\begin{aligned}
F\left(P^{\alpha}\right) &= \sum_{k=0}^{\alpha} h\left(P^{k}\right) g\left(P^{\alpha-k}\right) \mu\left(P^{\alpha-k}\right) \\
&= h\left(P^{\alpha}\right)-h\left(P^{\alpha-1}\right) g(P) \\
&= h(P)^{\alpha-1}[h(P)-g(P)] \\
&\neq 0.
\end{aligned}$$

Since $h(P)-g(P)\neq 0, h(P)\neq 0$.

■

3. Main Results

Theorem 3.1. *If $g(r)$ is multiplicative, $h(r)$ is completely multiplicative and for all prime $P, h(P)\neq 0, h(P)\neq g(P)$, then*

$$\sum_{\substack{d \mid r \\ (d,n)=1}} \frac{g(d)}{F(d)} \mu^{2}(d)=\frac{h(r)}{F(r)} \frac{F((n, r))}{h((n, r))}.$$

Proof. Denote

$$J(n, r)=\frac{h(r)}{F(r)} \frac{F((n, r))}{h((n, r))}. \tag{3.1}$$

$J(n,r)$ is properly defined since $F(r)\neq 0, h(n,r)\neq 0$.

By Lemma 2.2, we get,

$$\begin{aligned}
J(n, r) &= \frac{h(r) h\left(\frac{n, r}{\gamma(n, r)}\right) F(\gamma(n, r))}{h\left(\frac{r}{\gamma(r)}\right) F(\gamma(r)) h((n, r))} \\
&= \frac{h(r) h((n, r)) F(\gamma(n, r)) h(\gamma(r))}{h(\gamma(n, r)) h(r) F(\gamma(r)) h((n, r))} \\
&= \frac{h(\gamma(r)) F(\gamma(n, r))}{h(\gamma(n, r)) F(\gamma(r))} \\
&= \frac{h\left(\frac{\gamma(r)}{\gamma(n, r)}\right)}{F\left(\frac{\gamma(r)}{\gamma(n, r)}\right)} \\
&= \frac{h(m)}{F(m)},
\end{aligned}$$

where $m=\frac{\gamma(r)}{\gamma(n, r)}$.

Thus

$$J(n, r)=\frac{h(m)}{F(m)}, \quad \text{where } m=\frac{\gamma(r)}{\gamma(n, r)}. \tag{3.2}$$

Now we prove that

$$\mathrm{J}(n, \mathrm{r}) \quad \text{is completely primitive} \quad (\bmod r). \tag{3.3}$$

That is, we have to show that $J(n, r)=J\left(n^1, r^1\right)$ for all n, n^1, r, r^1 with

$$\frac{\gamma(r)}{\gamma(n, r)}=\frac{\gamma\left(r^1\right)}{\gamma\left(n^1, r^1\right)}$$

By (3.2), we have

$$\begin{aligned} J(n, r) & =\frac{h\left(\frac{\gamma(r)}{\gamma(n, r)}\right)}{F\left(\frac{\gamma(r)}{\gamma(n, r)}\right)} \\ & =\frac{h\left(\frac{\gamma\left(r^{\prime}\right)}{\gamma\left(n^{\prime}, r^{\prime}\right)}\right)}{F\left(\frac{\gamma\left(r^{\prime}\right)}{\gamma\left(n^{\prime}, r^{\prime}\right)}\right)} \\ & =J\left(n^{\prime}, r^{\prime}\right) . \end{aligned}$$

Therefore by Lemma 2.1, we have

$$J(n, r)=\sum_{\substack{d \mid \gamma(r) \\ (d, n)=1}} G(d) \Leftrightarrow G\left(r_1\right)=\sum_{d \mid r_1} J\left(\frac{r_1}{d}, r_1\right) \mu\left(\frac{r_1}{d}\right) .$$

Consider

$$\begin{aligned} G\left(r_1\right) & =\sum_{d \mid r_1} J\left(\frac{r_1}{d}, r_1\right) \mu\left(\frac{r_1}{d}\right) . \\ & =\sum_{d \mid r_1} \frac{h(d)}{F(d)} \mu\left(\frac{r_1}{d}\right) \quad \text{by} \quad (3.2) \end{aligned}$$

which by multiplicativity of $\mu(r)$ and $F(r)$ gives

$$\begin{aligned} & =\frac{\mu\left(r_1\right)}{F\left(r_1\right)} \sum_{d \mid r_1} h(d) \mu(d) F\left(\frac{r_1}{d}\right) \\ & =\frac{\mu\left(r_1\right)}{F\left(r_1\right)} \sum_{d \mid r_1} h(d) \mu(d) \sum_{D \delta=\frac{r_1}{d}} h(D) g(\delta) \mu(\delta), \end{aligned}$$

where $\mathrm{E}=\mathrm{Dd}$. But

$$\sum_{d \mid E} \mu(d)= \begin{cases}1 & \text{if } E=1 \\ 0 & \text{if } E>1 .\end{cases}$$

Therefore,

$$\begin{aligned} G\left(r_1\right) & =\frac{\mu\left(r_1\right)}{F\left(r_1\right)} g\left(r_1\right) \mu\left(r_1\right) \\ & =\frac{\mu^2\left(r_1\right) g\left(r_1\right)}{F\left(r_1\right)} . \end{aligned}$$

Now we have

$$
\begin{aligned}
J(n,r) &= \sum_{\substack{d \mid \gamma(r) \\ (d,n)=1}} G(d) \\
&= \sum_{\substack{d \mid \gamma(r) \\ (d,n)=1}} \frac{\mu^2(d) g(d)}{F(d)}.
\end{aligned}
$$

■

Theorem 3.2.

$$
F(r) \sum_{\substack{d \mid r \\ (d,n)=1}} \frac{h(d)}{F(d)} \cdot \mu\left(\frac{r}{d}\right) = \mu(r) \sum_{d \mid (n,r)} h(d) f\left(\frac{r}{d}\right),
$$

where $f(n) = g(n)\mu(n)$.

Proof. Let

$$
Q(n,r) = F(r) \sum_{\substack{d \mid r \\ (d,n)=1}} \frac{h(d)}{F(d)} \mu\left(\frac{r}{d}\right).
$$

Let

$$
Q(n,r) = F(r) \sum_{\substack{d \mid r \\ (d,n)=1}} \frac{h(d)}{F(d)} \mu\left(\frac{r}{d}\right).
$$

Let r_1 and r_2 be the uniquely determined positive integers such that $r = r_1 r_2$ where $(r_1, r_2) = 1, \gamma(r_2) = \gamma(n,r)$.

Then

$$
\begin{aligned}
Q(n,r) &= F(r)\mu(r_2) \sum_{d_1 \mid r_1} \frac{h(d)}{F(d)} \mu\left(\frac{r_1}{d}\right) \\
&= F(r)\mu(r_2) G(r_1) \\
&= F(r_1) F(r_2) \mu(r_2) \frac{\mu^2(r_1) g(r_1)}{F(r_1)} \\
&= \mu(r)\mu(r_1) g(r_1) \sum_{d \mid (n,r)} h(d) g\left(\frac{r_2}{d}\right) \mu\left(\frac{r_2}{d}\right).
\end{aligned}
$$

In view of the presence of $\mu(r)$ and the fact that $\gamma(r_2) = \gamma(n,r)$, we have

$$
Q(n,r) = \mu(r) \sum_{d \mid (n,r)} h(d) g\left(\frac{r}{d}\right) \mu\left(\frac{r}{d}\right).
$$

That is,

$$
F(r) \sum_{\substack{d \mid r \\ (d,n)=1}} \frac{h(d)}{F(d)} \mu\left(\frac{r}{d}\right) = \mu(r) \sum_{d \mid (n,r)} h(d) f\left(\frac{r}{d}\right),
$$

where $f(n) = \mu(n)g(n)$. ■

Remark 3.3. *Substituting $h(n) = n^k, f(n) = \mu(n)$ and $F(r) = J_k(r)$ in Theorem 3.2, we get a well known identity known as Brauer - Rademacher identity.*

$$J_k(r) \sum_{d|r} \frac{d^k}{J_k(d)} \mu\left(\frac{r}{d}\right) = \mu(r) \sum_{d|(n,r)} d^k \mu\left(\frac{r}{d}\right).$$

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On Pillai's problem involving two linear recurrent sequences: Padovan and Fibonacci

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Abstract. In this paper, we find all integers c having at least two representations as a difference between linear recurrent sequences. This problem is a Pillai problem involving Padovan and Fibonacci sequence. The proof of our main theorem uses lower bounds for linear forms in logarithms, properties of continued fractions, and a version of the Baker-Davenport reduction method in Diophantine approximation.

AMS Subject Classifications: 11B39, 11J86, 11D61.

Keywords: Linear forms in logarithm, Diophantine equations, Fibonacci sequence, Padovan sequence, Diophantine equation, Baker's theory, Reduction procedure.

Contents

1. Introduction

It is well-known that the sequence $\{\mathcal{P}_k\}_{k\geq 1}$ of Padovan numbers is defined by

$$\mathcal{P}_0 = \mathcal{P}_1 = \mathcal{P}_2 = 1, \quad \mathcal{P}_{k+3} = \mathcal{P}_{k+1} + \mathcal{P}_k, \quad k \geq 0.$$

The first Padovan numbers are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, 265 \ldots$$

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The sequence $\{F_k\}_{k\geq 1}$ of Fibonacci numbers is defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{k+2} = F_{k+1} + F_k, \quad k \geq 0.$$

The first Fibonacci numbers are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots$$

In this paper, we are interested in the Diophantine equation

$$\mathcal{P}_m - F_n = c \tag{1.1}$$

for a fixed c and variable m and n. In particular, we are interested in those integers c admitting at least two representations as a difference between a Padovan number and Fibonacci number. This is a variation of the equation

$$a^x - b^y = c, \tag{1.2}$$

in non-negative integers (x, y) where a, b, c are given fixed positive integers. The history of equation (1.2) is very rich and goes back to 1935 when Herschfeld [8], [9] studied the particular case $(a, b) = (2, 3)$. Extending Herschfeld's work, Pillai [12], [13] proved that if a, b are coprime positive integers then there exists $c_0(a, b)$ such that if $c > c_0(a, b)$ is an integer, then equation (1.2) has at most one positive integer solution (x, y). Since then, variations of equation (1.2) has been intensively studied. Some recent results related to equation (1.1) are obtained by the third author and his collaborators in which they replaced Fibonacci numbers $\mathcal{P}_n$ by the Fibonacci numbers F_n (see [5]), Tribonacci numbers (see [2]), and k-generalized Fibonacci numbers (see [6]). The equation solved in this paper is an exponential Diophantine equation. The similar problem has been solved recently by the authors (see [14–16]). The aim of this paper is to prove the following result.

Theorem 1.1. *The only integers c having at least two representations of the form $\mathcal{P}_m - F_n$ with $m > 3, n > 1$ are*

$$c \in \{-226, -82, -52, -30, -27, -18, -9, -6, -5, -4, -3, -1, 0, 1, 2, 3, 4, 6, \\ 7, 8, 10, 11, 13, 15, 16, 20, 25, 31, 32, 36, 44, 52, 62, 111, 262\}.$$

We organize this paper as follows. In Section 2, we recall some results useful for the proof of Theorem 1.1. The proof of Theorem 1.1 is done in the last section.

2. Auxiliary results

2.1. Some properties of Fibonacci and Padovan sequences

Here we recall a few properties of the Fibonacci sequence $\{P_k\}_{k\geq 0}$ and Padovan sequences $\{\mathcal{P}_k\}_{k\geq 0}$ which are useful to proof our theorem.
The characteristic equation of Padovan sequence is

$$x^3 - x - 1 = 0,$$

has roots $\alpha, \beta, \gamma = \overline{\beta}$, where

$$\alpha = \frac{r_1 + r_2}{6}, \quad \beta = \frac{-r_1 - r_2 + i\sqrt{3}(r_1 - r_2)}{12},$$

and

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \text{ and } r_2 = \sqrt[3]{108 - 12\sqrt{69}}.$$

Further, Binet's formula is

$$P_k = a\alpha^k + b\beta^k + c\gamma^k, \quad \text{for all } k \geq 0, \tag{2.1}$$

where

$$\begin{aligned} a &= \frac{(1-\beta)(1-\gamma)}{(\alpha-\beta)(\alpha-\gamma)} = \frac{1+\alpha}{-\alpha^2+3\alpha+1}, \\ b &= \frac{(1-\alpha)(1-\gamma)}{(\beta-\alpha)(\beta-\gamma)} = \frac{1+\beta}{-\beta^2+3\beta+1}, \\ c &= \frac{(1-\alpha)(1-\beta)}{(\gamma-\alpha)(\gamma-\beta)} = \frac{1+\gamma}{-\gamma^2+3\gamma+1} = \overline{b}. \end{aligned} \tag{2.2}$$

Numerically, we have

$$\begin{aligned} &1.32 < \alpha < 1.33, \\ &0.86 < |\beta| = |\gamma| = \alpha^{-1/2} < 0.87, \\ &0.72 < a < 0.73, \\ &0.24 < |b| = |c| < 0.25. \end{aligned} \tag{2.3}$$

Using induction, we can prove that

$$\alpha^{k-2} \leq \mathcal{P}_k \leq \alpha^{k-1}, \tag{2.4}$$

for all $k \geq 4$.

On the other hand, let $(\delta, \eta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$ be the roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci sequence $\{F_k\}_{n\geq 0}$. The Binet formula for F_k

$$F_k = \frac{\delta^k - \eta^k}{\sqrt{5}} \quad \text{holds for all } k \geq 0. \tag{2.5}$$

This implies easily that the inequality

$$\delta^{k-2} \leq F_k \leq \delta^{k-1} \tag{2.6}$$

holds for all positive integers k.

2.2. A lower bound for linear forms in logarithms

The next tools are related to the transcendental approach to solve Diophantine equations. For a non-zero algebraic number γ of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^{d}\left(X - \gamma^{(j)}\right)$, we denote by

$$h(\gamma) = \frac{1}{d}\left(\log|a| + \sum_{j=1}^{d} \log\max\left(1, \left|\gamma^{(j)}\right|\right)\right)$$

the usual absolute logarithmic height of γ.

Lemma 2.1. *Let $\gamma_1, \ldots, \gamma_s$ be a real algebraic numbers and let $b_1, \ldots, b_s$ be nonzero rational integer numbers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_s)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j = \max\{Dh(\gamma_j), |\log\gamma_j|, 0.16\} \text{ for } j = 1, \ldots, s.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_s|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_s^{b_s} \neq 1$, then

$$|\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1| \geq \exp(-C(s, D)(1 + \log B)A_1 \cdots A_s),$$

where $C(s, D) := 1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2(1 + \log D)$.

2.3. A generalized result of Baker-Davenport

Lemma 2.2. *Assume that τ and μ are real numbers and M is a positive integer. Let p/q be the convergent of the continued fraction of the irrational τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon = ||\mu q|| - M \cdot ||\tau q||$, where $||\cdot||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality*

$$0 < m\tau - n + \mu < AB^{-k}$$

in positive integers m, n and k with

$$m \leq M \quad \text{and} \quad k \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

3. Proof of Theorem 1.1

Assume that there exist positive integers n, m, n_1, m_1 such that $(n, m) \neq (n_1, m_1)$, and

$$F_n - \mathcal{P}_m = F_{n_1} - \mathcal{P}_{m_1}.$$

Because of the symmetry, we can assume that $m \geq m_1$. If $m = m_1$, then $F_n = F_{n_1}$, so $(n, m) = (n_1, m_1)$, contradicting our assumption. Thus, $m > m_1$. Since

$$F_n - F_{n_1} = \mathcal{P}_m - \mathcal{P}_{m_1}, \tag{3.1}$$

and the right-hand side is positive, we get that the left-hand side is also positive and so $n > n_1$. Thus, $n \geq 2$ and $n_1 \geq 1$. Using the Binet's formulas (2.5) and (2.1), the equation (3.1) implies that

$$\delta^{n-4} \leq F_{n-2} \leq F_n - F_{n_1} = \mathcal{P}_m - \mathcal{P}_{m_1} < \alpha^{m-1}, \tag{3.2a}$$

$$\delta^{n-1} \geq F_n > F_n - F_{n_1} = \mathcal{P}_m - \mathcal{P}_{m_1} = \mathcal{P}_{m-5} \geq \alpha^{m-7}, \tag{3.2b}$$

therefore

$$1 + \left(\frac{\log\alpha}{\log\delta}\right)(m-1) < n < \left(\frac{\log\alpha}{\log\delta}\right)(m-7) + 4, \tag{3.3}$$

where $\dfrac{\log\alpha}{\log\delta} = 0.5843\ldots$. If $n < 300$, then $m \leq 190$. We ran a computer program for $2 \leq n_1 < n \leq 300$ and $1 \leq m_1 < m < 190$ and found only the solutions listed in the (3.2) at the end of the paper. From now, we assume that $n \geq 300$.

Note that the inequality (3.3) implies that $m < 2n$. So, to solve equation (3.1), we need an upper bound for n.

3.1. Bounding n

Note that using the numerical inequalities (2.3) we have

$$\frac{|\eta|^n}{\sqrt{5}} + \frac{|\eta|^{n_1}}{\sqrt{5}} + |b||\beta|^m + |c||\gamma|^m + |b||\beta|^{m_1} + |c||\gamma|^{m_1} < 1.9. \tag{3.4}$$

Using the Binet formulas in the Diophantine equation (3.1), we get

$$\begin{aligned}
\left|\frac{\delta^n}{\sqrt{5}} - a\alpha^m\right| &= \left|\frac{\eta^n}{\sqrt{5}} + \frac{\delta^{n_1} - \eta^{n_1}}{\sqrt{5}} + (b\beta^m + c\gamma^m) - (a\alpha^{m_1} + b\beta^{m_1} + c\gamma^{m_1})\right| \\
&\leq \frac{\delta^{n_1}}{\sqrt{5}} + a\alpha^{m_1} + \frac{|\eta|^n}{\sqrt{5}} + \frac{|\eta|^{n_1}}{\sqrt{5}} + |b||\beta|^m + |c||\gamma|^m + |b||\beta|^{m_1} + |c||\gamma|^{m_1} \\
&< \frac{\delta^{n_1}}{\sqrt{5}} + a\alpha^{m_1} + 1.9 \\
&< 3.08 \max\{\delta^{n_1}, \alpha^{m_1}\}.
\end{aligned}$$

Dividing through by $a\alpha^m$ and using the relation (3.2a), we obtain

$$\left|(\sqrt{5}a)^{-1}\delta^n\alpha^{-m}-1\right| < \max\left\{\frac{3.08}{a\alpha^m}\delta^{n_1}, \frac{3.08}{a}\alpha^{m_1-m}\right\} < \max\left\{3.24\frac{\delta^{n_1}}{\delta^{n-4}}, 4.28\alpha^{m_1-m}\right\}.$$

Hence, we get

$$\left|(\sqrt{5}a)^{-1}\delta^n\alpha^{-m}-1\right| < \max\{\delta^{n_1-n+6}, \alpha^{m_1-m+3}\}. \tag{3.5}$$

For the left-hand side, we apply Theorem 2.1 with the data

$$s=3, \quad \gamma_1=\sqrt{5}a, \quad \gamma_2=\delta, \quad \gamma_3=\alpha, \quad b_1=-1, \quad b_2=n, \quad b_3=-m.$$

Throughout we work with $\mathbb{K} := \mathbb{Q}(\sqrt{5}, \alpha)$ with $D=6$. Since $\max\{1,n,m\} \leq 2n$ we take $B := 2n$. We have

$$h(\gamma_2)=\frac{\log\delta}{2} \quad \text{and} \quad h(\gamma_3)=\frac{\log\alpha}{3}.$$

Further, the minimal polynomial of γ_1 is $529x^6-1265x^4-250x^2-125$, then

$$h(\gamma_1) \approx 1.204.$$

Thus, we can take

$$A_1=7.23, \quad A_2=3\log\delta, \quad A_3=2\log\alpha.$$

Put

$$\Lambda=(\sqrt{5}a)^{-1}\delta^n\alpha^{-m}-1.$$

If $\Lambda=0$, then $\delta^n(\alpha^{-1})^m=\sqrt{5}a$, which is false, since $\delta^n(\alpha^{-1})^m \in \mathcal{O}_{\mathbb{K}}$ whereas $\sqrt{5}a$ does not, as can be observed immediately from its minimal polynomial. Thus, $\Lambda \neq 0$. Then, by Lemma 2.1, the left-hand side of (3.5) is bounded as

$$\log|\Lambda| > -1.4\cdot 30^6\cdot 3^{4.5}\cdot 6^2(1+\log 6)(1+\log 2n)(7.23)(3\log\delta)(2\log\alpha).$$

Comparing with (3.5), we get

$$\min\{(n-n_1-6)\log\delta, (m-m_1-3)\log\alpha\} < 8.45\times 10^{13}(1+\log 2n),$$

wich gives

$$\min\{(n-n_1)\log\delta, (m-m_1)\log\alpha\} < 8.45\times 10^{13}(1+\log 2n).$$

Now the argument splits into two cases.

Case 1. $\min\{(n-n_1)\log\delta, (m-m_1)\log\alpha\} = (n-n_1)\log\delta$.

In this case, we rewrite (3.1) as

$$\left|\left(\frac{\delta^{n-n_1}-1}{\sqrt{5}}\right)\delta^{n_1}-a\alpha^m\right| = \left|-a\alpha^{m_1}+\frac{\eta^n}{\sqrt{5}}-\frac{\eta^{n_1}}{\sqrt{5}}+(b\beta^m+c\gamma^m)-(b\beta^{m_1}+c\gamma^{m_1})\right|$$

by using (3.4) and dividing by α^m, we obtain

$$\left|\left(\frac{\delta^{n-n_1}-1}{\sqrt{5}a}\right)\delta^{n_1}\alpha^{-m}-1\right| < 3.65\alpha^{m_1-m}. \tag{3.6}$$

We put

$$\Lambda_1 = \left(\frac{\delta^{n-n_1}-1}{a\sqrt{5}}\right)\delta^{n_1}\alpha^{-m} - 1.$$

Clearly, $\Lambda_1 \neq 0$, for if $\Lambda_1 = 0$, then $\delta^n - \delta^{n_1} = \sqrt{5}a\alpha^m$. This is impossible if $\sqrt{5}a\alpha^m \in \mathbb{Q}(\sqrt{5}, \alpha)$ but $\notin \mathbb{Q}(\sqrt{5})$. Therefore, let us assume that $\sqrt{5}a\alpha^m \in \mathbb{Q}(\sqrt{5})$. Since $a\alpha^m \in \mathbb{Q}(\alpha)$ and $\mathbb{Q}(\alpha) \cap \mathbb{Q}(\sqrt{5}) = \mathbb{Q}$, we deduce from $\sqrt{5}a\alpha^m \in \mathbb{Q}(\sqrt{5})$ that we have $\sqrt{5}a\alpha^m = y\sqrt{5}$ for some $y \in \mathbb{Q}$. Let $\sigma \neq id$ be the unique non trivial $\mathbb{Q}$-automorphism over $\mathbb{Q}(\sqrt{5})$. Then, we get

$$\delta^n - \delta^{n_1} = \sqrt{5}a\alpha^m = y\sqrt{5} = -\sigma(\sqrt{5}a\alpha^m) = -\sigma(\delta^n - \delta^{n_1}) = \eta^{n_1} - \eta^n.$$

However, the absolute value of the left-hand side is at least $\delta^n - \delta^{n_1} \geq \delta^{n-2} \geq \delta^{\cdots} > 2$, while the absolute value of right-hand side is at most $|\eta^{n_1} - \eta^n| \leq |\eta|^{n_1} + |\eta|^n < 2$. By this obvious contradiction we conclude that $\Lambda_1 \neq 0$.

We apply Lemma 2.1 by taking $s = 3$, and

$$\gamma_1 = \frac{\delta^{n-n_1}-1}{\sqrt{5}a}, \quad \gamma_2 = \delta, \quad \gamma_3 = \alpha, \quad b_1 = 1, \quad b_2 = n_1, \quad b_3 = -m.$$

On the other hand, the minimal polynomial of a is $23x^3 - 23x^2 + 6x - 1$ and has roots a, b, c. Since $|b| = |c| < 1$ and $a < 1$, then $h(a) = \dfrac{\log 23}{3}$.

Thus, we obtain

$$\begin{aligned} h(\gamma_1) &\leq h\left(\frac{\delta^{n-n_1}+1}{\sqrt{5}}\right) + h(a) \\ &\leq (n-n_1)h(\delta) + h(\sqrt{5}) + h(a) + \log(2) \\ &< \frac{1}{2}(n-n_1)\log\delta + \log(\sqrt{5}) + \frac{\log 23}{3} + \log(2) \\ &< 4.22 \times 10^{13} \cdot (1 + \log 2n). \end{aligned} \tag{3.7}$$

So, we can take $A_1 := 2.53 \times 10^{14}(1 + \log 2n)$. Further, as before, we can take $A_2 := 3\log\delta$ and $A_3 := 2\log\alpha$. Finally, since $\max\{1, n_1, m\} \leq 2n$, we can take $B := 2n$. We then get that

$$\log|\Lambda_1| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 6^2(1 + \log 6)(1 + \log 2n) \times (2.53 \times 10^{14}(1 + \log 2n))(3\log\delta)(2\log\alpha).$$

Thus,

$$\log|\Lambda_1| > -2.96 \cdot 10^{27}(1 + \log 2n)^2.$$

Comparing this with (3.6), we get that

$$(m - m_1)\log\alpha < 2.96 \cdot 10^{27}(1 + \log 2n)^2.$$

Case 2. $\min\{(n-n_1)\log\delta, (m-m_1)\log\alpha\} = (m-m_1)\log\alpha$.

In this case, we rewrite (3.1) as

$$\left|\frac{\delta^n}{\sqrt{5}} - a\alpha^m + a\alpha^{m_1}\right| = \left|\frac{\eta^n}{\sqrt{5}} + \frac{\delta^{n_1} - \eta^{n_1}}{\sqrt{5}} + (b\beta^m + c\gamma^m) - (b\beta^{m_1} + c\gamma^{m_1})\right|$$

so

$$\left|\frac{\delta^n\alpha^{-m_1}}{\sqrt{5}a(\alpha^{m-m_1}-1)} - 1\right| < \frac{2.35}{\sqrt{5}a(1-\alpha^{m_1-m})\alpha}\frac{\delta^{n_1}}{\alpha^{m-1}} < 17\delta^{n_1-n+4}. \tag{3.8}$$

Let

$$\Lambda_2 = (\sqrt{5}a(\alpha^{m-m_1}-1))^{-1}\delta^n\alpha^{-m_1} - 1.$$

Clearly, $\Lambda_2 \neq 0$, for if $\Lambda_2 = 0$ implies $\delta^{2n} = 5\alpha^{2m_1}a^2(\alpha^{m-m_1}-1)^2$.,However, $\delta^{2n} \in \mathbb{Q}(\sqrt{5})\backslash\mathbb{Q}$, whereas $5\alpha^{2m_1}a^2(\alpha^{m-m_1}-1)^2 \in \mathbb{Q}(\alpha)$, which is not possible.

We apply again Lemma 2.1. In this application, we take again $s = 3$, and

$$\gamma_1 = \sqrt{5}a(\alpha^{m-m_1}-1), \quad \gamma_2 = \delta, \quad \gamma_3 = \alpha, \quad b_1 = -1, \quad b_2 = n, \quad b_3 = -m_1.$$

We have

$$\begin{aligned} h(\alpha^{m-m_1}-1) &\leq h(\alpha^{m-m_1}) + h(-1) + \log 2 = (m-m_1)h(\alpha) + \log 2 \\ &= \frac{(m-m_1)\log\alpha}{3} + \log 2 < 9.51 \times 10^{13}(1+\log 2n). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} h(\gamma_1) &< 2.82 \times 10^{13}(1+\log 2n) + \frac{\log 23}{3} + \log\sqrt{5} \\ &< 2.82 \times 10^{13}(1+\log 2n). \end{aligned}$$

So, we can take $A_1 := 1.69 \times 10^{14}(1+\log 2n)$. Further, as before, we can take $A_2 := 3\log\delta$ and $A_3 := 2\log\alpha$. Finally, since $\max\{1, n, m_1+1\} \leq 2n$, we can take $B := 2n$.

We then get that

$$\log|\Lambda_2| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 6^2(1+\log 6)(1+\log 2n) \times (1.69 \times 10^{14}(1+\log 2n))(3\log\delta)(2\log\alpha).$$

Thus,

$$\log|\Lambda_1| > -1.97 \cdot 10^{27}(1+\log 2n)^2.$$

Comparing this with (3.8), we get that

$$(n-n_1)\log\delta < 1.97 \cdot 10^{27}(1+\log 2n)^2.$$

Thus, in both Case 1 and Case 2, we have

$$\min\{(n-n_1)\log\delta, (m-m_1)\log\alpha\} < 8.45 \times 10^{13}(1+\log 2n) \tag{3.9a}$$

$$\max\{(n-n_1)\log\delta, (m-m_1)\log\alpha\} < 2.96 \cdot 10^{27}(1+\log 2n)^2. \tag{3.9b}$$

We now finally rewrite equation (3.1) as

$$\left|\frac{\delta^n}{\sqrt{5}} - \frac{\delta^{n_1}}{\sqrt{5}} - a\alpha^m + a\alpha^{m_1}\right| = \left|\frac{\delta^n}{\sqrt{5}} - \frac{\delta^{n_1}}{\sqrt{5}} + (b\beta^m + c\gamma^m) - (b\beta^{m_1} + c\gamma^{m_1})\right| < 1.9.$$

Dividing both sides by $a\alpha^{m_1}(\alpha^{m-m_1}-1)$, we get

$$\left|\left(\frac{\delta^{n-n_1}-1}{\sqrt{5}a(\alpha^{m-m_1}-1)}\right)\delta^{n_1}\alpha^{-m_1} - 1\right| < \frac{5.84}{a(1-\alpha^{m_1-m})\alpha}\frac{1}{\alpha^{m-1}} < 13.8\delta^{4-n}. \tag{3.10}$$

To find a lower-bound on the left-hand side, we use again Lemma 2.1 with $s = 3$, and

$$\gamma_1 = \frac{\delta^{n-n_1}-1}{\sqrt{5}a(\alpha^{m-m_1}-1)}, \quad \gamma_2 = \delta, \quad \gamma_3 = \alpha, \quad b_1 = 1, \quad b_2 = n_1, \quad b_3 = -m_1.$$

Using $h(x/y) = h(x) + h(y)$ for any two nonzero algebraic numbers x and y, we have

$$\begin{aligned} h(\gamma_1) &\leq h\left(\frac{\delta^{n-n_1}-1}{\sqrt{5}a}\right) + h(\alpha^{m-m_1}-1) \\ &< \frac{1}{2}(n-n_1)\log\delta + \log\sqrt{5} + \frac{\log 23}{3} + \frac{(m-m_1)\log\alpha}{3} + \log 2 \\ &< 2.47 \cdot 10^{27}(1+\log 2n)^2, \end{aligned}$$

where in the above chain of inequalities, we used the argument from (3.7) as well as the bound (3.9b). So, we can take $A_1 := 1.78 \cdot 10^{28}(1+\log 2n)^2$ and certainly $A_2 := 3\log\delta$ and $A_3 := 2\log\alpha$. Using similar arguments as in the proof that $\Lambda_1 \neq 0$ we show that if we put

$$\Lambda_3 = \left(\frac{\delta^{n-n_1}-1}{\sqrt{5}a(\alpha^{m-m_1}-1)}\right)\delta^{n_1}\alpha^{-m_1} - 1,$$

then $\Lambda_3 \neq 0$. Lemma 2.1 gives

$$\log|\Lambda_3| > -1.4\cdot 30^6 \cdot 3^{4.5}\cdot 6^2(1+\log 6)(1+\log 2n)\times(1.78\cdot 10^{28}(1+\log 2n)^2)(3\log\delta)(2\log\alpha),$$

which together with (3.10) gives

$$(n-4) < 2.08\cdot 10^{41}(1+\log 2n)^3,$$

leading to $n < 2.83\cdot 10^{47}$.

3.2. Reducing n

We now need to reduce the above bound for n and to do so we make use of Lemma 2.2 several times and each time $M := 2.83\cdot 10^{47}$. To begin with, we return to (3.5) and put

$$\Gamma := n\log\delta - m\log\alpha - \log(\sqrt{5}a).$$

For technical reasons we assume that $\min\{n-n_1, m-m_1\} \geq 20$. We go back to the inequalities for $\Lambda,\ \Lambda_1,\ \Lambda_2$. Since we assume that $\min\{n-n_1, m-m_1\} \geq 20$ we get $|e^\Gamma - 1| = |\Lambda| < \frac{1}{4}$. Hence, $|\Lambda| < \frac{1}{2}$ and since the inequality $|x| < 2|e^x - 1|$ holds for all $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, we get

$$|\Gamma| < 2\max\{\delta^{n_1-n+6}, \alpha^{m_1-m+3}\} \leq \max\{\delta^{n_1-n+8}, \alpha^{m_1-m+6}\}.$$

Assume $\Gamma > 0$. We then have the inequality

$$\begin{aligned} 0 < n\left(\frac{\log\delta}{\log\alpha}\right) - m - \frac{\log(1/(\sqrt{5}a))}{\log\alpha} &< \max\left\{\frac{\delta^8}{\log\alpha}\delta^{-(n-n_1)}, \frac{\alpha^6}{\log\alpha}\alpha^{-(m-m_1)}\right\} \\ &< \max\{170\cdot\delta^{-(n-n_1)}, 20\cdot\alpha^{-(m-m_1)}\}. \end{aligned}$$

We apply Lemma 2.2 with

$$\tau = \frac{\log\delta}{\log\alpha}, \quad \mu = \frac{\log(1/(\sqrt{5}a))}{\log\alpha}, \quad (A,B) = (170,\delta) \text{ or } (20,\alpha).$$

Let $\tau = [a_0, a_1, \ldots] = [1; 1, 2, 2, 6, 2, 1, 2, 1, 2, 1, 1, 11, 1, 2, 3, 1, 7, 37, 4, \ldots]$ be the continued fraction of τ. We choose consider the 98-th convergent

$$\frac{p}{q} = \frac{p_{98}}{q_{98}} = \frac{780930677042238317990327545345035018596353914355517}{456342430763874570970465280842084901475949683089 75}.$$

If satisfied $q = q_{98} > 6M$. Further, it yields $\varepsilon > 0.35$, and therefore either

$$n - n_1 \leq \frac{\log(170q/\varepsilon)}{\log\delta} < 250, \text{ or } m - m_1 \leq \frac{\log(20q/\varepsilon)}{\log\alpha} < 420.$$

In the case of $\Gamma < 0$, we consider the following inequality instead:

$$\begin{aligned} m\left(\frac{\log\alpha}{\log\delta}\right) - n + \frac{\log(\sqrt{5}a)}{\log\delta} &< \max\left\{\frac{\delta^9}{\log\delta}\alpha^{-(n-n_1)}, \frac{\alpha^{12}}{\log\delta}\alpha^{-(m-m_1)}\right\} \\ &< \max\{98\cdot\delta^{-(n-n_1)}, 12\cdot\alpha^{-(m-m_1)}\}, \end{aligned}$$

instead and apply Lemma 2.2 with

$$\tau = \frac{\log\alpha}{\log\delta}, \quad \mu = \frac{\log(\sqrt{5}a)}{\log\delta}, \quad (A,B) = (98,\delta) \text{ or } (12,\alpha).$$

Let $\tau = [a_0, a_1, \ldots] = [0; 1, 1, 2, 2, 6, 2, 1, 2, 1, 2, 1, 1, 11, 1, 2, 3, 1, 7, 37, \ldots]$ be the continued fraction of τ (note that the current τ is just the reciprocal of the previous τ). We consider the 98-th convergent

$$\frac{p}{q} = \frac{p_{98}}{q_{98}} = \frac{100054033487924293472614176116281329403488597772 2}{1712206861451396832387596141129961335575127483549}.$$

which satisfies $q = q_{98} > 6M$. This yields again $\varepsilon > 0.47$, and therefore either

$$n - n_1 \leq \frac{\log(98q/\varepsilon)}{\log\delta} < 242, \text{ or } m - m_1 \leq \frac{\log(12q/\varepsilon)}{\log\alpha} < 406.$$

These bounds agree with the bounds obtained in the case that $\Gamma > 0$. As a conclusion, we have either $n - n_1 \leq 250$ or $m - m_1 \leq 420$ whenever $\Gamma \neq 0$.

Now, we have to distinguish between the cases $n - n_1 \leq 250$ and $m - m_1 \leq 420$. First, let assume that $n - n_1 \leq 250$. In this case, we consider inequality (3.6) and assume that $m - m_1 \geq 20$. We put

$$\Gamma_1 = n_1 \log\delta - m\log\alpha + \log\left(\frac{\delta^{n-n_1} - 1}{\sqrt{5}a}\right).$$

Then inequality (3.6) implies that

$$|\Gamma_1| < 7.3\alpha^{m_1 - m}.$$

If we further assume that $\Gamma_1 > 0$, we then get

$$0 < n_1\left(\frac{\log\delta}{\log\alpha}\right) - m + \frac{\log((\delta^{n-n_1} - 1)/(\sqrt{5}a)}{\log\alpha} < 26 \cdot \alpha^{-(m-m_1)}.$$

Again we apply Lemma 2.2 with the same τ as in the case when $\Gamma > 0$. We use the 98-th convergent $p/q = p_{98}/q_{98}$ of τ as before. But in this case we choose $(A, B) := (26, \alpha)$ and use

$$\mu_k = \frac{\log((\delta^k - 1)/(\sqrt{5}a))}{\log\alpha},$$

instead of μ for each possible value of $k := n - n_1 \in [1, 2, \ldots 250]$. For the remaining values of k, we get $\varepsilon > 0.0004$. Hence, by Lemma 2.2, we get

$$m - m_1 < \frac{\log(26q/0.0004)}{\log\alpha} < 446.$$

Thus, $n - n_1 \leq 250$ implies $m - m_1 \leq 446$.

In the case that $\Gamma_1 < 0$ we follow the ideas from the case that $\Gamma_1 > 0$. We use the same τ as in the case that $\Gamma < 0$ but instead of μ we take

$$\mu_k = \frac{\log((\sqrt{5}a)/(\delta^k - 1))}{\log\delta},$$

for each possible value of $n - n_1 = k = 1, 2, \ldots, 250$. Using Lemma 2.2 with this setting we also obtain in this case that $n - n_1 \leq 250$ implies $m - m_1 \leq 429$.

Now let us turn to the case that $m - m_1 \leq 420$ and let us consider inequality (3.8). We put

$$\Gamma_2 = n\log\delta - m_1\log\alpha + \log(1/(\sqrt{5}a(\alpha^{m-m_1} - 1))),$$

and we assume that $n - n_1 \geq 20$. We then have

$$|\Gamma_2| < \frac{34\delta^4}{\alpha^{n-n_1}}.$$

Assuming $\Gamma_2 > 0$, we get

$$0 < n\left(\frac{\log\delta}{\log\alpha}\right) - m_1 + \frac{\log((1/(\sqrt{5}a(\alpha^{m-m_1}-1)))}{\log\alpha} < \frac{34\delta^4}{(\log\alpha)\alpha^{n-n_1}} < 830\delta^{-(n-n_1)}.$$

We apply again Lemma 2.2 with the same τ, q, M, $(A,B) := (830,\delta)$ and

$$\mu_k = \frac{\log((1/(\sqrt{5}a(\alpha^k-1)))}{\log\alpha} \quad \text{for } k = 1,2,\ldots 420.$$

We get $\varepsilon > 0.00077$, therefore

$$n - n_1 < \frac{\log(830q_{98}/0.00077)}{\log\delta} < 263.$$

A similar conclusion is reached when $\Gamma_2 < 0$. To conclude, we first get that either $n-n_1 \leq 250$ or $m-m_1 \leq 446$. If $n-n_1 \leq 250$, then $m-m_1 \leq 446$, and if $m-m_1 \leq 420$ then $n-n_1 \leq 263$. In conclusion, we always have $n-n_1 < 263$ and $m-m_1 < 446$.

Finally we go to (3.10). We put

$$\Gamma_3 = n_1\log\delta - m_1\log\alpha + \log\left(\frac{\delta^{n-n_1}-1}{\sqrt{5}a(\alpha^{m-m_1}-1)}\right).$$

Since $n \geq 300$, inequality (3.10) implies that

$$|\Gamma_3| < \frac{17}{\delta^{n-4}}.$$

Assume that $\Gamma_3 > 0$. Then

$$0 < n_1\left(\frac{\log\delta}{\log\alpha}\right) - m_1 + \frac{\log((\delta^k-1)/(\sqrt{5}a(\alpha^l-1)))}{\log\alpha} < 390\delta^n,$$

where $(k,l) := (n-n_1, m-m_1)$. We apply again Lemma 2.2 with the same τ, M, q, $(A,B) := (390,\delta)$ and

$$\mu_{k,l} = \frac{\log((\delta^k-1)/(\sqrt{5}a(\alpha^l-1)))}{\log\alpha} \quad \text{for } 1 \leq k \leq 264,\ 1 \leq l \leq 446.$$

We consider the 99th convergent $\frac{p_{99}}{q_{99}}$. For all pairs (k,l) we get that $\varepsilon > 2\times 10^{-5}$. Thus, Lemma 2.2 yields that

$$n < \frac{\log(390\times q_{99}/\varepsilon)}{\log\delta} < 274.$$

Theorem 1.1 is therefore proved.

On the next page is presented the table that gives the couples for which we obtain the different representations of c on the form $\mathcal{P}_m - F_n = c$.

4. Acknowledgement

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c	(m,n)
-226	$(8,13),(19,14)$
-82	$(8,11),(19,13)$
-52	$(5,10),(14,11)$
-30	$(6,9),(18,12)$
-27	$(8,9),(13,10)$
-18	$(5,8),(11,9),(14,10)$
-9	$(6,7),(10,8)$
-6	$(4,6),(8,7),(13,9),(15,10)$
-5	$(5,6),(11,8)$
-4	$(6,6),(9,7)$
-3	$(4,5),(7,6),(17,11)$
-1	$(4,4),(6,5),(8,6),(10,7)$
0	$(4,3),(5,4),(7,5),(12,8)$
1	$(4,2),(5,3),(6,4),(9,6)$
2	$(5,2),(6,3),(7,4),(8,5)$
3	$(6,2),(7,3),(11,7),(14,9)$
4	$(7,2),(8,4),(9,5),(10,6)$
6	$(8,2),(9,4),(24,15)$
7	$(9,3),(10,5),(13,8),(19,12)$
8	$(9,2),(11,6),(12,7)$
10	$(10,3),(16,10)$
11	$(10,2),(11,5)$
13	$(11,4),(12,6)$
15	$(11,2),(13,7),(15,9)$
16	$(12,5),(14,8)$
20	$(12,2),(13,6)$
25	$(13,4),(18,11)$
31	$(16,9),(17,10)$
32	$(14,5),(21,13)$
36	$(14,2),(15,7)$
44	$(15,5),(16,8)$
52	$(16,7),(17,9)$
62	$(16,4),(19,11)$
111	$(18,4),(20,11)$
262	$(21,4),(22,11)$

Table 1: Representations

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Interval-valued intuitionistic fuzzy linear transformation

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Abstract. In this paper we introduce the concept of interval-valued intuitionistic fuzzy relations(in biefly IVIFR) and composition of IVIFR-equations. Then we continued it to interval-valued intuitionistic fuzzy linear transformation(in brief IVIFL-transformation) and discussed its properties. Also introduced the concept of composition of IVIFL-transformations.

AMS Subject Classifications: 03E72, 15A03.

Keywords: IVIF-Relations, Composition of IVIF-Relations, IVIFL-transformations, $IVIF_0L$-transformations, $IVIF_IL$-transformations.

Contents

1. Introduction and Background

Zadeh [19] introduced fuzzy set and properties of fuzzy sets. At [1], Atanassov introduced the intuitionistic fuzzy set which was broadened to interval-valued intuitionistic fuzzy set by Atanassov and Gargov [2] whose membership and nonmembership functions are intervals. In 2011, Lin and Huang [5, 8] introduced the basic concepts of (T, S)-composition matrix and (T, S)- interval-valued intuitionistic fuzzy equivalence matrix. Initially, Shyamal and Pal [16] introduced interval-valued fuzzy matrix. Then Intuitionistic fuzzy matrices introduced by Madhumangal Pal et al. [9]. Pal and Susanta K. Khan [10] introduced some basic operators in interval-valued intuitionistic fuzzy matrices. Xu and Yager [18] introduced intuitionistic and interval-valued intuitionistic fuzzy preference relations and their measures of similarity in decision making methods. Also

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Ze-shui Xu and Jian chen [20], approaches the decision making methods using interval-valued intuitionistic judgement matrices.

In 1977, Katsaras and Liu [6] introduced fuzzy vector and fuzzy topological vector spaces. In 1994, Terao and Kitsunezaki [17] introduced fuzzy sets and linear mappings on vector spaces. Kim and Roush [7] presents some basic concepts of generalized fuzzy matrices. Bhowmik and Pal [4] described and studied the concept of generalized intuitionistic fuzzy matrices.

Narayanan et al. [13] introduced the notion of intuitionistic fuzzy continuous mappings and intuitionistic fuzzy bounded linear operators from one intuitionistic fuzzy n-normed linear space to another. Recently Moumitha Chiney and Samanta studied and introduced the concept of intuitionistic fuzzy vector spaces [12]. And Santhi and Udhayarani introduced and studied the concept of [15] interval-valued intuitionistic fuzzy vector spaces. Then intuitionistic fuzzy linear transformations described by Meenakshi and Gandhimathi [11] and Rajkumar Pradhan and Madhumangal Pal [14].

In this paper, we introduced the concept of interval-valued intuitionistic fuzzy linear transformations and scrutinized some of its properties. In section 2, some basic concepts and properties are reviewed. In section 3, we introduced the concept of $IVIFR$-equations and its composition. Also introduced the concept of $IVIFL$-transformations.

2. Preliminaries

This section briefly discussed about the basic concepts of IVIFS which were used in the following sections.

Definition 2.1. ***Interval-valued fuzzy vector:*** *An interval valued fuzzy vector is an n-tuple of elements from an interval -valued fuzzy algebra. That is, an IVFV is of the form $(x_1, x_2, ..., x_n)$, where each element $x_i \in F$, $i = 1, 2, ..., n$.*

Definition 2.2. ***Interval-valued fuzzy vector space:*** *An interval-valued fuzzy vector space(IVFV Space) is a pair $(E, A(x))$, if E is a vector space in crisp sense and $A : E \to D[0,1]$ with the property, that for all $a, b \in F$ and $x, y \in E$, then,*
$\underline{A}(ax + by) \geq \underline{A}(x) \wedge \underline{A}(y)$ *and*
$\overline{A}(ax + by) \geq \overline{A}(x) \wedge \overline{A}(y)$.

Definition 2.3. ***Interval-valued intuitionistic fuzzy set:*** *Let $D[0,1]$ be the set of closed subintervals of the interval $[0,1]$ and $X(\neq \phi)$ be a given set. An interval-valued intuitionistic fuzzy set in X is described as, $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$, where $\mu_A(x) : X \to D[0,1]$, $\nu_A(x) : X \to D[0,1]$ with the condition $0 \leq sup(\mu_A(x)) + sup(\nu_A(x)) \leq 1$ for any $x \in X$. The intervals $\mu_A(x)$ and $\nu_A(x)$ denotes the degree of belongingness and the degree of nonbelongingness of the element x to the set A. Thus for each $x \in X$, $\mu_A(x)$ and $\nu_A(x)$ are closed intervals and their lower and upper end points are denoted by $\mu_{A_L}(x), \mu_{A_U}(x), \nu_{A_L}(x)$ and $\nu_{A_U}(x)$. We can denote it by:*

$$A = \{\langle x, [\mu_{A_L}(x), \mu_{A_U}(x)], [\nu_{A_L}(x), \nu_{A_U}(x)]\rangle / x \in X\},$$

where $0 \leq \mu_{A_U}(x) + \nu_{A_U}(x) \leq 1$, $\mu_{A_L}(x) \geq 0$, $\nu_{A_L}(x) \geq 0$. For each element x, we can compute the unknown degree(hesitancy degree) of an intuitionistic fuzzy interval of $x \in X$ in A defined as follows:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) = [1 - \mu_{A_U}(x) - \nu_{A_U}(x), 1 - \mu_{A_L}(x) - \nu_{A_L}(x)]$$

Especially, if $\mu_A(x) = \mu_{A_U}(x) = \mu_{A_L}(x)$ and $\nu_A(x) = \nu_{A_U}(x) = \nu_{A_L}(x)$, then the given IVIFS A is reduced to an ordinary intuitionistic fuzzy set.

Definition 2.4. *For any two IVIFSs* $A = \{\langle x, [\mu_{A_L}(x), \mu_{A_U}(x)], [\nu_{A_L}(x), \nu_{A_U}(x)]\rangle / x \in X\}$ *and* $B = \{\langle x, [\mu_{B_L}(x), \mu_{B_U}(x)], [\nu_{B_L}(x), \nu_{B_U}(x)]\rangle / x \in X\}$ *then the following two relations are explained as:*

1. $A \subseteq B$ *if and only if*
   (a) $\mu_{A_U}(x) \leq \mu_{B_U}(x)$,
   (b) $\mu_{A_L}(x) \leq \mu_{B_L}(x)$,
   (c) $\nu_{A_U}(x) \geq \nu_{B_U}(x)$,
   (d) $\nu_{A_L}(x) \geq \nu_{B_L}(x)$, *for any* $x \in X$.
2. $A = B$ *if and only if*
   (a) $\mu_{A_U}(x) = \mu_{B_U}(x)$,
   (b) $\mu_{A_L}(x) = \mu_{B_L}(x)$,
   (c) $\nu_{A_U}(x) = \nu_{B_U}(x)$,
   (d) $\nu_{A_L}(x) = \nu_{B_L}(x)$, *for any* $x \in X$.

Definition 2.5. ***Interval-valued Intuitionistic Fuzzy Vector Space:*** *The Mathematical system of interval-valued intuitionistic fuzzy algebra is defined with two binary operations '+' and '.' on the set* $\widetilde{V}$ *satisfying the following properties:*
Let $[x_L, x_U]$, $[y_L, y_U]$ *and* $[z_L, z_U]$ *be the elements of* $\widetilde{V}$.

1. $Idempotent : [x_L, x_U] + [x_L, x_U] = max\{[x_L, x_U], [x_L, x_U]\} = [x_L, x_U]$
2. $Commutative : [x_L, x_U] + [y_L, y_U] = [y_L, y_U] + [x_L, x_U]$
3. $Associativity : [x_L, x_U] + ([y_L, y_U] + [z_L, z_U]) = ([x_L, x_U] + [y_L, y_U]) + [z_L, z_U]$
4. *Absorption:*
   (a) $[x_L, x_U] + ([x_L, x_U].[y_L, y_U]) = [x_L, x_U]$
   (b) $[x_L, x_U].([x_L, x_U] + [y_L, y_U]) = [x_L, x_U]$
5. *Universal Bounds:*
   (a) $[x_L, x_U] + \phi = [x_L, x_U]$
   (b) $[x_L, x_U] + I = I$
   (c) $[x_L, x_U].\phi = \phi$
   (d) $[x_L, x_U].I = [x_L, x_U]$

where $\phi = \langle [0,0], [1,1] \rangle$ *is the zero element and* $I = \langle [1,1], [0,0] \rangle$, *is the identity element.*

Definition 2.6. *The pair* $(V, \langle [\mu_L(x), \mu_U(x)], [\nu_L(x), \nu_U(x)] \rangle) = \widetilde{V}$ *is said to be an interval-valued intuitionistic fuzzy vector space, if* $\alpha_{\mu_U} : \widetilde{V} \to D[0,1]$, $\alpha_{\nu_U} : \widetilde{V} \to D[0,1]$, $\alpha_{\mu_L} \geq 0$ *and* $\alpha_{\nu_L} \geq 0$ *with the property that for all* $\alpha, \beta \in \widetilde{V}$ *and* $x, y \in F$, *then*

1. $\left\langle [(\alpha_{\mu_{L_1}} + \beta_{\mu_{L_1}}), (\alpha_{\mu_{U_1}} + \beta_{\mu_{U_1}})][(\alpha_{\nu_{L_1}} + \beta_{\nu_{L_1}}), (\alpha_{\nu_{U_1}} + \beta_{\nu_{U_1}})] \right\rangle \in \widetilde{V}$
2. $\{\langle [((\alpha_L \wedge \mu_L), (\alpha_U \wedge \mu_U)), (((1-\alpha_L) \vee \nu_L, (1-\alpha_U) \vee \nu_U))] \rangle\} \in \widetilde{V}$,

where $\alpha_{\mu_{L_1}} + \beta_{\mu_{L_1}} = \alpha_{\mu_{L_1}} \vee \beta_{\mu_{L_1}}$,
$\alpha_{\mu_{U_1}} + \beta_{\mu_{U_1}} = \alpha_{\mu_{U_1}} \vee \beta_{\mu_{U_1}}$,
$\alpha_{\nu_{L_1}} + \beta_{\nu_{L_1}} = \alpha_{\nu_{L_1}} \wedge \beta_{\nu_{L_1}}$,
$\alpha_{\nu_{U_1}} + \beta_{\nu_{U_1}} = \alpha_{\nu_{U_1}} \wedge \beta_{\nu_{U_1}}$.

3. Main Results

In this section, the concept of interval-valued intuitionistic fuzzy linear realtional equations(in briefly IVIFLR-equation) was defined and its basic properties were discussed. Also, this section defines interval-valued intuitionistic fuzzy linear transformations(in brief IVIFL- transformations) on IVIFV- space and discuss its properties.

Definition 3.1. *For interval-valued intuitionistic fuzzy relation $R(x,y)$ we register the lower and upper end points of membership and non-membership value of x in relation with y under R defined by $[\mu_{RL},\mu_{RU}] = [\alpha_{\mu L},\alpha_{\mu U}]$ and $[\nu_{RL},\nu_{RU}] = [\alpha_{\nu L},\alpha_{\nu U}]$ and represented as $\langle[\alpha_{\mu L},\alpha_{\mu U}],[\alpha_{\nu L},\alpha_{\nu U}]\rangle$.*

Note 3.2. *An interval-valued intuitionistic fuzzy binary relation R (in brief IVIF-binary relation) can be represented as an interval-valued intuitionistic fuzzy matrix M_R (in brief IVIF-Matrix).*

Example 3.3. *If R is an $IVIF$-Relation with $X = \{X_1,X_2,X_3\}$ and $Y = \{Y_1,Y_2\}$ that indicates the relational concept 'that the element of set X are different kind of air-conditioning systems to the physical structure of the computer lab of the set Y'. The upper and lower end points of the membership and nonmembership values can be represented by the following IVIF-matrix,*

$$[M_{R_1}] = \begin{bmatrix} \langle[0.3,0.5],[0.2,0.4]\rangle & \langle[0.2,0.5],[0.1,0.5]\rangle & \langle[0.3,0.6],[0.1,0.2]\rangle \\ \langle[0.1,0.3],[0.2,0.5]\rangle & \langle[0.2,0.4],[0.1,0.3]\rangle & \langle[0.1,0.3],[0.2,0.5]\rangle \end{bmatrix}$$

Definition 3.4. *If $R_1(X,Y)$ and $R_2(Y,Z)$ are two IVIF-relations then its IVIF-composition is described by the max-min operation and signified as $R_1 \circ R_2$ with respect to IVIF-matrices of R_1 and R_2. That is, if there is two IVIF-binary relations $R_1(X,Y)$, $R_2(Y,Z)$ then $R(X,Z)$ defined on the sets $X = \{[x_i]/i \in N_s\}, i = 1,2,...,s$, $Y = \{[y_j]/j \in N_m\}, j = 1,2,...,m$, and $Z = \{[z_k]/k \in N_n\}, k = 1,2,...,n$, where N is the set of all positive integers. Let the corresponding IVIF-matrices be denoted by,*

$R_1 = [a_{ij}] = \left\langle\left[\mu_{a_{ijL}},\mu_{a_{ijU}}\right],\left[\nu_{a_{ijL}},\nu_{a_{ijU}}\right]\right\rangle$

$R_2 = [b_{jk}] = \left\langle\left[\mu_{b_{jkL}},\mu_{b_{jkU}}\right],\left[\nu_{b_{jkL}},\nu_{b_{jkU}}\right]\right\rangle$

$R = [r_{ik}] = \langle[\mu_{r_{ikL}},\mu_{r_{ikU}}],[\nu_{r_{ikL}},\nu_{r_{ikU}}]\rangle$ *then the IVIF-composition $R(X,Z)$ of $R_1(X,Y)$ and $R_2(Y,Z)$ is given by*

$$R_1 \circ R_2 = R \tag{3.1}$$

That is,

$$\left\langle \max_j\{\min\left[\left(\mu_{a_{ijL}},\mu_{b_{jkL}}\right),\left(\mu_{a_{ijU}},\mu_{b_{jkU}}\right)\right]\}, \min_j\{\max\left[\left(\nu_{a_{ijL}},\nu_{b_{jkL}}\right),\left(\nu_{a_{ijU}},\nu_{b_{jkU}}\right)\right]\}\right\rangle = \langle[\mu_{r_{ikL}},\mu_{r_{ikU}}],[\nu_{r_{ikL}},\nu_{r_{ikU}}]\rangle$$

where $i \in N_s, j \in N_m$ and $k \in N_n$.

The above equations renders IVIF-relational equations and we get it from performing the max-min operations on R_1 and R_2.

Example 3.5. *Let R_1 and R_2 be two IVIF-matrices and*

$$[M_{R_1}] = \begin{bmatrix} \langle[0.3,0.5],[0.2,0.4]\rangle & \langle[0.2,0.5],[0.1,0.5]\rangle & \langle[0.1,0.5],[0.1,0.2]\rangle \\ \langle[0.1,0.3],[0.2,0.5]\rangle & \langle[0.2,0.6],[0.1,0.3]\rangle & \langle[0.1,0.3],[0.2,0.5]\rangle \end{bmatrix}$$

$$[M_{R_2}] = \begin{bmatrix} \langle [0.2,0.5],[0.1,0.2] \rangle & \langle [0.3,0.6],[0.1,0.3] \rangle \\ \langle [0.2,0.6],[0.1,0.4] \rangle & \langle [0.1,0.5],[0.1,0.3] \rangle \\ \langle [0.2,0.7],[0.1,0.3] \rangle & \langle [0.1,0.6],[0.1,0.2] \rangle \end{bmatrix}$$

Then the composition of these two IVIF-matrices is,

$$[M_{R_1}] \circ [M_{R_2}] = [M]$$

$$[M] = \begin{bmatrix} \langle [0.2,0.5],[0.1,0.3] \rangle & \langle [0.3,0.5],[0.1,0.2] \rangle \\ \langle [0.2,0.6],[0.1,0.4] \rangle & \langle [0.1,0.5],[0.1,0.3] \rangle \end{bmatrix}$$

Definition 3.6. *If M_2 and M are given IVIF-matrices in*

$$M_1 \circ M_2 = M \tag{3.2}$$

then we can determine particular IVIF-matrices for M_1 which will be satisfy (3.2). Each of this particular IVIF-matrix for M_1 that satisfies (3.2) is called its IVIF-solution and the set

$$\mathscr{M}(M_2, M) = \{M_1 / M_1 \circ M_2 = M\} \tag{3.3}$$

denotes the IVIF-solution set.

Definition 3.7. *An element p of $\mathscr{M}(A,b)$ is called an IVIF-solution of the equation $Ay = b$ if $p = \left[\left\langle \left[x_{j_{\mu L}}, x_{j_{\mu U}}\right], \left[x_{j_{\nu L}}, x_{j_{\nu U}}\right]\right\rangle / j \in N_m\right]^T$ be defined as,*

$$p = \min \sigma(a_{jk}, b_k) \tag{3.4}$$

where

$$\sigma(a_{jk}, b_k) = \begin{cases} b_k & \text{if } a_{jk} > b_k \\ I & \text{otherwise} \end{cases}$$

$I = \langle [1,1],[0,0] \rangle$, $a_{jk} = \left\langle \left[a_{jk_{\mu L}}, a_{jk_{\mu U}}\right], \left[a_{jk_{\nu L}}, a_{jk_{\nu U}}\right]\right\rangle$ *and* $b_k = \left\langle \left[b_{k_{\mu L}}, b_{k_{\mu U}}\right], \left[b_{k_{\nu L}}, b_{k_{\nu U}}\right]\right\rangle$.

Example 3.8. *Let A and b be two IVIF-matrices,*

$$[A] = \begin{bmatrix} \langle [0.2,0.7],[0.1,0.2] \rangle & \langle [0.3,0.6],[0.1,0.3] \rangle \\ \langle [0.2,0.6],[0.1,0.4] \rangle & \langle [0.1,0.5],[0.1,0.3] \rangle \end{bmatrix}$$

and

$$[b] = \begin{bmatrix} \langle [0.2,0.7],[0.1,0.3] \rangle & \langle [0.1,0.6],[0.1,0.2] \rangle \end{bmatrix}^T$$

Using the above definition of IVIF-solution set, we get,
$p_1 = \min\{\sigma(a_{11}, b_{11}), \sigma(a_{12}, b_{12})\}$
$p_1 = \langle [0.1,0.6],[0.1,0.2] \rangle$
$p_2 = \min\{\sigma(a_{21}, b_{11}), \sigma(a_{22}, b_{12})\}$
$p_2 = \langle [1,1],[0,0] \rangle$
Then one of the IVIF-solution is $p = [\langle [0.1,0.6],[0.1,0.2] \rangle, \langle [1,1],[0,0] \rangle]$

Definition 3.9. *An IVIF-transformation $\widetilde{T}$ of $\widetilde{U}$ into $\widetilde{V}$ is called IVIFL-transformations if for every $\alpha, \beta \in \widetilde{V}$ and $x \in F$ then it satisfies the following conditions:*

1. $\widetilde{T}(\alpha + \beta) = \widetilde{T}(\alpha) + \widetilde{T}(\beta)$,
2. $\widetilde{T}(x\alpha) = x.\widetilde{T}(\alpha)$

Where $\alpha = \langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle$ *and* $\beta = \langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle$

Example 3.10. *Let* $\widetilde{V}^3$ *and* $\widetilde{V}^2$ *be an IVIFV-spaces over* F. *The IVIF-transformation* $\widetilde{T} : \widetilde{V}^3 \to \widetilde{V}^2$ *defined as* $\widetilde{T}(\alpha_1, \alpha_2, \alpha_3) = (\alpha_1, \alpha_2)$ *is an IVIFL- transformation.*

Proposition 3.11. *Let* $\widetilde{T}_1$ *and* $\widetilde{T}_2$ *be two IVIFL- transformations in an IVIFV-space* $\widetilde{V}$ *over* F, *and* $L(\widetilde{V})$ *be the set of all IVIFL-transformations defined on* $\widetilde{V}$, *then* $L(\widetilde{V})$ *is closed under addition and multiplication defined by,*

1. $(\widetilde{T}_1 + \widetilde{T}_2)(\alpha) = \widetilde{T}_1(\alpha) + \widetilde{T}_2(\alpha)$
2. $(x\widetilde{T}_1)(\alpha) = x\widetilde{T}_1(\alpha), \forall \widetilde{T}_1, \widetilde{T}_2 \in \widetilde{V}$ *and* $x \in F$.

Proof. Let $\widetilde{T}_1$ and $\widetilde{T}_2$ be two IVIFL-transformations in an IVIFV- space $\widetilde{V}$ over F.
To prove (1): $\widetilde{T}_1, \widetilde{T}_2 \in \widetilde{V}$ and $x \in F$. Now consider,

$$\begin{aligned}
(\widetilde{T}_1 + \widetilde{T}_2)(\alpha + \beta) &= \widetilde{T}_1(\alpha + \beta) + \widetilde{T}_2(\alpha + \beta) \\
&= \widetilde{T}_1(\langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle + \langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle) \\
&\quad + \widetilde{T}_2(\langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle + \langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle) \\
&= \widetilde{T}_1(\langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle) + \widetilde{T}_1(\langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle) \\
&\quad + \widetilde{T}_2(\langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle) + \widetilde{T}_2(\langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle) \\
&= (\widetilde{T}_1 + \widetilde{T}_2)(\langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle) \\
&\quad + (\widetilde{T}_1 + \widetilde{T}_2)(\langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle) \\
&= (\widetilde{T}_1 + \widetilde{T}_2)(\alpha) + (\widetilde{T}_1 + \widetilde{T}_2)(\beta)
\end{aligned}$$

$\forall \widetilde{T}_1, \widetilde{T}_2 \in L(\widetilde{V})$.

Now consider,

$$\begin{aligned}
(\widetilde{T}_1 + \widetilde{T}_2)(x\alpha) &= \widetilde{T}_1(x\alpha) + \widetilde{T}_2(x\alpha) \\
&= x\widetilde{T}_1(\alpha) + x\widetilde{T}_2(\alpha) \\
&= x(\widetilde{T}_1(\alpha) + \widetilde{T}_2(\alpha)) \\
&= x(\widetilde{T}_1 + \widetilde{T}_2)(\alpha)
\end{aligned}$$

for every $\widetilde{T}_1, \widetilde{T}_2 \in L(\widetilde{V})$ and $x \in F$
To prove (2): For $\alpha \in \widetilde{V}$ and $\widetilde{T} \in L(\widetilde{V})$,

$$\begin{aligned}
(x\widetilde{T})(\alpha + \beta) &= x\widetilde{T}(\langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle) + (\langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle) \\
&= (\langle[x_{\mu_L}, x_{\mu_U}], [x_{\nu_L}, x_{\nu_U}]\rangle)(\widetilde{T}\langle[\alpha_{\mu_L}, \alpha_{\mu_U}], [\alpha_{\nu_L}, \alpha_{\nu_U}]\rangle) \\
&\quad + \widetilde{T}(\langle[\beta_{\mu_L}, \beta_{\mu_U}], [\beta_{\nu_L}, \beta_{\nu_U}]\rangle) \\
&= x\widetilde{T}(\alpha) + x\widetilde{T}(\beta)
\end{aligned}$$

Thus $L(\widetilde{V})$ is closed under addition and multiplication. ■

Definition 3.12. *If* $\widetilde{U}$ *and* $\widetilde{V}$ *are IVIFV- spaces then the IVIF-transformation* $\widetilde{T}$ *is defiened by* $\widetilde{T} : \widetilde{U} \to \widetilde{V}$, $\widetilde{T}(\alpha) = \Phi$, *for all* $\alpha \in \widetilde{V}$ *then* $\widetilde{T}$ *is said to be an interval-valued intuitionistic fuzzy zero linear(in briefly* $IVIF_0L$ *)- transformation.*

Definition 3.13. *If $\widetilde{V}$ is an IVIFV-space, then IVIF-transformation $\widetilde{T}$ is defined as,*

$$\widetilde{T}(\alpha) = \alpha, \forall \alpha \in \widetilde{V}$$

then $\widetilde{T}$ is called as an interval-valued intuitionistic fuzzy identity linear(in briefly, $IVIF_I L$)- transformation.

Definition 3.14. *If $\widetilde{U}, \widetilde{V}, \widetilde{W}$ be three IVIFV-spaces over the IVIF-field F such that $\widetilde{T_1} : \widetilde{U} \to \widetilde{V}$, $\widetilde{T_2} : \widetilde{V} \to \widetilde{W}$ be two IVIFL-transformations, then the composition of two IVIFL-transformations, $\widetilde{T_1 T_2}$ is defined by,*

$$(\widetilde{T_1 T_2})\alpha = \widetilde{T_1}(\widetilde{T_2}(\alpha)), \forall \alpha \in \widetilde{W}$$

Remark 3.15. *if Range $(\widetilde{T_2})$ = Domain $(\widetilde{T_1})$, then we can define $\widetilde{T_1 T_2}$. Also, $\widetilde{T_1 T_2} \neq \widetilde{T_2 T_1}$.*

Proposition 3.16. *Let $\widetilde{U}, \widetilde{V}$ and $\widetilde{W}$ be an IVIFV-spaces over the IVIF-field F and $\widetilde{T_1} : \widetilde{U} \to \widetilde{V}$, $\widetilde{T_2} : \widetilde{V} \to \widetilde{W}$ be two IVIFL-transformations, then $\widetilde{T_1 T_2}$ is an IVIFL- transformations from $\widetilde{U} \to \widetilde{W}$.*
Proof: *Let $\widetilde{T_1} : \widetilde{U} \to \widetilde{V}$ and $\widetilde{T_2} : \widetilde{V} \to \widetilde{W}$ be two IVIFL-transformations. Now define the IVIF-transformation $\widetilde{T_1 T_2} : \widetilde{U} \to \widetilde{W}$ as, $(\widetilde{T_1 T_2})\alpha = \widetilde{T_1}(\widetilde{T_2}(\alpha)), \forall \alpha \in \widetilde{W}$. Let $\alpha, \beta \in \widetilde{W}$ and $x, y \in F$ then,*

$$\begin{aligned} \widetilde{T_1 T_2}(\alpha + \beta) &= \widetilde{T_1}(\widetilde{T_2}(\alpha + \beta)) \\ &= \widetilde{T_1}(\widetilde{T_2}\alpha + \widetilde{T_2}\beta) \\ &= \widetilde{T_1}(\widetilde{T_2}(\alpha)) + \widetilde{T_1}(\widetilde{T_2}(\beta)) \\ &= \widetilde{T_1 T_2}(\alpha) + \widetilde{T_1 T_2}(\beta) \end{aligned}$$

Hence $\widetilde{T_1 T_2}$ is an IVIFL- transformations.

Proposition 3.17. *Product of $IVIF_0 L$-transformation with any other IVIFL- transformation is again an $IVIF_0 L$ - transformation.*
Proof: *Let $\widetilde{T_1}$ be any IVIFL-transformation and Φ be an $IVIF_0 L$-transformation. Given $\alpha \in \widetilde{V}$,*
$(\widetilde{T_1}\Phi)(\alpha) = \widetilde{T_1}(\Phi(\alpha)) = \widetilde{T_1}(\Phi) = \Phi = \Phi(\alpha)$
Similarly, $(\Phi\widetilde{T_1})(\alpha) = \Phi(\widetilde{T_1}(\alpha) = \Phi$.
Hence $(\widetilde{T_1}\Phi)(\alpha) = (\Phi\widetilde{T_1}(\alpha)) = \Phi$.

Proposition 3.18. *Product of $IVIF_I L$-transformation with any other IVIFL-transformation is an IVIFL-transformation.*
Proof: *Let $\widetilde{T_1}$ be an IVIFL-transformation and $\mathscr{I}$ be an $IVIF_I L$-transformation. Given $\alpha \in \widetilde{V}$,*
$(\widetilde{T_1}\mathscr{I})(\alpha) = \widetilde{T_1}(\mathscr{I}(\alpha)) = \widetilde{T_1}(\alpha)$
Like this manner, we can prove, $(\mathscr{I}\widetilde{T_1})(\alpha) = \widetilde{T_1}(\alpha)$.
Hence, $(\widetilde{T_1}\mathscr{I})(\alpha) = (\mathscr{I}\widetilde{T_1})(\alpha) = \widetilde{T_1}(\alpha)$.

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A new fixed point result in bipolar controlled fuzzy metric spaces with application

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Abstract. In this paper, we introduce the notion of bipolar controlled fuzzy metric spaces which is an extension of the result of Sezen [20]. The paper concerns our sustained efforts for the materialization of controlled fuzzy metric spaces. Further, we establish a Banach-type fixed point theorem. We provide suitable examples with graphics which validate our result. We also employ an application to substantiate the utility of our established result to find the unique solution of an integral equation arising in automobile suspension system.

AMS Subject Classifications: 54H25, 47H10.

Keywords: Fixed point, Control function, Controlled fuzzy metric spaces, Bipolar controlled fuzzy metric spaces.

Contents

1. Introduction and Background

In 1922, S. Banach [8] provided an important result to fixed point theory. This topic has been studied, presented and generalized by many researchers in many different spaces. Firstly, the work of Bakhtin [7], Bourbaki [10] and Czerwik [11] expanded the theory of fixed points for b-metric spaces. Also, many authors proved some important fixed point theorems in b-metric spaces ([3], [4], [5]). Later, Abdeljawad et al. [1] proved some fixed point results in partial b-metric spaces. Nabil Mlaiki et al. [18] introduced controlled metric spaces and proved some fixed point theorems. Abdeljawad et al. [2] modified controlled metric spaces called double controlled metric spaces.

On the other hand, after producing the fuzzy subject of Zadeh [22], Kramosil and Michalek [16] introduced the concept of fuzzy metric spaces, which can be regarded as a generalization of the statistical metric spaces. Subsequently, M. Grabiec [13] defined G-complete fuzzy metric spaces and extended the complete fuzzy metric spaces. Following Grabiec's work, many authors introduced and generalized the different types of fuzzy contractive mappings and investigated some fixed point theorems in fuzzy metric spaces. George and Veeramani [12] modified the notion of M-complete fuzzy metric spaces with the help of continuous t-norms.

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Nădăban [19] introduced the concept of fuzzy b-metric spaces. Kim et al. [15] established some fixed point results in fuzzy b-metric spaces. Recently, Mehmood et al. [17] has defined a new concept called extended fuzzy b-metric spaces, which is the generalization of fuzzy b-metric spaces. Most recently Müzeyyen Sangurlu Sezen [20] introduced controlled fuzzy metric spaces, which is a generalization of extended fuzzy b-metric spaces.

In [9], Ayush Bartwal et. al. introduced new generalization of the fuzzy metric space called bipolar fuzzy metric space and proved some fixed point results in this space. The objective of this work is to prove a Banach type fixed point theorem in bipolar controlled fuzzy metric spaces. Our result generalizes many recent fixed point theorems in the literature ([15],[17],[19]). We furnish an example to validate our result. An application is also provided in support of our result.

2. Preliminaries

Now, we begin with some basic concepts, notations and definitions. Let $\mathbb{R}$ represent the set of real numbers, $\mathbb{R}_+$ represent the set of all non-negative real numbers and $\mathbb{N}$ represent the set of natural numbers.

We start with the following definitions of a fuzzy metric space. Schweizer and Sklar introduced the continuous t- norm as follows:

Definition 2.1. *[21]. A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is called a continuous t- norm if for all* $r_1, r_2, r_3 \in [0,1]$, *the following conditions are hold:*
(T-1) $r_1 * r_2 = r_2 * r_1$ *and* $r_1 * (r_2 * r_3) = (r_1 * r_2) * r_3$,
(T-2) $r_1 * r_2 \leq r_3 * r_4$ *whenever* $r_1 \leq r_3$ *and* $r_2 \leq r_4$,
(T-3) $r_1 * 1 = r_1$,
(T-4) $*$ *is a continuous.*
The most commonly used t-norms are: $r_1 *_p r_2 = r_1 \Delta r_2$, $r_1 *_m r_2 = min\{r_1, r_2\}$ *and* $r_1 *_L r_2 = max\{r_1 + r_2 - 1, 0\}$ *which known as product, minimum and Lukasiewicz t-norms respectively.*

Kramosil and Michalek [16] introduced the notion of fuzzy metric space as follows:

Definition 2.2. *[16]. An ordered triple* $(X, M, *)$ *is called fuzzy metric space such that* X *is a nonempty set,* $*$ *defined a continuous* t*-norm and* M *is a fuzzy set on* $X \times X \times (0,\infty)$, *satisfying the following conditions, for all* $x, y, z \in X, s, t > 0$,
(KM-1) $M(x, y, 0) = 0$,
(KM-2) $M(x, y, t) = 1$ *iff* $x = y$,
(KM-3) $M(x, y, t) = M(y, x, t)$,
(KM-4) $(M(x, y, t) * M(y, z, s)) \leq M(x, z, t + s)$,
(KM-5) $M(x, y, \cdot) : (0,\infty) \to [0,1]$ *is left continuous.*

George and Veeramani[12] modified the definition of M-complete fuzzy metric spaces due to Kramosil and Michalek and the concept as follows:

Definition 2.3. *[12]. An ordered triple* $(X, M, *)$ *is called fuzzy metric space such that* X *is a nonempty set,* $*$ *defined a continuous* t*-norm and* M *is a fuzzy set on* $X \times X \times (0,\infty)$, *satisfying the following conditions:*
(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ *if and only if* $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $(M(x, y, t) * M(y, z, s)) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, \cdot) : (0,\infty) \to [0,1]$ *is left continuous,* $x, y, z \in X$ *and* $t, s > 0$.

In 2017, Nădăban [19] introduced the idea of a fuzzy b-metric space to generalize the notion of a fuzzy metric spaces introduced by Kramosil and Michalek [16].

Definition 2.4. *[19]. Let X is a non-empty set and $k \geq 1$ be a given real number and $*$ be a continuous t-norm. A fuzzy set M in $X^2 \times (0, \infty)$ is called fuzzy b-metric on X if for all $x, y, z \in X$, the following conditions hold.*
(bM-1) $M(x, y, 0) = 0$,
(bM-2) $[M(x, y, t) = 1, (\forall)t > 0]$ *if and only if* $x = y$,
(bM-3) $M(x, y, t) = M(y, x, t), (\forall)t > 0$,
(bM-4) $M(x, z, k(t + s)) \geq M(x, y, t) * M(y, z, s)$, *(∀)t,s ¿ 0,*
$(bM - 5)$*M(x, y, ·)* $: [0, \infty) \to [0, 1]$ *is left continuous and* $lim_{t\to\infty} M(x, y, t) = 1$.
The quadruple $(X, M, *, k)$ is said to be a fuzzy b-metric space.

Mehmood et al. [17] introduced the notion of an extended fuzzy b-metric space following the approach of Grabiec [13].

Definition 2.5. *[17]. Let X be a non-empty set, $\alpha : X \times X \to [1, \infty)$, $*$ is a continuous t-norm and M_α is a fuzzy set on $X^2 \times (0, \infty)$, is called extended fuzzy b-metric on X if for all $x, y, z \in X$ and $s, t > 0$, satisfying the following conditions.*
$(FbM_\alpha 1)$ $M_\alpha(x, y, 0) = 0$,
$(FbM_\alpha 2)$ $M_\alpha(x, y, t) = 1$ *iff* $x = y$,
$(FbM_\alpha 3)$ $M_\alpha(x, y, t) = M_\alpha(y, x, t)$,
$(FbM_\alpha 4)$ $M_\alpha(x, z, \alpha(x, z)(t + s)) \geq M_\alpha(x, y, t) * M_\alpha(y, z, s)$,
$(FbM_\alpha 5)$ $M_\alpha(x, y, \cdot) : (0, \infty) \to [0, 1]$ *is continuous.*
Then $(X, M_\alpha, *, \alpha(x, y))$ is an extended fuzzy b-metric space.

In [20], Sezen introduced the controlled fuzzy metric spaces, which is a generalization of extended fuzzy b-metric spaces.

Definition 2.6. *[20]. Let X be a non-empty set, $\lambda : X \times X \to [1, \infty)$, $*$ is a continuous t-norm and M_λ is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions, for all $a, c, d \in X, s, t > 0$:*
(FM-1) $M_\lambda(a, c, 0) = 0$,
(FM-2) $M_\lambda(a, c, t) = 1$ *iff* $a = c$,
(FM-3) $M_\lambda(a, c, t) = M_\lambda(c, a, t)$,
(FM-4) $M_\lambda(a, d, t + s) \geq M_\lambda(a, c, \frac{t}{\lambda(a,c)}) * M_\lambda(c, d, \frac{s}{\lambda(c,d)})$,
(FM-5) $M_\lambda(a, c, \cdot) : [0, \infty) \to [0, 1]$ *is continuous,*
Then, the triple $(X, M_\lambda, *)$ is called a controlled fuzzy metric space on X.

In [9], Ayush Bartwal et. al. introduced new generalization of the fuzzy metric space called fuzzy bipolar metric space and prove some fixed point results in this space.

Definition 2.7. *[9]. Let X and Y be two non-empty sets. A quadruple $(X, Y, M_b, *)$ is said to be fuzzy bipolar metric space (FB-space), where $*$ is continuous t-norm and M_b is a fuzzy set on $X \times Y \times (0, \infty)$, satisfying the following conditions for all $t, s, r > 0$:*
(FB-1) $M_b(x, y, t) > 0$ *for all* $(x, y) \in X \times Y$,
(FB-2) $M_b(x, y, t) = 1$ *if and only if* $x = y$ *for* $x \in X$ *and* $y \in Y$,
(FB-3) $M_b(x, y, t) = M_b(y, x, t)$ *for all* $x, y \in X \cap Y$,
(FB-4) $M_b(x_1, y_2, t + s + r) > M_b(x_1, y_1, t) * M_b(x_2, y_1, s) * M_b(x_2, y_2, r)$
for all $x_1, x_2 \in X$ *and* $y_1, y_2 \in Y$,
(FB-5) $M_b(x, y, .) : [0, \infty) \to [0, 1]$ *is left continuous,*
(FB-6) $M_b(x, y, .)$ *is non-decreasing for all* $x \in X$ *and* $y \in Y$.

3. Main Results

In this section, we introduce some new definitions and establish a fixed point theorem in bipolar controlled fuzzy metric spaces.

Definition 3.1. *Let X and Y be two non-empty sets, A quadruple $(X, Y, M_{b\lambda}, *)$ is said to be bipolar controlled fuzzy metric space, where $*$ is continuous t-norm defined as $a * b = min\{a, b\}$ and $\lambda : X \times X \to [1, \infty)$, and $M_{b\lambda}$ is a fuzzy set on $X^2 \times (0, \infty)$. If for all $x \in X,\ y \in Y$ and $s, t, r > 0$. $M_{b\lambda}$ satisfying the following conditions:*
($FM_{b\lambda}$-1) $M_{b\lambda}(x, y, 0) = 0$.
($FM_{b\lambda}$-2) $M_{b\lambda}(x, y, t) = 1$ iff $x = y$. for $x \in X$ and $y \in Y$.
($FM_{b\lambda}$-3) $M_{b\lambda}(x, y, t) = M_{b\lambda,\mu}(y, x, t)$. for all $x, y \in X \cap Y$.
($FM_{b\lambda}$-4) $M_{b\lambda}(x_1, y_2, t + s + r) \geq M_{b\lambda}(x_1, y_1, \frac{t}{\lambda(x_1,y_1)}) * M_{b\lambda}(x_2, y_1, \frac{s}{\lambda(x_2,y_1)}) * M_{b\lambda}(x_2, y_2, \frac{r}{\lambda(x_2,y_2)})$. for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.
($FM_{b\lambda}$-5) $M_{b\lambda}(x, y, \cdot) : [0, \infty) \to [0, 1]$ is continuous.
($FM_{b\lambda}$-6) $M_{b\lambda}(x, y, \cdot)$ is non decreasing for all $x \in X,\ y \in Y$.

Definition 3.2. *Let $(X, Y, M_{b\lambda}, *)$ be a fuzzy bipolar controlled metric spaces. Then*

1. *A sequence $\{x_n, y_n\}$ in $X \times Y$ is said to be BPC-convergent sequence to x in $X \times Y$, if both x_n and y_n converge to same point.*
2. *A sequence $\{x_n, y_n\}$ in $X \times Y$ said to be BPC-Cauchy sequence if and only if $\lim_{n\to\infty} M_\lambda(x_n, y_m, t) = 1$ for any $n, m > 0$ and for all $t > 0$.*
3. *The fuzzy bipolar controlled metric space $(X, Y, M_{b\lambda}, *)$ is called BPC-complete if every BPC- Cauchy sequence is convergent in it.*

Now, we display an example to verify our definition 3.1.

Example 3.3. *Let $X = (0, 2],\ \ Y = [2, \infty)$. Define $M_{b\lambda}$ is a fuzzy set on $X^2 \times (0, \infty)$, as*

$$M_{b\lambda}(x, y, t) = \begin{cases} 1 & \text{if } x = y \\ txy & \text{if } x \neq y \text{ and } t \geq 0 \end{cases},$$

With the continuous t-norm $*$ such that $t_1 * t_2 = min\{t_1, t_2\}$. Define $\lambda : X \times X \to [1, \infty)$, as

$$\lambda(x, y) = \begin{cases} 1 & \text{if } x \in X \text{ and } y \in Y \\ 1 + \frac{1}{a} & \text{otherwise,} \end{cases}$$

Let us show that $(X, Y, M_{b\lambda}, *)$ is a bipolar controlled fuzzy metric spaces. It is easy to prove conditions ($FM_{b\lambda}$-1), ($FM_{b\lambda}$-2) and ($FM_{b\lambda}$-3). We have to examine the following case to show that condition ($FM_{b\lambda}$-4) holds.
For $x \neq y$ **and** $t \geq 0$*: By assuming $x_1 = 2,\ \ y_1 = 3,\ \ x_2 = 1$ and $y_2 = 4$, we obtain a non-trivial sequence as $(x_n, y_n) = \{(2, 3), (1, 4), (\frac{1}{2}, 5), \cdots\}$ and taking $t = 1,\ s = 2, r = 3$.*

$$\begin{aligned} M_{b\lambda}(x_1, y_2, t + s + r) &= M_\lambda(2, 4, 6) = 48 \\ &\geq M_{b\lambda}(2, 3, \frac{1}{\lambda(2,3)}) * M_{b\lambda}(1, 3, \frac{2}{\lambda(1,3)}) * M_{b\lambda}(1, 4, \frac{3}{\lambda(1,4)}) = 6 \\ &= M_{b\lambda}(x_1, y_1, \frac{t}{\lambda(x_1,y_1)}) * M_{b\lambda}(x_2, y_1, \frac{s}{\lambda(x_2,y_1)}) * M_{b\lambda}(x_2, y_2, \frac{r}{\lambda(x_2,y_2)}) \end{aligned}$$

Which satisfies condition of bipolar controlled fuzzy metric spaces. But, if we take $x_1 = \frac{1}{2},\ x_2 = 2,\ \ x_3 = \frac{1}{3}$ and $t = 1,\ s = 2$ for all $x_1, x_2, x_3 \in X$ and $t, s > 0$ in the definition [20], we have

$$\begin{aligned} M_\lambda(x_1, x_3, t + s) &= M_\lambda(\frac{1}{2}, \frac{1}{3}, 2) = 0.33 \\ &\leq M_\lambda(\frac{1}{2}, 2, \frac{1}{\lambda(\frac{1}{2}, 2)}) * M_\lambda(2, \frac{1}{3}, \frac{2}{\lambda(2, \frac{1}{3})}) = 0.8, \\ &= M_\lambda(x_1, x_2, \frac{t}{\lambda(x_1, x_2)}) * M_\lambda(x_2, x_3, \frac{s}{\lambda(x_2, x_3)}), \end{aligned}$$

which not satisfies the condition [20] of controlled fuzzy metric spaces.

Example 3.4. *Let* $X = [0,1), \;\; Y = [1,\infty)$. *Define* $M_{b\lambda}$ *is a fuzzy set on* $X^2 \times (0,\infty)$, *as*

$$M_{b\lambda}(x,y,t) = \frac{t}{t + d(x,y)}$$

With the continuous t-norm $*$ *such that* $t_1 * t_2 = min\{t_1, t_2\}$. *Define* $\lambda : X \times X \to [1,\infty)$, *as*

$$\lambda(x,y) = \begin{cases} 1 & \textit{if } x \in X \textit{ and } y \in Y \\ max\{x,y\} & \textit{otherwise}, \end{cases}$$

Then $(X, Y, M_{b\lambda}, *)$ *is bipolar controlled fuzzy metric space.*

Now, we present our main result as follows:

Theorem 3.5. *Let* $(X, Y, M_{b\lambda}, *)$ *be a Bipolar controlled fuzzy metric spaces with* $\lambda : X \times X \to [1,\infty)$ *and suppose that*

$$lim_{n\to\infty} M_{b\lambda}(x,y,t) = 1, \tag{3.1}$$

for all $x \in X, \;\; y \in Y$ *and* $t > 0$. *If* $T : X \cup Y \to X \cup Y$ *satisfies that:*

1. $T(X) \subseteq X$ *and* $T(Y) \subseteq Y$,

2. $$M_{b\lambda}(Tx, Ty, kt) \geq M_{b\lambda}(x,y,t), \tag{3.2}$$

   where $k \in (0,1)$. *Also, we assume that for every* $x_n \in X$,

$$lim_{n\to\infty} \lambda(x_n, y) \tag{3.3}$$

exist and are finite. Then T *has a unique fixed point.*

Proof. Let $x_0 \in X$ and $y_0 \in Y$ and define (x_n, y_n) as a sequence by $x_n = Tx_{n-1}$ and $y_n = Ty_{n-1}$ for all $n \in \mathbb{N}$ on bipolar controlled fuzzy metric space $(X, Y, M_{b\lambda}, *)$. If $x_n = x_{n-1}$ then x_n is a fixed point of T. Suppose that $x_n \neq x_{n-1}$ for all $t > 0$ and $n \in \mathbb{N}$. Successively applying inequality (3.2), we get

$$\begin{aligned} M_{b\lambda}(x_n, y_{n+1}, t) &= M_{b\lambda}(Tx_{n-1}, Ty_n, t) \\ &\geq M_{b\lambda}(x_{n-2}, y_{n-1}, \frac{t}{k}) \\ &\vdots \\ &\geq M_{b\lambda}(x_0, x_1, \frac{t}{k^{n-1}}). \end{aligned} \tag{3.4}$$

Now, using the condition ($FM_{b\lambda}$-4), we have

$$\begin{aligned}
M_{b\lambda}(x_n, y_{n+m}, t) &\geq M_{b\lambda}(x_n, y_{n+1}, \frac{t}{3\lambda(x_n, y_{n+1})}) * M_{b\lambda}(x_{n+1}, y_{n+2}, \frac{t}{3\lambda(x_{n+1}, y_{n+2})}) \\
&\quad * M_{b\lambda}(x_{n+2}, y_{n+m}, \frac{t}{3\lambda(x_{n+2}, y_{n+m})}) \\
&\geq M_{b\lambda}(x_n, y_{n+1}, \frac{t}{3\lambda(x_n, y_{n+1})}) \\
&\quad * M_{b\lambda}(x_{n+1}, y_{n+2}, \frac{t}{3\lambda(x_{n+1}, y_{n+2})} \\
&\quad * M_{b\lambda}(x_{n+2}, y_{n+3}, \frac{t}{(3)^2\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+2}, y_{n+3})}) \\
&\quad * M_{b\lambda}(x_{n+3}, y_{n+4}, \frac{t}{(3)^2\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+3}, y_{n+4})}) \\
&\quad * M_{b\lambda}(x_{n+4}, y_{n+m}, \frac{t}{(3)^2\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+4}, y_{n+m})}) \\
&\geq M_{b\lambda}(x_n, y_{n+1}, \frac{t}{3\lambda(x_n, y_{n+1})}) \\
&\quad * M_{b\lambda}(x_{n+1}, y_{n+2}, \frac{t}{3\lambda(x_{n+1}, y_{n+2})} \\
&\quad * M_{b\lambda}(x_{n+2}, y_{n+3}, \frac{t}{(3)^2\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+2}, y_{n+3})}) \\
&\quad * M_{b\lambda}(x_{n+3}, y_{n+4}, \frac{t}{(3)^2\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+3}, y_{n+4})}) \\
&\quad * M_{b\lambda}(x_{n+4}, y_{n+5}, \frac{t}{(3)^3\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+4}, y_{n+5})}) \\
&\quad * M_{b\lambda}(x_{n+5}, y_{n+6}, \frac{t}{(3)^3\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+4}, y_{n+6})}) \\
&\quad * M_{b\lambda}(x_{n+6}, y_{n+7}, \frac{t}{(3)^3\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+6}, y_{n+7})}) \\
&\quad \vdots \\
&\geq M_{b\lambda}(x_0, x_1, \frac{t}{3k^{n-1}\lambda(x_n, x_{n+1})}) \\
&\quad * [*_{i=n+1}^{n+m-2} M_{b\lambda}(x_0, y_1, \frac{t}{(3)^{m-1}k^{i-1}(\prod_{j=n+1}^{i}\lambda(x_j, y_{n+m}))\lambda(x_i, y_{i+1})})] \\
&\quad * [M_{b\lambda}(x_0, y_1, \frac{t}{(3)^{m-1}k^{n+m-1}(\prod_{i=n+1}^{n+m-1}\lambda(x_i, y_{n+m}))})].
\end{aligned} \tag{3.5}$$

Therefore, by taking limit as $n \to \infty$ in (3.5), from (3.4) together with (3.1) we have

$$\lim_{n\to\infty} M_{b\lambda}(x_n, y_{n+m}, t) \geq 1 * 1 * \cdots * 1 = 1 \ \ for\ all\ \ t > 0,\ n < m\ and\ n, m \in \mathbb{N}.$$

Thus, (x_n, y_n) is a BPC-Cauchy sequence in X. From the completeness of $(X, Y, M_{b\lambda}, *)$, there exists $u \in X \cap Y$ which is a limit of the both sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} M_{b\lambda}(Tu, u, t) = 1, \tag{3.6}$$

for all $t > 0$. Now we show that u is a fixed point of T. For any $t > 0$ and from the condition ($FM_{b\lambda}$-4), we have

$$\begin{aligned} M_{b\lambda}(Tu, u, t) \geq M_{b\lambda}(Tu, Ty_n, \frac{t}{3\lambda(Tu, Ty_n)}) * M_{b\lambda}(Tx_n, Ty_{n+1}, \frac{t}{3\lambda(Tx_n, Ty_{n+1})}) \\ * M_{b\lambda}(Tx_{n+1}, u, \frac{t}{3\lambda(Tx_{n+1}, u)}) \end{aligned} \tag{3.7}$$

Letting $n \to \infty$ in (3.7) and using (3.6), we get $M_{b\lambda}(Tu, u, t) = 1$ for all $t > 0$, that is, $Tu = u$.
For uniqueness, let $w \in X \cap Y$ is another fixed point of T and there exists $t > 0$ such that $M_{b\lambda}(u, w, t) \neq 1$, then it follows from (3.2) that

$$\begin{aligned} M_{b\lambda}(u, w, t) &= M_{b\lambda}(Tu, Tw, t) \\ &\geq M_{b\lambda}(u, w, \frac{t}{k}) \\ &\geq M_{b\lambda}(u, w, \frac{t}{k^2}) \\ &\vdots \\ &\geq M_{b\lambda}(u, w, \frac{t}{k^n}), \end{aligned} \tag{3.8}$$

for all $n \in \mathbb{N}$. By taking limit as $n \to \infty$ in (3.8), $M_{b\lambda}(u, w, t) = 1$ for all $t > 0$, that is, $u = w$. This completes the proof. ■

Now we furnish an example to validate Theorem 3.5.

Example 3.6. *Let $X = [0, 2)$ and $Y = [2, \infty)$. Define*
$M_{b\lambda} : X \times X \times [0, \infty) \to [0, 1]$ as

$$M_{b\lambda}(x, y, t) = \begin{cases} 1 & \text{if } x = y \\ \frac{t}{(t+\frac{2}{y})} & \text{if } x \in X \text{ and } y \in Y \\ \frac{t}{(t+\frac{2}{x})} & \text{if } x \in Y \text{ and } y \in X \\ \frac{1}{(t+1)} & \text{otherwise.} \end{cases}$$

With the continuous t-norm $*$ such that $t_1 * t_2 = min\{t_1, t_2\}$. Define $\lambda : X \times Y \to [1, \infty)$, as

$$\lambda(x, y) = \begin{cases} 1 & \text{if } x, y \in X \\ max\{x, y\} & \text{otherwise.} \end{cases}$$

Clearly $(X, Y, M_{b\lambda}, *)$ is a bipolar controlled fuzzy metric space. Consider $T : X \cup Y \to X \cup Y$ by

$$T(u) = \begin{cases} u & \text{if } u \in X \\ u^2 + 1 & \text{if } u \in Y, \end{cases}$$

for all $x \in X,\ \ y \in Y$ and $k = 0.5$. We have to examine the inequality (3.2) for all the four cases given below.
Case I. *If $x = y$ then we have $Tx = Ty$. In this case:*

$$M_{b\lambda}(Tx, Ty, kt) = 1 = M_{b\lambda}(x, y, t). \tag{3.9}$$

Case II. *Let $x \in X$ and $y \in Y$, then we have $Tx \in X$ and $Ty \in Y$. In this case:*

$$\begin{aligned} M_{b\lambda}(Tx, Ty, kt) &= \frac{kt}{(kt + \frac{2}{Ty})} \\ &= \frac{0.5t}{(0.5t + \frac{2}{y^2+1})} \\ &\geq \frac{t}{(t + \frac{2}{y})} \\ &= M_{b\lambda}(x, y, t). \end{aligned} \tag{3.10}$$

Figure $1(a)$ shows the illustration of above case on $2D$ view, in which the variation of $M_{b\lambda}(Tx, Ty, kt)$ as a

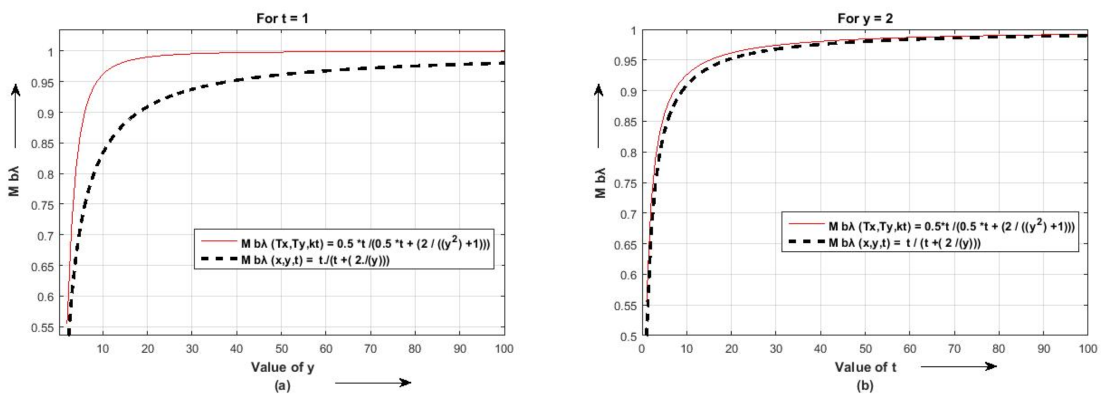

Figure 1: Variation of $M_{b\lambda}(Tx, Ty, kt)$ with $M_{b\lambda}(x, y, t)$ of Example 3.6, case-II on $2D$ view, for:
(a)$M_{b\lambda}(Tx, Ty, kt)$ vs $M_{b\lambda}(x, y, t)$ at $t = 1$ and $y \in (2, 100)$.
(b)$M_{b\lambda}(Tx, Ty, kt)$ vs $M_{b\lambda}(x, y, t)$ at $t \in (1, 100)$ and $y = 2$.

function of y with fixed values of t, is shown as a red colored curve. A dotted curved line represents the variation of $M_{b\lambda}(x, y, t)$ as a function of y relative to t, the variation of this curve is similar to the red colored line with little smaller values of $M_{b\lambda}(x, y, t)$.
Figure $1(b)$ shows the variation of $M_{b\lambda}(Tx, Ty, kt)$ as a function of t with fixed values of y, is shown as a red colored curve. A dotted curved line represents the variation of $M_{b\lambda}(x, y, t)$ as a function of t fixed to y, the variation of this curve is similar to the red colored line with little smaller values of $M_{b\lambda}(x, y, t)$.

Figure $2(a)$ shows the illustration of case II of Example 3.6 on $3D$ view, in which the variation of $M_{b\lambda}(Tx, Ty, kt)$ as a function of y with different values of t, is shown as a red-yellow surface and a blue-black surface represents the variation of $M_{b\lambda}(x, y, t)$ as a function of y relative to t, the variation of this curve is similar to the red-yellow surface with little smaller values of $M_{b\lambda}(x, y, t)$.

Figure $2(b)$ is similar to the variation of $M_{b\lambda}(Tx, Ty, kt)$ as a function of y with different values of t, is shown as a yellow surface curve and a green surface represents the variation of $M_{b\lambda}(x, y, t)$ as a function of y relative to t.

Table 1 and 2 show the variation between $M_{b\lambda}(Tx, Ty, kt)$ and $M_{b\lambda}(x, y, t)$ as a function of y with relative to t, this table justifies inequality (3.10)*, which observed in both the curves for the value of t is a higher than 50 as a function of y. At $t = 50$, $M_{b\lambda}(Tx, Ty, kt)$ becomes 1 and after higher value of t, it remains constant ($= 1$). $M_{b\lambda}(x, y, t)$ doesn't become to 1 till $t = 100$, but it approached nearby to 1.*

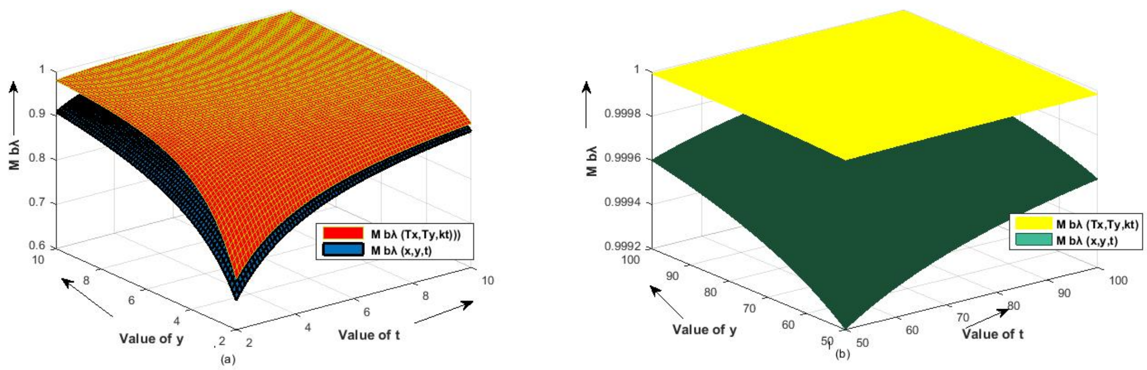

Figure 2: Variation of $M_{b\lambda}(Tx, Ty, kt)$ with $M_{b\lambda}(x, y, t)$ of Example 3.6, case-II on $3D$ view, for:
(a)$M_{b\lambda}(Tx, Ty, kt)$ vs $M_{b\lambda}(x, y, t)$ at $t \in (1, 10)$ and $y \in (2, 10)$.
(b)$M_{b\lambda}(Tx, Ty, kt)$ vs $M_{b\lambda}(x, y, t)$ at $t \in (50, 100)$ and $y \in (50, 100)$.

Value of t	Value of y	$M_{b\lambda}(Tx, Ty, kt)$	$M_{b\lambda}(x, y, t)$
1	2	0.5556	0.5000
	20	0.9901	0.9091
	50	0.9984	0.9615
	100	0.9996	0.9804
50	2	0.9843	0.9804
	20	0.9998	0.9980
	50	1.0000	0.9992
	100	1.0000	0.9996

Table 1: Variation of $M_{b\lambda}(Tx, Ty, kt)$ with $M_{b\lambda}(x, y, t)$ of inequality (3.10), as a function of y with fixed value of $t = 1$ and $t = 50$.

Value of y	Value of t	$M_{b\lambda}(Tx, Ty, kt)$	$M_{b\lambda}(x, y, t)$
2	1	0.7143	0.6667
	20	0.9615	0.9524
	50	0.9843	0.9804
	100	0.9921	0.9901
50	1	0.9984	0.9615
	20	0.9999	0.9980
	50	1.0000	0.9992
	100	1.0000	0.9996

Table 2: Variation of $M_{b\lambda}(Tx, Ty, kt)$ with $M_{b\lambda}(x, y, t)$ of inequality (3.10) as a function of t with fixed value of $y = 2$ and $y = 50$.

Case III. *Let $x \in Y$ and $y \in X$, then we have $Tx \in Y$ and $Ty \in X$. In this case:*

$$\begin{aligned} M_{b\lambda}(Tx, Ty, kt) &= \frac{kt}{(kt + \frac{2}{Tx})} \\ &= \frac{0.5t}{(0.5t + \frac{2}{x^2+1})} \\ &\geq \frac{t}{(t + \frac{2}{x})} \\ &= M_{b\lambda}(x, y, t). \end{aligned} \tag{3.11}$$

Figure $3(a)$ Shows the illustration of case III of example 3.6, on $2D$ view, in which the variation of $M_{b\lambda}(Tx, Ty, kt)$ as a function of x with fixed values of t, is shown as red colored dotted curve and the variation of $M_{b\lambda}(x, y, t)$ as a function of x fixed to t shown as a blue colored curve.
Figure $3(b)$ is the variation of $M_{b\lambda}(Tx, Ty, kt)$ as a function of t with fixed values of x, is shown as red colored dotted curve and the variation of $M_{b\lambda}(x, y, t)$ as a function of t fixed to x shown as a blue colored curve.

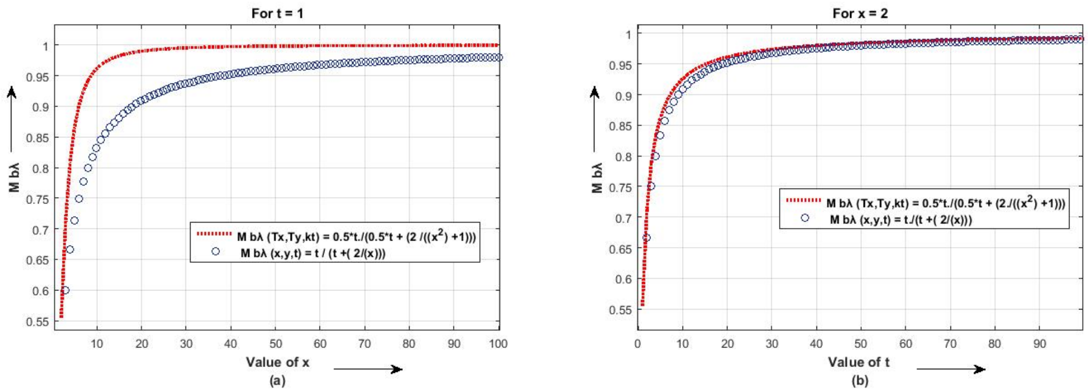

Figure 3: Variation of $M_{b\lambda}(Tx, Ty, kt)$ with $M_{b\lambda}(x, y, t)$ of Example 3.6, Case III on $2D$ view, for:
(a)$M_{b\lambda}(Tx, Ty, kt)$ vs $M_{b\lambda}(x, y, t)$ at $t = 1$ and $x \in (3, 50)$.
(b)$M_{b\lambda}(Tx, Ty, kt)$ vs $M_{b\lambda}(x, y, t)$ at $t \in (1, 50)$ and $x = 2$.

Case IV. *For other states of $x,\ y$ and similarly $Tx,\ Ty$. $M_{b\lambda}(Tx, Ty, kt)$ and $M_{b\lambda}(x, y, t)$ depends on only t. we have,*

$$\begin{aligned} M_{b\lambda}(Tx, Ty, kt) &= \frac{1}{(kt+1)} \\ &= \frac{1}{(0.5t+1)} \\ &\geq \frac{1}{(t+1)} \\ &= M_{b\lambda}(x, y, t). \end{aligned} \tag{3.12}$$

Therefore, all the conditions of Theorem 3.5 hold and T has a unique fixed point $x = 1$.

Remark 3.7. *By taking $\lambda = 1$, in Theorem 3.5 we infer the Theorem 2 in [20].*

Theorem 3.8. *Let $(X, Y, M_{b\lambda}, *)$ be bipolar controlled fuzzy metric spaces and $T : X \cup Y \to X \cup Y$ be a mapping satisfying $lim_{n\to\infty} M_{b\lambda}(x, y, t) = 1$. Suppose there exists a constant*
$k \in (0, 1)$ such that

$$\int_0^{M_{b\lambda}(Tx,Ty,kt)} \varphi(t)dt \geq \int_0^{M_{b\lambda}(x,y,t)} \varphi(t)dt, \tag{3.13}$$

for all $x \in X$ and $y \in Y$. Then T has a fixed point.

Proof. By taking $\varphi(t) = 1$ in equation (3.13), we obtain Theorem 3.5. ■

4. Application

It is well known that the realistic application for the spring mass system in engineering difficulties is an automobile suspension system. Consider the motion of a car's spring as it travels down a rough and pitted road, where the forcing term is the rough road and the damping is provided by shock absorbers. Gravity, ground vibrations, earthquakes, tension force, and other external forces may work on the system.

The critically damped motion of this system subjected to the external force F is governed by the following initial value problem, Let m be the mass of the spring and F be the external force acting on it [14].

$$\begin{cases} m\frac{d^2u}{dt^2} + l\frac{du}{dt} - mF(t, u(t)) = 0, \\ u(0) = 0, \\ u'(0) = 0, \end{cases} \tag{4.1}$$

where $l > 0$ is the damping constant and is a continuous function. It is easy to show that the problem (4.1) is equivalent to the integral equation:

$$u(t) = \int_0^T \zeta(t, r)F(t, r, u(r))dr. \tag{4.2}$$

where $\zeta(t, r)$ is Green's function given by

$$\zeta(t, r) = \begin{cases} \frac{1-e^{\mu(t-r)}}{\mu} & \text{if } 0 \leq r \leq t \leq T. \\ 0 & \text{if } 0 \leq t \leq r \leq T. \end{cases} \tag{4.3}$$

where $\mu = l/m$ is a constant. In this section, by using Theorem 3.5, we will show the existence of a solution to the integral equation:

$$u(t) = \int_0^T G(t, r, u(r))dr. \tag{4.4}$$

Let $X = C([0, T])$ be the set of real continuous functions defined on $[0, T]$. For $k \in (0, 1)$ we define

$$M_{b\lambda}(x, y, t) = sup_{t\in[0,T]} \frac{t}{t + (|x(t) - y(t)|)}. \tag{4.5}$$

for all $x \in X$ and $y \in Y$. Define $\lambda : X \times X \to [1, \infty)$, as

$$\lambda(x, y) = \begin{cases} 1 & \text{if } x \in X \text{ and } y \in Y \\ max\{x, y\} & \text{otherwise,} \end{cases}$$

It is easy to prove that $(X, Y, M_{b\lambda}, *)$ is a bipolar controlled fuzzy metric spaces. Consider the mapping $S : X \cup Y \to X \cup Y$ defined by

$$fx(t) = \int_0^T G(t, r, x(r))dr. \tag{4.6}$$

Suppose that

1. there exist a continuous function $\zeta : [0, T] \times [0, T] \to \mathbb{R}^+$ such that

$$sup_{t\in[0,T]} \int_0^T \zeta(t, r)dr \leq 1, \tag{4.7}$$

2. $G : [0, T] \times [0, T] \times \mathbb{R} \to \mathbb{R}^+$ is continuous such that

$$|G(t, r, x(r)) - G(t, r, y(r))| \geq |x(r) - y(r)|, \tag{4.8}$$

for all $k \in (0,1)$. Then, the integral equation (4.4) has a unique solution. **Proof** Let $x \in X$ and $y \in Y$, by using of assumptions $(1)-(2)$, we have

$$\begin{aligned} M_{b\lambda}(Sx, Sy, kt) &= sup_{t\in[0,T]} \frac{kt}{kt + (|Sx(t) - Sy(t)|)} \\ &= sup_{t\in[0,T]} \frac{kt}{kt + (|\int_0^T G(t,r,x(r))dr - \int_0^T G(t,r,y(r))dr|)} \\ &= sup_{t\in[0,T]} \frac{kt}{kt + (\int_0^T |G(t,r,x(r))dr - G(t,r,y(r))|dr)} \\ &\geq sup_{t\in[0,T]} \frac{kt}{kt + (\int_0^T |x(r) - y(r)|dr)} \\ &\geq sup_{t\in[0,T]} \frac{t}{t + (\int_0^T |x(r) - y(r)|dr)} \\ &\geq M_{b\lambda}(x, y, t). \end{aligned} \tag{4.9}$$

Therefore all the condition of Theorem 3.5 are satisfied. As a result, the mapping S has a unique fixed point $x \in X \cap Y$, which is a solution of the integral equation (4.4).

Conclusion

In this article, we extend the controlled fuzzy metric spaces of Sezen [20] by introducing bipolar controlled fuzzy metric spaces. We prove a Banach-type fixed point theorem. Our investigations and results obtained were supported by suitable examples with graphics. We also provide an application of our result to the existence of solution to an integral equation. This work provides a new path for researchers in the concerned field.

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Geometrical approach on set theoretical solutions of Yang-Baxter equation in Lie algebras

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Abstract. In this paper, we handle set-theoretical solutions of Yang-Baxter equation and Lyubashenko set theoretical solutions in Lie algebras. We present a new commutative binary operation on these structures, and we obtain new set theoretical solutions including this operation by using property of commutativity of it. Also, we show that some set theoretical solutions of Yang-Baxter equation corresponds to the Lyubashenko set theoretical solutions on these structures. Additionally, we give some relations to verify set theoretical solution of Yang-Baxter equation. Moreover, we put an interpretation for these solutions from the point of geometrical view in Euclidean space, Minkowski space and differentiable manifolds by using Lie algebras.

AMS Subject Classifications: 16T25, 17B66, 53C99, 81R99

Keywords: Yang-Baxter equation, Lyubashenko solution, Lie algebras, Set theoretical solutions, Differentiable manifolds, Minkowski space.

Contents

1. Introduction

Yang-Baxter equation introduced by the Nobel laureate C.N. Yang in theoretical physics [18] and by R.J. Baxter in statistical mechanics [1, 2]. Yang-Baxter equation has been recently attracted more attention among researchers in a wide range of disciplines such as knot theory, link invariants, quantum computing, braided categories, geometrical structures, quantum groups, the analysis of integrable systems, quantum mechanics, physics and etc. For example, Boucetta and Medina worked on solutions of the Yang-Baxter equations on quadratic Lie group by

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using the case of oscillator groups [25]. Berceanu et al. [4] examined algebraic structures arising from Yang-Baxter Systems. Oner, Senturk et. al constructed new set theoretical solutions of Yang-Baxter equation in MV-algebras [14]. Massuyeau and Nichita considered the problem of constructing knot invariants from Yang-Baxter operators associated to (unitary associative) algebra structures [15]. Abedin and Maximov classified the classical twists of standard Lie bialgebra structure on a loop algebra. Besides all these works, Gateva-Ivanova examined set theoretical solutions of the Yang-Baxter equation on braces and symmetric groups [7]. Wang and Ma provided a new framework of obtaining singular solutions of the quantum Yang-Baxter equation by constructing weak quasi-triangular structures [8]. Belavin and Drinfeld worked on solutions of the classical Yang-Baxter equation for simple Lie algebras [5]. Burban and Henrich handled semi-stable vector bundles on elliptic curves and their relation with associative Yang-Baxter equation [26]. Nichita and Parashar studied Spectral-parameter dependent Yang-Baxter operators and Yang-Baxter systems from algebraic structures [12], and etc. Moreover, Lyubashenko set theoretical solution of Yang-Baxter equation was given in [20]. This solution method have been used in many works such as [21–24].

In accordance with these works, we consider a geometrical approach of set theoretical solutions of Yang-Baxter equation in Lie algebras by defining a new operator. On the other hand, Lie algebra were introduced for the first time by Marius Sophus Lie in the 1870s to study the concept of infinitesimal transformations [13]. Moreover, this algebraic structure has widely served for many areas in science especially physics and geometry, such as [6, 9, 16].

In this study, we handle set theoretical solutions of Yang-Baxter equation in Lie algebras. For this aim, we define a new operator to find new set theoretical solutions of Yang-Baxter equation in this structure. Moreover, we reach that some set-theoretical solutions of Yang-Baxter equation corresponds to the Lyubashenko set theoretical solutions on these structures. We verify that some solutions are preserved under Lie homomorphism, additional homomorphism and $\circledast$-homomorphism. Besides, we deal with geometrical interpretation of set theoretical solution in Lie algebras which are defined in Euclidean space, Minkowski Space and differentiable manifolds. This paper is organized as follows: In Section 2, we recall basic notions which are going to be needed. In Section 3, we give set theoretical solutions of Yang-Baxter equation in Lie algebras. Moreover, we define a new operator to get new solutions which verifies Yang-Baxter condition. Moreover, we examine geometrical interpretation of some solutions in 3-dimensional Euclidean space. In Section 4, we interpret of geometrical meaning of set theoretical solutions of Yang-Baxter equation in Minkowski Space. Finally in Section 5, we construct a bridge between differentiable manifolds and set theoretical solutions of Yang-Baxter equation.

2. Preliminaries

In this section, we recall some definitions which are used during this work.

Definition 2.1. *[17] A Lie algebra over $\mathbb{R}$ is a real vector space U with a bilinear map*

$$[,]: U \times U \to U$$

such that:
• $[X,Y] = -[Y,X]$ *and*
• $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ *(Jacoby identity) for all* $X, Y, Z \in U$.

Definition 2.2. *[10] A Lie homomorphism is a linear map from a Lie algebra ϱ_1 to a Lie algebra ϱ_2 such that it is compatible with the Lie bracket*

$$\Psi : \varrho_1 \to \varrho_2, \quad \Psi([l,m]) = [\Psi(l), \Psi(m)]$$

where $l, m \in \varrho_1$.

Definition 2.3. *[19] Let M be a Hausdorff space. A differentiable structure on M of dimension n is a collection of open charts $(U_i, \phi_i)_{i\varepsilon\Lambda}$ on M where $\phi_i(U_i)$ is an open subset of $\mathbb{R}^n$ such that the following conditions are satisfied:*

(a) $M = \cup_{i\varepsilon\Lambda} U_i$,

(b) For each pair $i, j\varepsilon\Lambda$ the mapping $\phi_i \cdot \phi_j^{-1}$ is a differentiable mapping of $\phi_i(U_i \cap U_j)$ onto $\phi_j(U_i \cap U_j)$,

(c) The collection $(U_i, \phi_i)_{i\varepsilon\Lambda}$ is a maximal family of open charts for which (a) and (b) hold.

Definition 2.4. *[19] A differentiable manifold of dimension n is a Hausdorff space with differentiable structure of dimension n.*

Definition 2.5. *[17] Let M be a real $n-$dimensional differentiable manifold and $\chi(M)$ the module of differentiable vector fields of M and $f \in C^\infty(M, \mathbb{R})$. If X and Y are in $\chi(M)$, then the Lie bracket $[X, Y]$ is defined as a mapping from the ring of functions on M into itself by*

$$[X, Y]f = X(Y(f)) - Y(X(f))$$

where $X(f)$ is the directional derivative of f function in direction X.

Definition 2.6. *[11] The Minkowski Space is the metric space $E_1^3 = (\mathbb{R}^3, \langle , \rangle)$, where the metric is given by*

$$\langle u, v\rangle = u_1v_1 + u_2v_2 - u_3v_3$$

which is called the Minkowski metric for $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3$.

Definition 2.7. *[11] A vector $v \in E_1^3$ is called*

(1) spacelike if $\langle v, v\rangle > 0$ or $v = 0$,

(2) timelike if $\langle v, v\rangle < 0$,

(3) null (lightlike) if $\langle v, v\rangle = 0$ and $v \neq 0$.

3. Set Theoretical Solutions of Yang-Baxter Equation in Lie Algebras and Geometrical View in Euclidean Space

In this part of the paper, we give some set theoretical solutions of Yang-Baxter equation in Lie algebras. And also, we determine which of them are corresponding to Lyubashenko set theoretical solutions of Yang-Baxter equation on these structures. Along with these, we discuss the geometrical interpretations of some set theoretical solutions of Yang-Baxter equation in Euclidean space.

Let F be a field where tensor products are defined and W be a F-space. The mapping over $W \otimes W$ is denoted by μ. The twist map on this structure is given by $\mu(w_1 \otimes w_2) = w_2 \otimes w_1$ and the identity map on F is defined by $I : W \to W$; for a $F-$linear map $S : W \otimes W \to W \otimes W$, let $S^{12} = S \otimes I$, $S^{13} = (I \otimes \mu)(S \otimes I)(\mu \otimes I)$ and $S^{23} = I \otimes S$.

Definition 3.1. *[3] A Yang-Baxter operator is an invertible $F-$linear map $S : W \otimes W \to W \otimes W$ that verifies the braid condition (called the Yang-Baxter equation):*

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}. \tag{3.1}$$

If S verifies Equation (3.1), then both $S \circ \mu$ and $\mu \circ S$ verify the quantum Yang–Baxter equation (QYBE):

$$S^{12} \circ S^{13} \circ S^{23} = S^{23} \circ S^{13} \circ S^{12}. \tag{3.2}$$

Lemma 3.2. *[3] Equations (3.1) and (3.2) are equivalent to each other.*

To construct set theoretical solutions of Yang-Baxter equation in Lie algebras, we need the following definition.

Definition 3.3. *[3] Let L be a set and $S : L \times L \to L \times L$, $S(l,m) = (f(l), g(m))$ be a map. The map S is set theoretical solution of Yang-Baxter equation if it verifies the following equality for $l, m, n \in L$:*

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}, \tag{3.3}$$

which is also equivalent to

$$S^{12} \circ S^{13} \circ S^{23} = S^{23} \circ S^{13} \circ S^{12}, \tag{3.4}$$

where

$$\begin{aligned} S^{12} &: L^3 \to L^3, \;\; S^{12}(l,m,n) = (f(l), g(m), n), \\ S^{23} &: L^3 \to L^3, \;\; S^{23}(l,m,n) = (l, f(m), g(n)), \\ S^{13} &: L^3 \to L^3, \;\; S^{13}(l,m,n) = (f(l), m, g(n)). \end{aligned}$$

First of all, we handle some fundamental set theoretical solutions of Yang-Baxter equation in Lie algebras.

Lemma 3.4. *Let $(L, [,])$ be a Lie algebra. Then, the mapping*

$$S(l,m) = ([c,m], 0)$$

is a set theoretical solution of Yang-Baxter equation for any constant element $c \in L$ and $l, m \in L$ on this structure.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(l,m,n) &= ([c,m], 0, n), \\ S^{23}(l,m,n) &= (l, [c,n], 0). \end{aligned}$$

We need to satisfy the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $(l,m,n) \in L^3$ as below:

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(l,m,n) &= S^{12}(S^{23}(S^{12}(l,m,n))) \\ &= S^{12}(S^{23}([c,m], 0, n)) \\ &= S^{12}([c,m], [c,n], 0) \\ &= ([c,[c,n]], 0, 0) \\ &= S^{23}([c,[c,n]], 0, 0) \\ &= S^{23}(S^{12}(l, [c,n], 0)) \\ &= S^{23}(S^{12}(S^{23}(l,m,n))) = (S^{23} \circ S^{12} \circ S^{23})(l,m,n). \end{aligned}$$

Therefore, the mapping $S(l,m) = ([c,m], 0)$ is a set theoretical solution of Yang-Baxter equation in Lie algebras. ■

Lemma 3.5. *Let $(L, [,])$ be a Lie algebra. Then, the mapping*

$$S(l,m) = (0, [l,c])$$

is a set theoretical solution of Yang-Baxter equation for any constant element $c \in L$ and $l, m \in L$ on this structure.

Proof. It follows from similar procedure in the proof of Lemma 3.4. ■

Lemma 3.6. *Let $(L, [,])$ be a Lie algebra. Then, the mapping*

$$S(l,m) = ([c,m],[l,c])$$

is a set theoretical solution of Yang-Baxter equation for any constant element $c \in L$ and $l,m \in L$ on this structure.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(l,m,n) &= ([c,m],[l,c],n), \\ S^{23}(l,m,n) &= (l,[c,n],[m,c]). \end{aligned}$$

We need to satisfy the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $(l,m,n) \in L^3$.

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(l,m,n) &= S^{12}(S^{23}(S^{12}(l,m,n))) \\ &= S^{12}(S^{23}([c,m],[l,c],n)) \\ &= S^{12}([c,m],[c,n],[[l,c],c]) \\ &= ([c,[c,n]],[[c,m],c],[[l,c],c]) \\ &= ([c,[c,n]],[c,[m,c]],[[l,c],c]). \end{aligned}$$

By the using the property $[c,[m,c]] = [[c,m],c]$ of Lie brackets, then we obtain

$$\begin{aligned} &= S^{23}([c,[c,n]],[l,c],[m,c]) \\ &= S^{23}(S^{12}(l,[c,n],[m,c])) \\ &= S^{23}(S^{12}(S^{23}(l,m,n))) = (S^{23} \circ S^{12} \circ S^{23})(l,m,n). \end{aligned}$$

Therefore, the mapping $S(l,m) = ([c,m],[l,c])$ is a set theoretical solution of Yang-Baxter equation in Lie algebras. ■

Lemma 3.7. *Let $(L, [,])$ be a Lie algebra. Then, the mapping*

$$S(l,m) = ([c,m]+c,c)$$

is a set theoretical solution of Yang-Baxter equation for any constant element $c \in L$ and $l,m \in L$ on this structure.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(l,m,n) &= ([c,m]+c,c,n), \\ S^{23}(l,m,n) &= (l,[c,n]+c,c). \end{aligned}$$

We have to verify the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $(l, m, n) \in L^3$.

$$\begin{aligned}
(S^{12} \circ S^{23} \circ S^{12})(l, m, n) &= S^{12}(S^{23}(S^{12}(l, m, n))) \\
&= S^{12}(S^{23}([c, m] + c, c, n)) \\
&= S^{12}([c, m] + c, [c, n] + c, c) \\
&= ([c, [c, n] + c], c, c) \\
&= ([c, [c, n]], c, c) \\
&= ([c, [c, n]], [c, c] + c, c) \\
&= S^{23}([c, [c, n]], c, c) \\
&= S^{23}([c, [c, n] + c], c, c) \\
&= S^{23}(S^{12}(l, [c, n] + c, c)) \\
&= S^{23}(S^{12}(S^{23}(l, m, n))) = (S^{23} \circ S^{12} \circ S^{23})(l, m, n).
\end{aligned}$$

Then, the mapping $S(l, m) = ([c, m] + c, c)$ is a set theoretical solution of Yang–Baxter equation for any $c \in L$ in Lie algebras. ∎

Lemma 3.8. *Let $(L, [,])$ be a Lie algebra. Then, the mapping*

$$S(l, m) = (c, [l, c] + c)$$

is a set theoretical solution of Yang-Baxter equation for any constant element $c \in L$ and $l, m \in L$ on this structure.

Proof. It is clearly obtained by using similar method in the proof of Lemma 3.7. ∎

Definition 3.9. *[20] The mappings*

$$\begin{aligned}
S : L \times L &\to L \times L \\
(l, m) &\to S(l, m) = (f(l), g(m))
\end{aligned}$$

or

$$\begin{aligned}
S : L \times L &\to L \times L \\
(l, m) &\to S(l, m) = (f(m), g(l))
\end{aligned}$$

are called Lyubashenko set theoretical solutions for $l, m \in L$ where $f : L \to L$ and $g : L \to L$ are functions such that $f(g(x)) = g(f(x))$ for each $x \in L$.

Corollary 3.10. *Using Definition 3.9, we obtain the following results:*

- *In Lemma 3.4, we show that the mapping $S(l, m) = ([c, m], 0)$ is a set theoretical solution of Yang-Baxter equation for Lie algebras. Besides, if we take $f(m) = [c, m]$ and $g(l) = 0$, then we obtain*

  $$f(g(x)) = f(0) = [c, 0] = 0 = g([c, x]) = g(f(x))$$

  for each $x \in L$. So, we conclude that the mapping $S(l, m) = ([c, m], 0)$ is also a Lyubashenko set theoretical solution.

- *In Lemma 3.5, we show that the mapping $S(l, m) = (0, [l, c])$ is a set theoretical solution of Yang-Baxter equation for Lie algebras. Besides, if we take $f(m) = 0$ and $g(l) = [l, c]$, then we obtain*

  $$f(g(x)) = f([x, c]) = 0 = [0, c] = g(0) = g(f(x))$$

  for each $x \in L$. So, we conclude that the mapping $S(l, m) = (0, [l, c])$ is also a Lyubashenko set theoretical solution.

- *In Lemma 3.6, we show that the mapping* $S(l,m) = ([c,m],[l,c])$ *is a set theoretical solution of Yang-Baxter equation for Lie algebras. Besides, if we take* $f(m) = [c,m]$ *and* $g(l) = [l,c]$*, then we have* $f(x) = [c,x] = -[x,c] = -g(x)$ *for each* $x \in L$*. By using this relation and since* $f(x)$ *is an odd function, we obtain*
$$f(g(x)) = f(-f(x)) = -f(f(x)) = g(f(x))$$
for each $x \in L$*. So, we conclude that the mapping* $S(l,m) = ([c,m],[l,c])$ *is also a Lyubashenko set theoretic solution.*

- *In Lemma 3.7, we show that the mapping* $S(l,m) = ([c,m]+c,c)$ *is a set theoretical solution of Yang-Baxter equation for Lie algebras. Besides, if we take* $f(m) = [c,m]+c$ *and* $g(l) = c$ *where* c *is any constant element in* L*, then we obtain*
$$f(g(x)) = f(c) = [c,c]+c = c = g(f(x))$$
for each $x \in L$*. So, we conclude that the mapping* $S(l,m) = ([c,m]+c,c)$ *is also a Lyubashenko set theoretical solution.*

- *In Lemma 3.8, we obtain that the mapping* $S(l,m) = (c,[l,c]+c)$ *is also a Lyubashenko set theoretical solution on* L *by using similar procedure as above.*

Now, we give a theorem which gives us a general set theoretical solution of Yang-Baxter equation for Lie Algebras in 3-dimensional Euclidean space (E^3).

Theorem 3.11. *Let* E *be a Euclidean space and let* $(E^3,[,])$ *be a Lie algebra. Then, the mapping*
$$S(l,m) = ([l,m],0)$$
is a set theoretical solution of Yang-Baxter equation for $l,m \in E^3$*.*

Proof. Let S^{12} and S^{23} be defined as follows:
$$S^{12}(l,m,n) = ([l,m],0,n),$$
$$S^{23}(l,m,n) = (l,[m,n],0)$$
where l,m and n are linearly dependent with Euclidean bases e_1,e_2 and e_3, respectively.

We satisfy the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $l,m,n \in E^3$.
$$\begin{aligned} &(S^{12} \circ S^{23} \circ S^{12})(l,m,n) \\ =\ & S^{12}(S^{23}(S^{12}(l,m,n))) \\ =\ & S^{12}(S^{23}([l,m],0,n)) \\ =\ & S^{12}([l,m],0,0) \\ =\ & (0,0,0) \end{aligned} \tag{3.5}$$
and
$$\begin{aligned} &(S^{23} \circ S^{12} \circ S^{23})(l,m,n) \\ =\ & S^{23}(S^{12}(S^{23}(l,m,n))) \\ =\ & S^{23}(S^{12}(l,[m,n],0)) \\ =\ & S^{23}([l,[m,n]],0,0) \\ =\ & ([l,[m,n]],0,0). \end{aligned} \tag{3.6}$$

From the Equations (3.5) and (3.6), we have $([l,[m,n]],0,0) = (0,0,0)$. As it seen, the condition is satisfied when $[l,[m,n]] = 0$. In 3-dimensional Euclidean space, Lie bracket corresponds to cross product. Therefore, the vector l is parallel to $[m,n]$. So, the equality $[l,[m,n]] = 0$ is verified in 3-dimensional Euclidean space. Hence, the mapping $S(l,m) = ([l,m],0)$ is a set theoretical solution of Yang-Baxter equation in 3-dimensional Euclidean space. ■

By Theorem 3.11, we attain a more general case as follows.

Corollary 3.12. *The mapping*

$$S(l,m) = (f(l,m),0)$$

is a set theoretical solution of Yang-Baxter equation in 3-dimensional Euclidean space where $f(l,0) = f(0,m) = 0$ for each $l,m \in E^3$ since the condition $f(l,f(m,n)) = 0$ is verified for each $l,m,n \in E^3$ in 3-dimensional Euclidean space.

Using similar method in Theorem 3.11, we obtain the following theorem.

Theorem 3.13. *Let $(E^3,[,])$ be a Lie algebra. The mapping*

$$S(l,m) = (0,[m,l])$$

is a set theoretical solution of Yang-Baxter equation for $l,m \in E^3$ in 3-dimensional Euclidean space.

By the help of Theorem 3.13, we also get a more general case as follows.

Corollary 3.14. *The mapping*

$$S(l,m) = (0,g(l,m))$$

is a set theoretical solution of Yang-Baxter equation in 3-dimensional Euclidean space where $g(l,0) = g(0,m) = 0$ for each $l,m \in E^3$ since the condition $g(g(l,m),n)) = 0$ is verified for each $l,m,n \in E^3$ in 3-dimensional Euclidean space.

Now, we introduce a new binary operation in Lie algebras. This operation has advantages to find set theoretical solutions of Yang-Baxter equation on these structures.

Definition 3.15. *Let $(L,[,])$ be a Lie algebra. The binary operation $\circledast$-operation defined as*

$$l \circledast m := [l,m] + l + m$$

for each $l,m \in L$.

Lemma 3.16. *Let $(L,[,])$ be a Lie algebra. Then, the identities*

(i) $l \circledast l = 2l,$

(ii) $l \circledast 0_L = l,$

(iii) $l \circledast (-l) = 0_L,$

(iv) $(l \circledast m) \circledast (m \circledast l) = 2([[l,m],l] + [[l,m],m] + l + m),$

(v) $l \circledast (m - l) = (l \circledast m) - l,$

(vi) $(l \circledast m) + (m \circledast l) = 2(l + m)$

are verified for each $l,m \in L$.

Theorem 3.17. *Let $(E^3, [,])$ be a Lie algebra. Then, the mapping*

$$S(l, m) = (l \circledast m, 0)$$

is a set theoretical solution of Yang-Baxter equation for $l, m \in E^3$ in 3-dimensional Euclidean space.

Proof. Let S^{12} and S^{23} be defined as follows:

$$S^{12}(l, m, n) = (l \circledast m, 0, n) = ([l, m] + l + m, 0, n),$$
$$S^{23}(l, m, n) = (l, m \circledast n, 0) = (l, [m, n] + m + n, 0)$$

where l, m and n are linearly dependent to Euclidean bases e_1, e_2 and e_3, respectively.

We have to satisfy the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $l, m, n \in E^3$.

$$\begin{aligned}
&(S^{12} \circ S^{23} \circ S^{12})(l, m, n) \\
&= S^{12}(S^{23}(S^{12}(l, m, n))) \\
&= S^{12}(S^{23}([l, m] + l + m, 0, n)) \\
&= S^{12}([l, m] + l + m, [0, n] + 0 + n, 0) \\
&= S^{12}([l, m] + l + m, n, 0) \\
&= ([[l, m] + l + m, n] + [l, m] + l + m + n, 0, 0) \\
&= ([[l, m], n] + [l, n] + [m, n] + [l, m] + l + m + n, 0, 0)
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&(S^{23} \circ S^{12} \circ S^{23})(l, m, n) \\
&= S^{23}(S^{12}(S^{23}(l, m, n))) \\
&= S^{23}(S^{12}(l, [m, n] + m + n, 0)) \\
&= S^{23}([l, [m, n] + m + n] + l + [m, n] + m + n, 0, 0) \\
&= ([l, [m, n] + m + n] + l + [m, n] + m + n, 0, 0) \\
&= ([l, [m, n]] + [l, m] + [l, n] + [m, n] + l + m + n, 0, 0).
\end{aligned} \tag{3.8}$$

Using Equation (3.7) and (3.8), we have

$$\begin{aligned}
&([[l, m], n] + [l, n] + [m, n] + [l, m] + l + m + n, 0, 0) \\
&= ([l, [m, n]] + [l, m] + [l, n] + [m, n] + l + m + n, 0, 0).
\end{aligned}$$

By the help of above equality, we need to satisfy the following condition

$$[[l, m], n] = [l, [m, n]].$$

From the point of geometrical view, since we can correspond cross product to Lie bracket, we obtain

$$l \wedge m = [l, m].$$

Hence, we get

$$[[l, m], n] = ((l \wedge m) \wedge n) = 0. \quad \text{(Since } (l \wedge m) \text{ is parallel to } n)$$
$$[l, [m, n]] = (l \wedge (m \wedge n)) = 0. \quad \text{(Since } (m \wedge n) \text{ is parallel to } l)$$

So, the mapping $S(l, m) = ([l, m] + l + m, 0)$ is a set theoretical solution of Yang-Baxter equation in 3-dimensional Euclidean space. ■

After these theorems and lemmas, we can give the following examples in E^3.

Example 3.18. *Let* $(E^3, [,])$ *be a Lie algebra. Then, the mappings*
(i) $S(l,m) = (l \cdot (l \wedge m), 0)$,
(ii) $S(l,m) = (l \cdot (l \wedge m), m)$,
(ii) $S(l,m) = (l \cdot (l \wedge m), m \cdot (l \wedge m))$
are set theoretical solutions of Yang-Baxter equation in 3-dimensional Euclidean space where the operation "$\cdot$"corresponds dot product such that $l \cdot (m \wedge n) = (l \wedge m) \cdot n = det(l,m,n)$ *where* $l, m, n \in E^3$.

Theorem 3.19. *Let* $(L, [,])$ *be a Lie algebra. Then, the mapping*

$$S(l,m) = (\frac{1}{2}((l \circledast m) + (m \circledast l)), 0)$$

is a set theoretical solution of Yang-Baxter equation for $l, m \in L$ *in Lie algebras.*

Proof. Let S^{12} and S^{23} be defined as follows:

$$S^{12}(l,m,n) = (\tfrac{1}{2}((l \circledast m) + (m \circledast l)), 0, n),$$
$$S^{23}(l,m,n) = (l, \tfrac{1}{2}((m \circledast n) + (n \circledast m)), 0).$$

We need to verify the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $l, m, n \in L$.

$$\begin{aligned}
&(S^{12} \circ S^{23} \circ S^{12})(l,m,n) \\
&= S^{12}(S^{23}(S^{12}(l,m,n))) \\
&= S^{12}(S^{23}(\frac{1}{2}((l \circledast m) + (m \circledast l)), 0, n)) \\
&= S^{12}(S^{23}(\frac{1}{2}(2(l+m), 0, n)), && \text{(By Lemma 3.16 } (vi)) \\
&= S^{12}(S^{23}(l+m, 0, n)) \\
&= S^{12}(l+m, \frac{1}{2}((0 \circledast n) + (n \circledast 0)), 0) \\
&= S^{12}(l+m, \frac{1}{2}(2n), 0) && \text{(By Lemma 3.16 } (ii)) \\
&= S^{12}(l+m, n, 0) \\
&= (\frac{1}{2}(((l+m) \circledast n) + (n \circledast (l+m))), 0, 0) \\
&= (\frac{1}{2}(2(l+m+n), 0, 0)) && \text{(By Lemma 3.16 } (vi)) \\
&= (l+m+n, 0, 0)
\end{aligned} \tag{3.9}$$

and we have

$$\begin{aligned}
& (S^{23} \circ S^{12} \circ S^{23})(l,m,n) \\
= & S^{23}(S^{12}(S^{23}(l,m,n))) \\
= & S^{23}(S^{12}(l, \frac{1}{2}((m \circledast n) + (n \circledast m)), 0)) \\
= & S^{23}(S^{12}(l, \frac{1}{2}(2(m+n), 0)) && \text{(By Lemma 3.16 } (vi)) \\
= & S^{23}(S^{12}(l, m+n, 0)) \\
= & S^{23}(\frac{1}{2}((l \circledast (m+n)) + ((m+n) \circledast l)), 0, 0) \\
= & S^{23}(\frac{1}{2}(2(l+m+n)), 0, 0) && \text{(By Lemma 3.16 } (vi)) \\
= & S^{23}(l+m+n, 0, 0) \\
= & (l+m+n, 0, 0).
\end{aligned} \tag{3.10}$$

From the equality of (3.9) and (3.10), the mapping $S(l,m) = (\frac{1}{2}((l \circledast m) + (m \circledast l)), 0)$ is satisfied Yang-Baxter equation in Lie algebras. ■

By Theorem 3.17 and Theorem 3.19, we attain a more general case as follows.

Corollary 3.20. *The mapping*

$$S(l,m) = (f(l,m), 0)$$

is a set theoretical solution of Yang-Baxter equation in 3-dimensional Euclidean space where $f(l,0) = f(0,m) = 0$ *for each* $l, m \in E^3$ *since the condition* $f(f(l,m),n) = f(l, f(m,n))$ *is verified for each* $l,m,n \in E^3$ *in 3-dimensional Euclidean space.*

Definition 3.21. *Let* L *be a Lie algebra.Then, the mapping* Ψ *is called a* $\circledast$*-homomorphism if the equality*

$$\Psi(l \circledast m) = \Psi(l) \circledast \Psi(m)$$

is satisfied for each $l, m \in L$.

Lemma 3.22. *Let* Ψ *be a Lie homomorphism and additional homomorphism, then* Ψ *is also a* $\circledast-$*homomorphism in Lie algebras.*

Proof. Let L be a Lie algebra and $l, m \in L$. Then, we obtain

$$\begin{aligned}
\Psi(l \circledast m) &= \Psi([l,m] + l + m) \\
&= \Psi([l,m]) + \Psi(l) + \Psi(m) \\
&= ([\Psi(l), \Psi(m)]) + \Psi(l) + \Psi(m) \\
&= \Psi(l) \circledast \Psi(m).
\end{aligned}$$

■

Lemma 3.23. *Let* L *be a Lie algebra and the mapping* $f : L^2 \to L$ *only consist of the combinations of binary operations* "$[,]$", "$+$" *and* "$\circledast$". *Besides, the mapping* Ψ *be a Lie homomorphism and additional homomorphism. Then, the mapping*

$$\Psi(f(l,m)) = \begin{cases} f(\Psi(l), \Psi(m)), & \text{if } f(l,m) \text{ does not contain any constant} \\ f(\Psi(l), \Psi(m)) + \Psi(c) - c, & \text{if } f(l,m) \text{ contains any constant element } c \end{cases}$$

is verified for each $l, m \in L$.

Proof. We make induction on the number of the operations of the f mapping.

• Assume that the mapping $f(l, m)$ consists of only the binary operation " $+$ " and it does not contain any constant element. Then, we use induction on the number of " $+$ " operators as logical induction on the number of connectives:

- If $f(l, m) = p_{i_1} + p_{i_2}$ such that $p_{i_1}, p_{i_2} \in \{l, m\}$. $\Psi(f(l, m)) = \Psi(p_{i_1} + p_{i_2}) = \Psi(p_{i_1}) + \Psi(p_{i_2}) = f(\Psi(l), \Psi(m))$.

- Let $f(l, m) = p_{i_1} + p_{i_2} + ... + p_{i_n}$ such that $p_{i_1}, p_{i_2}, ..., p_{i_n} \in \{l, m\}$. Assume that $\Psi(f(l, m)) = \Psi(p_{i_1} + p_{i_2} + ... + p_{i_n}) = \Psi(p_{i_1}) + \Psi(p_{i_2}) + ... + \Psi(p_{i_n}) = f(\Psi(l), \Psi(m))$ is verified for each $p_{i_1}, p_{i_2}, ..., p_{i_n} \in \{l, m\}$. Let $g(l, m) = p_{i_1} + p_{i_2} + ... + p_{i_n} + p_{i_{n+1}}$ such that $p_{i_{n+1}} \in \{l, m\}$. Then, we show that $\Psi(g(l, m)) = g(\Psi(l), \Psi(m))$ as follows:

$$\begin{aligned}\Psi(g(l, m)) &= \Psi(p_{i_1} + p_{i_2} + ... + p_{i_n} + p_{i_{n+1}}) \\ &= \Psi(p_{i_1} + p_{i_2} + ... + p_{i_n}) + \Psi(p_{i_{n+1}}) \\ &= \Psi(p_{i_1}) + \Psi(p_{i_2}) + ... + \Psi(p_{i_n}) + \Psi(p_{i_{n+1}}) \\ &= g(\Psi(l), \Psi(m)).\end{aligned}$$

• We assume that the mapping $f(l, m)$ consists of only the binary operation " $+$ " and it contains any constant element such as c. Then we have the following conditions:

- If $f(l, m) = c$ is any constant funtion, then we obtain clearly $\Psi(f(l, m)) = \Psi(c) = \Psi(c) + c - c = f(\Psi(l), \Psi(m)) + \Psi(c) - c$.

- Let $f(l, m) = p_{i_1} + p_{i_2} + ... + p_{i_n} + c$ such that $p_{i_1}, p_{i_2}, ..., p_{i_n} \in \{l, m\}$ and c be any constant element. Then we obtain

$$\begin{aligned}\Psi(f(l, m)) &= \Psi(p_{i_1} + p_{i_2} + ... + p_{i_n} + c) \\ &= \Psi(p_{i_1}) + \Psi(p_{i_2}) + ... + \Psi(p_{i_n}) + \Psi(c) \\ &= \Psi(p_{i_1}) + \Psi(p_{i_2}) + ... + \Psi(p_{i_n}) + \Psi(c) + c - c \\ &= (\Psi(p_{i_1}) + \Psi(p_{i_2}) + ... + \Psi(p_{i_n}) + c) + \Psi(c) - c \\ &= f(\Psi(l), \Psi(m)) + \Psi(c) - c\end{aligned}$$

So, the equality $f(l, m) = f(\Psi(l), \Psi(m))$ is satisfied when the mapping consists of only "$+$"operation.
By using similar procedure as the "$+$"operation, we verify this equality for another operations and their combinations on Lie algebras. ∎

Theorem 3.24. *Let Ψ be a Lie homomorphism and an additional homomorphism. If $S(l, m) = (f(l, m), g(l, m))$ is a set theoretical solution of Yang-Baxter equation in Lie algebras where $f(l, m)$ and $g(l, m)$ do not contain any constant element and consist of only " $[,]$", " $+$"and " $\circledast$"operations, then $\Psi(S(l, m))$ is also a set theoretical solution of Yang-Baxter equation in Lie algebras.*

Proof. Let L be a Lie algebra. Assume that $S(l, m) = (f(l, m), g(l, m))$ is a set theoretical solution of Yang-Baxter equation in Lie algebras where $f(l, m)$ and $g(l, m)$ do not contain a constant element and consist of only " $[,]$", "$+$"and " $\circledast$"operations. Then, we have the following equality:

$$(S^{12} \circ S^{23} \circ S^{12})(l, m, n) = (S^{23} \circ S^{12} \circ S^{23})(l, m, n) \tag{3.11}$$

for each l, m and $n \in L$. So, we obtain the following conclusions:

$$\begin{aligned}
&(S^{12} \circ S^{23} \circ S^{12})(l,m,n) \\
&= S^{12}(S^{23}(f(l,m), g(l,m), n)) \\
&= S^{12}(f(l,m), f(g(l,m),n), g(g(l,m),n)) \\
&= (f(f(l,m), f(g(l,m),n)), g(f(l,m), f(g(l,m),n)), g(g(l,m),n))
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&(S^{23} \circ S^{12} \circ S^{23})(l,m,n) \\
&= S^{23}(S^{12}(l, f(m,n), g(m,n))) \\
&= S^{23}(f(l, f(m,n)), g(l, f(m,n)), g(m,n)) \\
&= (f(l, f(m,n)), f(g(l, f(m,n)), g(m,n)), g(g(l, f(m,n)), g(m,n))).
\end{aligned} \tag{3.13}$$

From the Equality (3.11), we achieve the Equation (3.12) and the Equation (3.13). Moreover, we get

$$\Psi(S(l,m)) = \Psi((f(l,m), g(l,m))) = (f(\Psi(l), \Psi(m)), g(\Psi(l), \Psi(m))).$$

If we show the accuracy of the below equation

$$(S^{12} \circ S^{23} \circ S^{12})(\Psi(l), \Psi(m), \Psi(n)) = (S^{23} \circ S^{12} \circ S^{23})(\Psi(l), \Psi(m), \Psi(n)) \tag{3.14}$$

then we prove that $\Psi(S(l,m))$ is a set theoretical solution of Yang-Baxter equation in Lie algebras.
Now, we verify the Equality (3.14) by the help of Lemma 3.23,the Equations (3.12) and (3.13), respectively.

$$\begin{aligned}
&(S^{12} \circ S^{23} \circ S^{12})(\Psi(l), \Psi(m), \Psi(n)) \\
&= (f(f(\Psi(l), \Psi(m)), f(g(\Psi(l), \Psi(m)), \Psi(n))), g(f(\Psi(l), \Psi(m)), \\
&\quad f(g(\Psi(l), \Psi(m)), \Psi(n))), g(g(\Psi(l), \Psi(m)), \Psi(n))) \\
&= \Psi((f(f(l,m), f(g(l,m),n)), g(f(l,m), f(g(l,m),n)), g(g(l,m),n))) \\
&= \Psi(f(f(l,m), f(g(l,m),n)), g(f(l,m), f(g(l,m),n)), g(g(l,m),n)) \\
&= \Psi(f(l, f(m,n)), f(g(l, f(m,n)), g(m,n)), g(g(l, f(m,n)), g(m,n))) \\
&= f(\Psi(l), f(\Psi(m), \Psi(n))), f(g(\Psi(l), f(\Psi(m), \Psi(n))), g(\Psi(m), \Psi(n))), \\
&\quad g(g(\Psi(l), f(\Psi(m), \Psi(n))), g(\Psi(m), \Psi(n))).
\end{aligned}$$

Then, we reach

$$(S^{12} \circ S^{23} \circ S^{12})(\Psi(l), \Psi(m), \Psi(n)) = (S^{23} \circ S^{12} \circ S^{23})(\Psi(l), \Psi(m), \Psi(n))$$

for each $l, m, n \in L$. So, $\Psi(S(l,m))$ is a set theoretical solution of Yang-Baxter equation in Lie algebras. ∎

Theorem 3.25. *Let Ψ be a Lie homomorphism and an additional homomorphism. If $S(l,m) = (f(l), g(m))$ is a Lyubashenko set theoretical solution of Yang-Baxter equation in Lie algebras where the funtions f and g consist of only "$[,]$", "$+$"and "$\circledast$"operations, then $\Psi(S(l,m))$ is also a Lyubashenko set theoretical solution of Yang-Baxter equation in Lie algebras.*

Proof. It follows from Lemma 3.23 and Theorem 3.24. ∎

4. Set Theoretical Solutions of Yang-Baxter Equation in Minkowski Space

In the previous section, we study in Euclidean space and realize that we need more flexible space to obtain new solutions. For this reason, we decide to study in Minkowski space. The following theorems correspond to geometrical interpretations of set theoretical solutions of Yang-Baxter equation in Minkowski space via Lie algebras.

Theorem 4.1. *Let* $(E_1^3, [,])$ *be a Lie algebra. Then, the mapping*

$$S(l, m) = ([m, l] - l, 0)$$

is a set theoretical solution of Yang-Baxter equation for $l, m \in E_1^3$ *in Minkowski space where* l *is a null vector and* m *is a spacelike, timelike or null vector.*

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(l, m, n) &= ([l, m], [m, l], n), \\ S^{23}(l, m, n) &= (l, [m, n], [n, m]). \end{aligned}$$

We need to reach the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $l, m, n \in E_1^3$. Then, we get

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(l, m, n) &= S^{12}(S^{23}(S^{12}(l, m, n))) \\ &= S^{12}(S^{23}([m, l] - l, 0, n)) \\ &= S^{12}([m, l] - l, 0, 0) \\ &= (-[m, l] + l, 0, 0) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} (S^{23} \circ S^{12} \circ S^{23})(l, m, n) &= S^{23}(S^{12}(S^{23}(l, m, n))) \\ &= S^{23}(S^{12}(l, [n, m] - m, 0)) \\ &= S^{23}([[n, m] - m, l] - l, 0, 0) \\ &= ([[n, m] - m, l] - l, 0, 0) \\ &= ([[n, m], l] - [m, l] - l, 0, 0). \end{aligned} \tag{4.2}$$

From the Equations (4.1) and (4.2), we have

$$(-[m, l] + l, 0, 0) = ([[n, m], l] - [m, l] - l, 0, 0).$$

Thus, we need to satisfy the following condition

$$[[n, m], l] = 2l. \tag{4.3}$$

If we use cross product instead of Lie bracket, then we get

$$(n \wedge m) \wedge l = 2l. \tag{4.4}$$

From the geometrical meaning of cross product, we know that if $a \wedge b = c$ then c is both orthogonal to a and b where $a, b, c \in E_1^3$. So, the Equation (4.4) is not achieved because a vector can not orthogonal to itself in Euclidean space. However in Minkowski Space a null vector is orthogonal to itself so if l is a null vector and $(n \wedge m)$ is a spacelike vector, then the Equation (4.3) is satisfied. ■

Theorem 4.2. *Let $(E_1^3, [,])$ be a Lie algebra. Then, the mapping*

$$S(l, m) = (0, [l, m] + l)$$

is a set theoretical solution of Yang-Baxter equation for $l, m \in E_1^3$ in Minkowski space where l is a null or spacelike vector and m is a spacelike vector.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(l, m, n) &= ([l, m], [m, l], n), \\ S^{23}(l, m, n) &= (l, [m, n], [n, m]). \end{aligned}$$

We have to get the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $l, m, n \in E_1^3$. Then

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(l, m, n) &= S^{12}(S^{23}(S^{12}(l, m, n))) \\ &= S^{12}(S^{23}([l, m], [m, l], n)) \\ &= S^{12}(S^{23}(0, [l, m] + l, n)) \\ &= S^{12}(0, 0, [[l, m] + l, n] + [l, m] + l) \\ &= (0, 0, [[l, m] + l, n] + [l, m] + l) \\ &= (0, 0, [[l, m], n] + [l, n] + [l, m] + l) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} (S^{23} \circ S^{12} \circ S^{23})(l, m, n) &= S^{23}(S^{12}(S^{23}(l, m, n))) \\ &= S^{23}(S^{12}(l, [m, n], [n, m])) \\ &= S^{23}(S^{12}(l, 0, [m, n] + m)) \\ &= S^{23}(0, l, [m, n] + m) \\ &= (0, 0, [l, [m, n] + m] + l) \\ &= (0, 0, [l, [m, n]] + [l, m] + l). \end{aligned} \tag{4.6}$$

From the Equations (4.5) and (4.6), we have

$$(0, 0, [[l, m], n] + [l, n] + [l, m] + l) = (0, 0, [l, [m, n]] + [l, m] + l).$$

Thus, we need to satisfy the following condition

$$[[l, m], n] + [l, n] = [l, [m, n]]. \tag{4.7}$$

With the help of Jacobi identity of Lie bracket definition, we know that

$$[l, [m, n]] = -[m, [n, l]] - [n, [l, m]] \tag{4.8}$$

and on the other hand from anti-symmetry identity of Lie bracket definition, we have

$$-[n, [l, m]] = [[l, m], n]. \tag{4.9}$$

Finally, from the Equations (4.7), (4.8) and (4.9) , we attain

$$[l, n] = [m, [l, n]]. \tag{4.10}$$

When we examine the Equation (4.10) in geometric terms via cross product, we get

$$(l \wedge n) = m \wedge (l \wedge n).$$

So, if $(l \wedge n)$ is a null vector and m is a spacelike vector, then the Equation (4.7) is verified. ■

Theorem 4.3. *The mappings in Theorem 4.1 and 4.2 preserve the Yang-Baxter condition in Minkowski space under the Ψ mapping where it is a Lie and additional homomorphism.*

Proof. It follows from Theorem 3.24. ■

5. Set Theoretical Solutions of Yang-Baxter Equation in Differentiable Manifolds via Lie Algebras

Until this section, we have discussed some set theoretical solutions of Yang-Baxter equations in Lie algebras, 3-dimensional Euclidean space and Minkowski space. Finally, we give a definition of Yang-Baxter equation in differentiable manifolds and we attain some solutions for this equation on this structure. Additionally, we present the definition of manifold quantum Yang-Baxter equation.

Definition 5.1. *Let M be a differentiable manifold, U, V and W be vector fields on M and f be a smooth function. Then, the mapping*

$$((S^{12} \circ S^{23} \circ S^{12})(U, V, W))_f = ((S^{23} \circ S^{12} \circ S^{23})(U, V, W))_f$$

is called manifold quantum Yang-Baxter equation.

Theorem 5.2. *Let M be an n-dimensional differentiable manifold and its local coordinate system denoted by $(x_1, x_2, ..., x_n)$. Assume that $U = g_1 \frac{\partial}{\partial x_1}$ and $V = g_2 \frac{\partial}{\partial x_2}$ are vector fields on M where f, g_1 and g_2 are smooth functions such that the functions g_1 and g_2 depend on the variables x_1 or x_3 and x_2 or x_3, respectively. Then, the mapping*

$$\begin{aligned} S(U, V) &= ([U, V]f, 0), \\ &= ([g_1 \frac{\partial}{\partial x_1}, g_2 \frac{\partial}{\partial x_2}]f, 0) \end{aligned}$$

is a set theoretical solution of Yang-Baxter equation on manifold M.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(U, V, W) &= ([U, V]f, 0, W), \\ S^{23}(U, V, W) &= (U, [V, W]f, 0) \end{aligned}$$

where $U = g_1 \frac{\partial}{\partial x_1}, V = g_2 \frac{\partial}{\partial x_2}$ and $W = g_3 \frac{\partial}{\partial x_3}$ are vector fields on M such that the functions g_1, g_2 and g_3 depend on the variables x_1 or x_3; x_2 or x_3 and x_1 or x_2 or x_3, respectively.

We satisfy the equation

$$((S^{12} \circ S^{23} \circ S^{12})(g_1 \frac{\partial}{\partial x_1}, g_2) \frac{\partial}{\partial x_2}, g_3 \frac{\partial}{\partial x_3}))_f = ((S^{23} \circ S^{12} \circ S^{23})(g_1 \frac{\partial}{\partial x_1}, g_2 \frac{\partial}{\partial x_2}, g_3 \frac{\partial}{\partial x_3}))_f$$

for each g_1, g_2 and g_3 functions with the help of the Definition 2.5.

$$\begin{aligned}
&((S^{12} \circ S^{23} \circ S^{12})(g_1\frac{\partial}{\partial x_1}, g_2\frac{\partial}{\partial x_2}, g_3\frac{\partial}{\partial x_3}))_f \\
&= (S^{12}(S^{23}(S^{12}(g_1\frac{\partial}{\partial x_1}, g_2\frac{\partial}{\partial x_2}, g_3\frac{\partial}{\partial x_3}))_f)_f)_f \\
&= (S^{12}(S^{23}(([g_1\frac{\partial}{\partial x_1}, g_2\frac{\partial}{\partial x_2}]f, 0, g_3\frac{\partial}{\partial x_3}))_f)_f \\
&= (S^{12}(S^{23}((g_1\frac{\partial}{\partial x_1}(g_2\frac{\partial}{\partial x_2}(f)) - g_2\frac{\partial}{\partial x_2}(g_1\frac{\partial}{\partial x_1}(f)), 0, g_3\frac{\partial}{\partial x_3}))_f)_f \\
&= (S^{12}(S^{23}((g_1\frac{\partial g_2}{\partial x_1}\frac{\partial f}{\partial x_2} + g_1 g_2\frac{\partial^2 f}{\partial x_1 \partial x_2} - g_2\frac{\partial g_1}{\partial x_2}\frac{\partial f}{\partial x_1} - g_2 g_1\frac{\partial^2 f}{\partial x_2 \partial x_1}, 0, g_3\frac{\partial}{\partial x_3}))_f)_f \\
&= (S^{12}(S^{23}(g_1 g_2\frac{\partial^2 f}{\partial x_1 \partial x_2} - g_2 g_1\frac{\partial^2 f}{\partial x_2 \partial x_1}, 0, g_3\frac{\partial}{\partial x_3}))_f)_f \\
&= (S^{12}(S^{23}(0, 0, g_3\frac{\partial}{\partial x_3}))_f)_f \\
&= (S^{12}(0,0,0))_f \\
&= (0,0,0)
\end{aligned}$$

and

$$\begin{aligned}
&((S^{23} \circ S^{12} \circ S^{23})(g_1\frac{\partial}{\partial x_1}, g_2\frac{\partial}{\partial x_2}, g_3\frac{\partial}{\partial x_3}))_f \\
&= (S^{23}(S^{12}(S^{23}(g_1\frac{\partial}{\partial x_1}, g_2\frac{\partial}{\partial x_2}, g_3\frac{\partial}{\partial x_3}))_f)_f)_f \\
&= (S^{23}(S^{12}(g_1\frac{\partial}{\partial x_1}, [g_2\frac{\partial}{\partial x_2}, g_3\frac{\partial}{\partial x_3}]f, 0))_f)_f \\
&= (S^{23}(S^{12}(g_1\frac{\partial}{\partial x_1}, g_2\frac{\partial}{\partial x_2}(g_3\frac{\partial}{\partial x_3}(f)) - g_3\frac{\partial}{\partial x_3}(g_2\frac{\partial}{\partial x_2}(f)), 0))_f)_f \\
&= (S^{23}(S^{12}((g_1\frac{\partial}{\partial x_1}, g_2\frac{\partial g_3}{\partial x_2}\frac{\partial f}{\partial x_3} + g_2 g_3\frac{\partial^2 f}{\partial x_2 \partial x_3} - g_3\frac{\partial g_2}{\partial x_3}\frac{\partial f}{\partial x_2} - g_3 g_2\frac{\partial^2 f}{\partial x_3 \partial x_2}, 0))_f)_f \\
&= (S^{23}(S^{12}(g_1\frac{\partial}{\partial x_1}, g_2 g_3\frac{\partial^2 f}{\partial x_2 \partial x_3} - g_3 g_2\frac{\partial^2 f}{\partial x_3 \partial x_2}, 0))_f)_f \\
&= (S^{23}(S^{12}(g_1\frac{\partial}{\partial x_1}, 0, 0))_f)_f \\
&= (S^{23}(0,0,0))_f \\
&= (0,0,0).
\end{aligned}$$

■

Example 5.3. *Let M be an n-dimensional differentiable manifold and two vector fields are defined as $U = x_1^2 x_3\frac{\partial}{\partial x_1}$ and $V = tan(x_3)\frac{\partial}{\partial x_2}$ on M. Then, the mapping*

$$\begin{aligned}
S(U,V) &= ([U,V]f, 0), \\
&= ([x_1^2 x_3\frac{\partial}{\partial x_1}, tan(x_3)\frac{\partial}{\partial x_2}]f, 0)
\end{aligned}$$

is a set theoretical solution of Yang-Baxter equation on manifold M.

Each mapping must not satisfy manifold quantum Yang-Baxter condition. For example, we can define a mapping which does not verify this condition as follows.

Example 5.4. *Let M be a differentiable manifold, U and V be vector fields on M and f be smooth function. Then the mapping*

$$\begin{aligned} S(U,V) &= ([U,V]f,[V,U]f), \\ &= (-V_f, V_f) \end{aligned}$$

is not a set theoretical solution of Yang–Baxter equation on manifold M.
Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(U,V,W) &= ([U,V]f,[V,U]f,W), \\ S^{23}(U,V,W) &= (U,[V,W]f,[W,V]f) \end{aligned}$$

where W is a vector field on M. We need to satisfy the equation

$$((S^{12} \circ S^{23} \circ S^{12})(U,V,W)_f = ((S^{23} \circ S^{12} \circ S^{23})(U,V,W)_f$$

for each U,V,W vector fields. Then, we obtain

$$\begin{aligned} &(S^{12} \circ S^{23} \circ S^{12})(U,V,W)_f \\ =\ &(S^{12}(S^{23}(S^{12}(U,V,W)_f)_f)_f \\ =\ &(S^{12}(S^{23}([U,V]f,[V,U]f,W)_f)_f \\ =\ &(S^{12}(S^{23}(-V,V,W)_f)_f \\ =\ &(S^{12}(-V,[V,W]f,[W,V]f))_f \\ =\ &(S^{12}(-V,-W,W))_f \\ =\ &(S^{12}([-V,-W]f,[-W,-V]f,W))_f \\ =\ &(W,-W,W) \end{aligned}$$

and

$$\begin{aligned} &((S^{23} \circ S^{12} \circ S^{23})(U,V,W))_f \\ =\ &(S^{23}(S^{12}(S^{23}(U,V,W)_f)_f)_f \\ =\ &(S^{23}(S^{12}(U,[V,W]f,[W,V]f))_f)_f \\ =\ &(S^{23}(S^{12}(U,-W,W))_f)_f \\ =\ &(S^{23}(([U,-W]f,[-W,U]f,W))_f)_f \\ =\ &(S^{23}(W,-W,W))_f)_f \\ =\ &(S^{23}(W,[-W,W]f,[W,-W]f))_f \\ =\ &(W,0,0). \end{aligned}$$

As we can see

$$(W,-W,W) \neq (W,0,0)$$

by this way the mapping does not verify manifold quantum Yang-Baxter condition.

Furthermore we have the following theorem for preserving of set theoretical solution of Yang-Baxter equation under the $\Psi-$homomorphism.

Theorem 5.5. *The mappings in Theorem 5.2 preserve the Yang-Baxter condition in differentiable manifolds under the Ψ mapping where it is a Lie and additional homomorphism.*

Proof. It follows from Theorem 3.24. ■

6. Conclusion

Lie algebra is one of the important algebraic structure which has been extensively investigated by many researchers. This structure has an important role for different areas such as physics and geometry. In this study, we examine fundamental set theoretical solutions of Yang-Baxter equation in Lie algebras. We indicate that some set theoretical solutions of Yang-Baxter equation corresponds to the Lyubashenko set theoretical solutions on these structures. Then, we define $\circledast$-operation on this structure. In accordance with this, we prove that all set theoretical solutions which do not contain any constant element are preserved under the homomorphism. Moreover, we give an interpretation for these solutions from the point of geometrical view in Euclidean space, Minkowski space and differentiable manifolds by constructing a bridge among Lie algebras and these all geometrical structures. As a result of this study, we think that further researchers should focus on new set theoretical solutions of Yang-Baxter equation in many algebraic structures with geometrical approach such as MV-algebras, C-algebras, Jordan algebras and etc. Furthermore, they should use applications of these solutions in different areas such as physics, statistical mechanics, quantum groups, quantum mechanics, knot theory and etc.

Compliance with ethical standarts

Conflict of interest The author declares that he has no conflict of interests.
Human and animal participants This article does not contain any studies with human participants performed by any of the authors.

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A common fixed point theorem for four weakly compatible self maps of a S-metric space using (CLR) property

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Abstract. In this paper, by employing a contractive condition of integral type, we obtain a unique common fixed point for four weakly compatible self maps of a S-metric space which satisfy common limit range property.

AMS Subject Classifications: 54H25, 47H10.

Keywords: S-metric space, Fixed point, Weakly compatibility, Common limit range property.

Contents

1. Introduction

Gerald Jungck [7] introduced the concept of compatibility to generalized the notion of commutative property. Further Jungck and Rhoades [8] proposed weakly compatibility of mappings. Also they proved that for a pair of mappings compatibility always implies weakly compatibility but not conversely.

To prove common fixed point theorems, Sintunavarat et al [15] initiated common limit range (CLR) property, which generalized the (E.A) property proposed M. Aamri, D. El Moutawakil [1].

Several authors Dhage, Gahler, Sedghi, Mustafa [3–5, 14, 16] generalized the notion of metric space by introducing 2-metric space, D^*-metric spaces and G-metric spaces.

Shaban Sedghi et al [13] proposed S-metric space as further generalization of metric spaces. This concept of S-metric spaces generated lot of interest among many researches.

In this paper, we prove a common fixed point theorem for four weakly compatible self maps of S-metric space satisfying common limit range property along with an integral type contractive condition [2]. Our result generalizes the results already proved in literature [6]. A suitable example is provided to validate our theorem.

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2. Preliminaries

Definition 2.1. *[13] Let M be non empty set. A function $S : M^3 \longrightarrow [0, \infty)$ is said to be a S-metric on M, if for each $\nu, \omega, \vartheta, \lambda \in M$*

1. $S(\nu, \omega, \vartheta) \geq 0$
2. $S(\nu, \omega, \vartheta) = 0 \Leftrightarrow \nu = \omega = \vartheta$
3. $S(\nu, \omega, \vartheta) \leq S(\nu, \nu, \lambda) + S(\omega, \omega, \lambda) + S(\vartheta, \vartheta, \lambda)$

then (M, S) is called a S-metric space.

Lemma 2.2. *[11] In a S-metric space we have $S(\nu, \nu, \omega) = S(\omega, \omega, \nu)$ for all $\nu, \omega \in M$.*

Definition 2.3. *[12] Let (M, S) be a S- metric space. A sequence (ν_n) in M is said to be convergent if there is a $\nu \in M$ such that $S(\nu_n, \nu_n, \nu) \to 0$ as $n \to \infty$, that is, for each $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(\nu_n, \nu_n, \nu) < \epsilon$ and we denote this by writing $\lim_{n\to\infty} \nu_n = \nu$.*

Definition 2.4. *[12] Let (M, S) be a S- metric space. A sequence (ν_n) in M is said to be Cauchy sequence if for each $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $S(\nu_n, \nu_n, \nu_m) \to 0$ for each $n, m \geq n_0$.*

Definition 2.5. *[12] A S- metric space (M, S) is said to be complete if for every Cauchy sequence converges to some point in it.*

Lemma 2.6. *[12] In a S-metric space (M, S), if there exist two sequences (ν_n) and (ω_n) such that $\lim_{n\to\infty} \nu_n = \nu$ and $\lim_{n\to\infty} \omega_n = \omega$, then $\lim_{n\to\infty} S(\nu_n, \nu_n, \omega_n) = S(\nu, \nu, \omega)$.*

Definition 2.7. *[8] The self mappings H, J of a S-metric space (M, S) are called weakly compatible if $HJ\nu = JH\nu$ whenever $H\nu = J\nu$ for any ν in M.*

Definition 2.8. *[9] In a S-metric space (M, S), the two pairs of self mappings (H, K) and (J, L) of M are said to satisfy common (E.A) property if there exist two sequences (ν_n) and (ω_n) in M such that*

$$\lim_{n\to\infty} H\nu_n = \lim_{n\to\infty} K\nu_n = \lim_{n\to\infty} J\omega_n = \lim_{n\to\infty} L\omega_n = \gamma, \text{ where } \gamma \in M.$$

Definition 2.9. *[15] In a S-metric space (M, S), the two pairs of self mappings (H, K) and (J, L) on M are said to satisfy common limit range property with respect to K and L, denoted by (CLR_{KL}) if there exists two sequences (ν_n) and (ω_n) in M such that*

$$\lim_{n\to\infty} H\nu_n = \lim_{n\to\infty} K\nu_n = \lim_{n\to\infty} J\omega_n = \lim_{n\to\infty} L\omega_n = \gamma, \text{ where } \gamma \in K(M) \cap L(M).$$

Remark 2.10. *Throughout this paper $f : [0, \infty) \longrightarrow [0, \infty)$ is a Lebesgue integrable function which is summable on compact subset of $[0, \infty)$ with $\int_0^\epsilon f(\gamma) d\gamma > 0$, for any $\epsilon > 0$.*

Remark 2.11. *Throughout this paper $g : [0, \infty) \longrightarrow [0, \infty)$ is a nondecreasing continuous function on $(0, \infty)$ and $g(\gamma) = 0 \Leftrightarrow \gamma = 0$.*

Remark 2.12. *Throughout this paper $h : [0, \infty) \longrightarrow [0, \infty)$ is a upper semicontinuous function on $(0, \infty)$ with $h(0) = 0$ and $h(\gamma) < \gamma$, for any $\gamma > 0$*

Lemma 2.13. *[10] Let $f : [0, \infty) \longrightarrow [0, \infty)$ is a Lebesgue integrable function which is summable on compact subset of $[0, \infty)$ with $\int_0^\epsilon f(\gamma) d\gamma > 0$, for any $\epsilon > 0$ and $\{\rho_n\}_{n\geq 1}$ be a non negative sequence with $\lim_{n\to\infty} \rho_n = \theta$. Then we have*

$$\lim_{n\to\infty} \int_0^{\rho_n} f(\gamma) d\gamma = \int_0^{\theta} f(\gamma) d\gamma.$$

3. Main Result

Now we state our main theorem.

Theorem 3.1. *In a S-metric space (M, S), suppose H, J, K and L are self mappings of M satisfying the following conditions*

(i) The pairs (H, K) and (J, L) satisfy (CLR_{KL}) property

(ii) The pairs (H, K) and (J, L) are weakly compatible

(iii)

$$g(\int_0^{S(H\nu,H\nu,J\omega)} f(\gamma)d\gamma) \leq g(\int_0^{p(\nu,\nu,\omega)} f(\gamma)d\gamma) - \int_0^{h(p(\nu,\nu,\omega))} f(\gamma)d\gamma$$

where

$$\begin{aligned} p(\nu,\nu,\omega) = max\{&S(K\nu,K\nu,L\omega), S(H\nu,H\nu,K\nu), S(J\omega,J\omega,L\omega),\\ &\frac{S(K\nu,K\nu,J\omega)+S(L\omega,L\omega,H\nu)}{2}, \frac{S(H\nu,H\nu,K\nu)S(J\omega,J\omega,L\omega)}{1+S(K\nu,K\nu,L\omega)},\\ &\frac{S(H\nu,H\nu,L\omega)S(J\omega,J\omega,K\nu)}{1+S(K\nu,K\nu,L\omega)},\\ &S(H\nu,H\nu,K\nu)(\frac{1+S(K\nu,K\nu,J\omega)+S(L\omega,L\omega,H\nu)}{1+S(H\nu,H\nu,K\nu)+S(L\omega,L\omega,J\omega)})\}\end{aligned}$$

then H, J, K and L have a unique common fixed point in M.

Proof. From the (CLR_{KL}) property of the pairs (H, K) and (J, L), we have two sequences (ν_n) and (ω_n) in M such that

$$\lim_{n\to\infty} H\nu_n = \lim_{n\to\infty} K\nu_n = \lim_{n\to\infty} J\omega_n = \lim_{n\to\infty} L\omega_n = \gamma,\ where\ \gamma \in K(M)\cap L(M). \tag{3.1}$$

Also there exists a point $\eta \in M$ such that $K\eta = \gamma$, from (3.1), we have

$$\lim_{n\to\infty} H\nu_n = \lim_{n\to\infty} K\nu_n = \lim_{n\to\infty} J\omega_n = \lim_{n\to\infty} L\omega_n = \gamma = K\eta.$$

We now claim that $H\eta = K\eta$, for if $H\eta \neq K\eta$ then $S(H\eta, H\eta, K\eta) > 0$.
Keeping $\nu = \eta$ and $\omega = \omega_n$ in condition (iii) of Theorem 3.1, we get

$$g(\int_0^{S(H\eta,H\eta,J\omega_n)} f(\gamma)d\gamma) \leq g(\int_0^{p(\eta,\eta,\omega_n)} f(\gamma)d\gamma) - \int_0^{h(p(\eta,\eta,\omega_n))} f(\gamma)d\gamma. \tag{3.2}$$

Then

$$\begin{aligned} p(\eta,\eta,\omega_n) = max\{&S(K\eta,K\eta,L\omega_n), S(H\eta,H\eta,K\eta), S(J\omega_n,J\omega_n,L\omega_n),\\ &\frac{S(K\eta,K\eta,J\omega_n)+S(L\omega_n,L\omega_n,H\eta)}{2}, \frac{S(H\eta,H\eta,K\eta)S(J\omega_n,J\omega_n,L\omega_n)}{1+S(K\eta,K\eta,L\omega_n)},\\ &\frac{S(H\eta,H\eta,L\omega_n)S(J\omega_n,J\omega_n,K\eta)}{1+S(K\eta,K\eta,L\omega_n)},\\ &S(H\eta,H\eta,K\eta)(\frac{1+S(K\eta,K\eta,J\omega_n)+S(L\omega_n,L\omega_n,H\eta)}{1+S(H\eta,H\eta,K\eta)+S(L\omega_n,L\omega_n,J\omega_n)})\}.\end{aligned}$$

Now

$$\lim_{n\to\infty} p(\eta,\eta,\omega_n) = max\{S(\gamma,\gamma,\gamma), S(H\eta,H\eta,\gamma), S(\gamma,\gamma,\gamma),$$
$$\frac{S(\gamma,\gamma,\gamma)+S(\gamma,\gamma,H\eta)}{2}, \frac{S(H\eta,H\eta,\gamma)S(\gamma,\gamma,\gamma)}{1+S(\gamma,\gamma,\gamma)},$$
$$\frac{S(H\eta,H\eta,\gamma)S(\gamma,\gamma,\gamma)}{1+S(\gamma,\gamma,\gamma)},$$
$$S(H\eta,H\eta,\gamma)(\frac{1+S(\gamma,\gamma,\gamma)+S(\gamma,\gamma,H\eta)}{1+S(H\eta,H\eta,\gamma)+S(\gamma,\gamma,\gamma)})\}$$

$$\lim_{n\to\infty} p(\eta,\eta,\omega_n) = max\{0, S(H\eta,H\eta,\gamma), 0, \frac{S(\gamma,\gamma,H\eta)}{2}, 0, 0, S(H\eta,H\eta,\gamma)\}$$
$$\lim_{n\to\infty} p(\eta,\eta,\omega_n) = S(H\eta,H\eta,\gamma).$$

On taking the limit in (3.2), we get

$$g(\int_0^{S(H\eta,H\eta,\gamma)} f(\gamma)d\gamma) = \limsup_{n\to\infty} g(\int_0^{S(H\eta,H\eta,J\omega_n)} f(\gamma)d\gamma)$$
$$\leq \limsup_{n\to\infty}\{g(\int_0^{p(\eta,\eta,\omega_n)} f(\gamma)d\gamma) - \int_0^{h(p(\eta,\eta,\omega_n))} f(\gamma)d\gamma\}$$
$$\leq \limsup_{n\to\infty}(g(\int_0^{p(\eta,\eta,\omega_n)} f(\gamma)d\gamma)) - \liminf_{n\to\infty}\int_0^{h(p(\eta,\eta,\omega_n))} f(\gamma)d\gamma.$$

From Lemma 2.13, we get

$$g(\int_0^{S(H\eta,H\eta,\gamma)} f(\gamma)d\gamma) \leq g(\int_0^{S(H\eta,H\eta,\gamma)} f(\gamma)d\gamma) - \int_0^{h(S(H\eta,H\eta,\gamma))} f(\gamma)d\gamma$$
$$< g(\int_0^{S(H\eta,H\eta,\gamma)} g(\gamma)d\gamma).$$

Which is a contradiction and hence $H\eta = K\eta$.
Therefore we get

$$H\eta = K\eta = \gamma. \tag{3.3}$$

Similarly there exists a point $\xi \in M$ such that $L\xi = \gamma$, from (3.1), we have

$$\lim_{n\to\infty} H\nu_n = \lim_{n\to\infty} K\nu_n = \lim_{n\to\infty} J\omega_n = \lim_{n\to\infty} L\omega_n = \gamma = L\xi.$$

We now claim that $J\xi = L\xi$, for if $J\xi \neq L\xi$ then $S(J\xi, J\xi, L\xi) > 0$.
Keeping $\nu = \nu_n$ and $\omega = \xi$ in condition (iii) of Theorem 3.1, we get

$$g(\int_0^{S(H\nu_n,H\nu_n,J\xi)} f(\gamma)d\gamma) \leq g(\int_0^{p(\nu_n,\nu_n,\xi)} f(\gamma)d\gamma) - \int_0^{h(p(\nu_n,\nu_n,\xi))} f(\gamma)d\gamma. \tag{3.4}$$

Then

$$p(\nu_n,\nu_n,\xi) = max\{S(K\nu_n,K\nu_n,L\xi), S(H\nu_n,H\nu_n,K\nu_n), S(J\xi,J\xi,L\xi),$$
$$\frac{S(K\nu_n,K\nu_n,J\xi)+S(L\xi,L\xi,H\nu_n)}{2}, \frac{S(H\nu_n,H\nu_n,K\nu_n)S(J\xi,J\xi,L\xi)}{1+S(K\nu_n,K\nu_n,L\xi)},$$
$$\frac{S(H\nu_n,H\nu_n,L\xi)S(J\xi,J\xi,K\nu_n)}{1+S(K\nu_n,K\nu_n,L\xi)},$$
$$S(H\nu_n,H\nu_n,K\nu_n)(\frac{1+S(K\nu_n,K\nu_n,J\xi)+S(L\xi,L\xi,H\nu_n)}{1+S(H\nu_n,H\nu_n,K\nu_n)+S(L\xi,L\xi,J\xi)})\}.$$

Now

$$\lim_{n\to\infty} p(\nu_n,\nu_n,\xi) = max\{S(\gamma,\gamma,\gamma), S(\gamma,\gamma,\gamma), S(J\xi,J\xi,\gamma), \frac{S(\gamma,\gamma,J\xi)+S(\gamma,\gamma,\gamma)}{2}, \frac{S(\gamma,\gamma,\gamma)S(J\xi,J\xi,\gamma)}{1+S(\gamma,\gamma,\gamma)}, \frac{S(\gamma,\gamma,\gamma)S(J\xi,J\xi,\gamma)}{1+S(\gamma,\gamma,\gamma)}, S(\gamma,\gamma,\gamma)(\frac{1+S(\gamma,\gamma,J\xi)+S(\gamma,\gamma,\gamma)}{1+S(\gamma,\gamma,\gamma)+S(\gamma,\gamma,J\xi)})\}$$

$$\lim_{n\to\infty} p(\nu_n,\nu_n,\xi) = max\{0,0,S(J\xi,J\xi,\gamma),\frac{S(\gamma,\gamma,J\xi)}{2},0,0,0\}$$

$$\lim_{n\to\infty} p(\nu_n,\nu_n,\xi) = S(J\xi,J\xi,\gamma).$$

On taking the limit in (3.4), we get

$$\begin{aligned} g(\int_0^{S(\gamma,\gamma,J\xi)} f(\gamma)d\gamma) &= \limsup_{n\to\infty} g(\int_0^{S(H\nu_n,H\nu_n,J\xi)} f(\gamma)d\gamma) \\ &\le \limsup_{n\to\infty}\{g(\int_0^{p(\nu_n,\nu_n,\xi)} f(\gamma)d\gamma) - \int_0^{h(p(\nu_n,\nu_n,\xi))} f(\gamma)d\gamma\} \\ &\le \limsup_{n\to\infty}(g(\int_0^{p(\nu_n,\nu_n,\xi)} f(\gamma)d\gamma)) - \liminf_{n\to\infty}\int_0^{h(p(\nu_n,\nu_n,\xi))} f(\gamma)d\gamma. \end{aligned}$$

From Lemma 2.13, we get

$$\begin{aligned} g(\int_0^{S(\gamma,\gamma,J\xi)} f(\gamma)d\gamma) &\le g(\int_0^{S(\gamma,\gamma,J\xi)} f(\gamma)d\gamma) - \int_0^{h(S(\gamma,\gamma,J\xi))} f(\gamma)d\gamma \\ &< g(\int_0^{S(\gamma,\gamma,J\xi)} f(\gamma)d\gamma). \end{aligned}$$

Which is a contradiction and hence $J\xi = L\xi$.
Therefore we get

$$J\xi = L\xi = \gamma. \tag{3.5}$$

From (3.3) and (3.5), we get

$$H\eta = K\eta = J\xi = L\xi = \gamma.$$

Now we establish γ is a common fixed point of H, J, L and K.
Clearly $HK\eta = KH\eta$
from which we get

$$H\gamma = K\gamma$$

and
$JL\xi = LJ\xi$ which implies

$$J\gamma = L\gamma.$$

Now we prove that $H\gamma = \gamma$, for if $H\gamma \neq \gamma$ then $S(H\gamma, H\gamma, \gamma) > 0$.
Substituting $\nu = \gamma$ and $\omega = \xi$ in condition (iii) of Theorem 3.1, we get

$$g(\int_0^{S(H\gamma,H\gamma,J\xi)} f(\gamma)d\gamma) \le g(\int_0^{p(\gamma,\gamma,\xi)} f(\gamma)d\gamma) - \int_0^{h(p(\gamma,\gamma,\xi))} f(\gamma)d\gamma. \tag{3.6}$$

Then

$$p(\gamma,\gamma,\xi) = max\{S(K\gamma,K\gamma,L\xi), S(H\gamma,H\gamma,K\gamma), S(J\xi,J\xi,L\xi), \frac{S(K\gamma,K\gamma,J\xi)+S(L\xi,L\xi,H\gamma)}{2}, \frac{S(H\gamma,H\gamma,K\gamma)S(J\xi,J\xi,L\xi)}{1+S(K\gamma,K\gamma,L\xi)}, \frac{S(H\gamma,H\gamma,L\xi)S(J\xi,J\xi,K\gamma)}{1+S(K\gamma,K\gamma,L\xi)}, S(H\gamma,H\gamma,K\gamma)(\frac{1+S(K\gamma,K\gamma,J\xi)+S(L\xi,L\xi,H\gamma)}{1+S(H\gamma,H\gamma,K\gamma)+S(L\xi,L\xi,J\xi)})\}$$

$$p(\gamma,\gamma,\xi) = max\{S(H\gamma,H\gamma,\gamma), S(H\gamma,H\gamma,H\gamma), S(\gamma,\gamma,\gamma), \frac{S(H\gamma,H\gamma,\gamma)+S(\gamma,\gamma,H\gamma)}{2}, \frac{S(H\gamma,H\gamma,H\gamma)S(\gamma,\gamma,\gamma)}{1+S(H\gamma,H\gamma,\gamma)}, \frac{S(H\gamma,H\gamma,\gamma)S(\gamma,\gamma,H\gamma)}{1+S(H\gamma,H\gamma,\gamma)}, S(H\gamma,H\gamma,H\gamma)(\frac{1+S(H\gamma,H\gamma,\gamma)+S(\gamma,\gamma,H\gamma)}{1+S(H\gamma,H\gamma,H\gamma)+S(\gamma,\gamma,\gamma)})\}$$

$$p(\gamma,\gamma,\xi) = max\{S(H\gamma,H\gamma,\gamma), 0, 0, S(H\gamma,H\gamma,\gamma), 0, \frac{S(H\gamma,H\gamma,\gamma)S(H\gamma,H\gamma,\gamma)}{1+S(H\gamma,H\gamma,\gamma)}, 0\}$$

$$p(\gamma,\gamma,\xi) = S(H\gamma,H\gamma,\gamma).$$

From (3.6), we get

$$g(\int_0^{S(H\gamma,H\gamma,\gamma)} f(\gamma)d\gamma) \leq g(\int_0^{S(H\gamma,H\gamma,\gamma)} f(\gamma)d\gamma) - \int_0^{h(S(H\gamma,H\gamma,\gamma))} f(\gamma)d\gamma < g(\int_0^{S(H\gamma,H\gamma,\gamma)} f(\gamma)d\gamma).$$

Which is a contradiction and hence $H\gamma = \gamma$.
Therefore we get

$$H\gamma = K\gamma = \gamma. \tag{3.7}$$

Similarly we can prove that

$$J\gamma = L\gamma = \gamma. \tag{3.8}$$

From (3.7) and (3.8), we get

$$H\gamma = K\gamma = J\gamma = L\gamma = \gamma.$$

Proving γ is a fixed point of H, J, K and L.
For if $\zeta(\zeta \neq \gamma)$ is in M such that

$$H\zeta = K\zeta = J\zeta = L\zeta = \zeta.$$

On taking $\nu = \gamma$ and $\omega = \zeta$ in condition (iii) of Theorem 3.1, we get

$$g(\int_0^{S(H\gamma,H\gamma,J\zeta)} f(\gamma)d\gamma) \leq g(\int_0^{p(\gamma,\gamma,\zeta)} f(\gamma)d\gamma) - \int_0^{h(p(\gamma,\gamma,\zeta))} f(\gamma)d\gamma. \tag{3.9}$$

Then

$$p(\gamma,\gamma,\zeta) = max\{S(K\gamma,K\gamma,L\zeta), S(H\gamma,H\gamma,K\gamma), S(J\zeta,J\zeta,L\zeta), \frac{S(K\gamma,K\gamma,J\zeta)+S(L\zeta,L\zeta,H\gamma)}{2}, \frac{S(H\gamma,H\gamma,K\gamma)S(J\zeta,J\zeta,L\zeta)}{1+S(K\gamma,K\gamma,L\zeta)}, \frac{S(H\gamma,H\gamma,L\zeta)S(J\zeta,J\zeta,K\gamma)}{1+S(K\gamma,K\gamma,L\zeta)}, S(H\gamma,H\gamma,K\gamma)(\frac{1+S(K\gamma,K\gamma,J\zeta)+S(L\zeta,L\zeta,H\gamma)}{1+S(H\gamma,H\gamma,K\gamma)+S(L\zeta,L\zeta,J\zeta)})\}$$

$$p(\gamma,\gamma,\zeta) = max\{S(\gamma,\gamma,\zeta), S(\gamma,\gamma,\gamma), S(\zeta,\zeta,\zeta), \frac{S(\gamma,\gamma,\zeta)+S(\zeta,\zeta,\gamma)}{2}, \frac{S(\gamma,\gamma,\gamma)S(\zeta,\zeta,\zeta)}{1+S(\gamma,\gamma,\zeta)}, \frac{S(\gamma,\gamma,\zeta)S(\zeta,\zeta,\gamma)}{1+S(\gamma,\gamma,\zeta)}, S(\gamma,\gamma,\gamma)(\frac{1+S(\gamma,\gamma,\zeta)+S(\zeta,\zeta,\gamma)}{1+S(\gamma,\gamma,\gamma)+S(\zeta,\zeta,\zeta)})\}$$

$$p(\gamma,\gamma,\zeta) = max\{S(\gamma,\gamma,\zeta), 0, 0, S(\gamma,\gamma,\zeta), 0, \frac{S(\gamma,\gamma,\zeta)S(\gamma,\gamma,\zeta)}{1+S(\gamma,\gamma,\zeta)}, 0\}$$

$$p(\gamma,\gamma,\zeta) = S(\gamma,\gamma,\zeta).$$

From (3.9), we get

$$g(\int_0^{S(\gamma,\gamma,\zeta)} f(\gamma)d\gamma) \le g(\int_0^{S(\gamma,\gamma,\zeta)} f(\gamma)d\gamma) - \int_0^{h(S(\gamma,\gamma,\zeta))} f(\gamma)d\gamma$$
$$< g(\int_0^{S(\gamma,\gamma,\zeta)} f(\gamma)d\gamma).$$

Which is a contradiction and hence $\gamma = \zeta$.
Proving that H, J, K and L have a unique common fixed point in M. ■

As an illustration we have the following example.

Example 3.2. *Let $M = (0,1]$. Define $S(\nu,\omega,\vartheta) = |\nu-\vartheta| + |\omega-\vartheta|$, where $\nu,\omega,\vartheta \in M$, then S is a S-metric on M. Now let H, J, K and L be self maps on M, defined by*

$$H(\nu) = \begin{cases} \frac{1}{2}, & \textit{if}\, \nu \in (0,\frac{1}{2}], \\ \frac{1}{5}, & \textit{if}\, \nu \in (\frac{1}{2},1]. \end{cases} \qquad J(\nu) = \begin{cases} \frac{1}{2}, & \textit{if}\, \nu \in (0,\frac{1}{2}], \\ \frac{1}{3}, & \textit{if}\, \nu \in (\frac{1}{2},1]. \end{cases}$$

$$K(\nu) = \begin{cases} \frac{1}{2}, & \text{if } \nu \in (0,\frac{1}{2}], \\ \frac{1}{7}, & \text{if } \nu \in (\frac{1}{2},1]. \end{cases} \qquad L(\nu) = \begin{cases} \frac{1}{2}, & \text{if } \nu \in (0,\frac{1}{2}], \\ \frac{1}{9}, & \text{if } \nu \in (\frac{1}{2},1]. \end{cases}$$

Also take $f(\gamma) = 3\gamma$, $g(\gamma) = \dfrac{\gamma}{3}$ and h(γ) as floor function.
Let (ν_n) and (ω_n) be sequences in M with $\nu_n = \dfrac{1}{n+1}$ and $\omega_n = \dfrac{1}{n+3}$, where $n \geq 1$, then

$$\lim_{n\to\infty} H\nu_n = \lim_{n\to\infty} H(\frac{1}{n+1}) = \frac{1}{2},$$

$$\lim_{n\to\infty} K\nu_n = \lim_{n\to\infty} K(\frac{1}{n+1}) = \frac{1}{2},$$

$$\lim_{n\to\infty} J\omega_n = \lim_{n\to\infty} J(\frac{1}{n+3}) = \frac{1}{2},$$

$$\lim_{n\to\infty} L\omega_n = \lim_{n\to\infty} L(\frac{1}{n+3}) = \frac{1}{2}.$$

$$Thus \lim_{n\to\infty} H\nu_n = \lim_{n\to\infty} K\nu_n = \lim_{n\to\infty} J\omega_n = \lim_{n\to\infty} L\omega_n = \frac{1}{2}\ and\ \frac{1}{2} \in K(M) \cap L(M).$$

Proving (H, K) and (J, L) satisfy (CLR_{KL}) property.
Also $H\nu = K\nu$, for all $\nu \in (0, \frac{1}{2}]$
H(K(ν)) $= \frac{1}{2} = K(H(\nu))$
therefore (H, K) is weakly compatible.
Similarlry (J, L) is also weakly compatible.
Now we verify the condition (iii) of Theorem 3.1 in different cases.
Case(i): Let $\nu, \omega \in (0, \frac{1}{2}]$
then $H\nu = K\nu = J\omega = L\omega = \frac{1}{2}$ and $p(\nu,\nu,\omega) = 0$, $S(H\nu, H\nu, J\omega) = 0$ from condition (iii) of Theorem 3.1, we get
$g(\int_0^{S(H\nu,H\nu,J\omega)} f(\gamma)d\gamma) = 0$ also $g(\int_0^{p(\nu,\nu,\omega)} f(\gamma)d\gamma) - \int_0^{h(p(\nu,\nu,\omega))} f(\gamma)d\gamma = 0$.
Case(ii): Let $\nu, \omega \in (\frac{1}{2}, 1]$
$H\nu = \frac{1}{5}, K\nu = \frac{1}{7}, J\omega = \frac{1}{3}, L\omega = \frac{1}{9}$ and

$$p(\nu,\nu,\omega) = max\{\frac{4}{63}, \frac{4}{35}, \frac{4}{9}, \frac{88}{315}, \frac{16}{335}, \frac{64}{1005}, \frac{4}{35}\} = \frac{4}{9}$$

$S(H\nu, H\nu, J\omega) = \frac{4}{15}$, then condition (iii) of Theorem 3.1, we get
$g(\int_0^{S(H\nu,H\nu,J\omega)} f(\gamma)d\gamma) = g(\int_0^{\frac{4}{15}} 3\gamma d\gamma) = \frac{8}{225}$ and

$$g(\int_0^{p(\nu,\nu,\omega)} f(\gamma)d\gamma) - \int_0^{h(p(\nu,\nu,\omega))} f(\gamma)d\gamma = g(\int_0^{\frac{4}{9}} 3\gamma d\gamma) - \int_0^{h(\frac{4}{9})} 3\gamma d\gamma$$
$$= \frac{8}{81}.$$

Thus $\frac{8}{225} < \frac{8}{81}$.
Case(iii): Let $\nu \in (0, \frac{1}{2}], \omega \in (\frac{1}{2}, 1]$

$H\nu = K\nu = \frac{1}{2}, J\omega = \frac{1}{3}, L\omega = \frac{1}{9}$ and

$$p(\nu,\nu,\omega) = max\{\frac{7}{9}, 0, \frac{4}{9}, \frac{5}{9}, 0, \frac{7}{48}, 0\} = \frac{7}{9}$$

$S(H\nu, H\nu, J\omega) = \frac{1}{3}$, then condition (iii) of Theorem 3.1, we get

$g(\int_0^{S(H\nu,H\nu,J\omega)} f(\gamma)d\gamma) = g(\int_0^{\frac{1}{3}} 3\gamma d\gamma) = \frac{1}{18}$ and

$$g(\int_0^{p(\nu,\nu,\omega)} f(\gamma)d\gamma) - \int_0^{h(p(\nu,\nu,\omega))} f(\gamma)d\gamma = g(\int_0^{\frac{7}{9}} 3\gamma d\gamma) - \int_0^{h(\frac{7}{9})} 3\gamma d\gamma$$
$$= \frac{49}{162}.$$

Thus $\frac{1}{18} < \frac{49}{162}$.
From above cases
$g\int_0^{S(H\nu,H\nu,J\omega)} f(\gamma)d\gamma) \leq g(\int_0^{p(\nu,\nu,\omega)} f(\gamma)d\gamma) - \int_0^{h(p(\nu,\nu,\omega))} f(\gamma)d\gamma.$
Similarly we can check condition (iii) of Theorem 3.1 in case if $\nu \in (\frac{1}{2}, 1]$, $\omega \in (0, \frac{1}{2}]$.
Hence condition (iii) of Theorem 3.1 is satisfied in different cases.
Thus all conditions of Theorem 3.1 are satisfied and clearly $\frac{1}{2}$ is the unique common fixed point of H, J, K and L.

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Generating functions for generalized tribonacci and generalized tricobsthal polynomials

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Abstract. In this work, we consider generating functions which are generalized tribonacci polynomials $T_n(x)$ and generalized tricobsthal polynomials $\mathbf{J}_n(x)$ which are defined in [7]. We derive generating functions for $(m+n)-th$ order of generalized tribonacci polynomials and generalized tricobsthal polynomials for $m \geq 2$. Furthermore, we obtain various families of bilinear and bilateral generating functions and give their special cases for these polynomials. Also, we obtain the summation formula of generalized tribonacci polynomials and generalized tricobsthal polynomials.

AMS Subject Classifications: 11B83, 11C08, 33C45.

Keywords: Generalized tricobsthal, generalized tribonacci polynomials, bilinear and bilateral generating functions.

Contents

1. Introduction

There are so many studies in the literature that concern about the special polynomials. In [10], they introduced generalized Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials and various families of multilinear and multilateral generating functions for these polynomials are derived. In [11], authors derived various families of multilinear and multilateral generating functions for generalized bivariate Fibonacci and Lucas polynomials. In [13], Mansour and Shattuck investigated some properties of polynomials whose coefficients are generalized tribonacci numbers. Recently, Kocer and Gedikce [12], has obtained some properties of the trivariate Fibonacci and Lucas polynomials by using these properties they gave some results for the tribonacci numbers and tribonacci polynomials. Also different types of polynomials are studied in [14], [15].

In [7], authors defined new kinds of polynomials called as generalized tribonacci polynomials and generalized tricobsthal polynomials. For these classes of polynomials, they found various results including recurrence relations and Binet's formulas, which can be useful also related our problem. Because in our work, we give the families of bilinear and bilateral generating functions which are generalized tribonacci polynomials $T_n(x)$ and generalized tricobsthal polynomials $\mathbf{J}_n(x)$ and are give their special cases. In addition to, we

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formulate the summation formula for these polynomials. Furthermore we give the exponential generating functions for generalized tribonacci polynomials and generalized tricobsthal polynomials.

Tribonacci numbers [5] which are defined by,

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad \text{for} \quad n \geq 4, \tag{1.1}$$

with initial conditions $T_1 = 1$, $T_2 = 1$ and $T_3 = 2$. In [5], they present tribonacci polynomials defined by recurrence relation

$$t_n(x) = x^2 t_{n-1}(x) + x t_{n-2}(x) + t_{n-3}(x) \quad \text{for} \quad n \geq 4,$$

with initial conditions

$$t_1(x) = 1,\ t_2(x) = x^2,\ t_3(x) = x^4 + x \tag{1.2}$$

and property $t_n(1) = T_n$.

Definition 1.1. *[7] Generalized tribonacci polynomials are defined by recurrence relation*

$$T_n(x) = x^2 T_{n-1}(x) + x T_{n-2}(x) + T_{n-3}(x) \quad \textit{for} \quad n \geq 4, \tag{1.3}$$

with initial conditions

$$\begin{aligned} T_1(x) &= a, \\ T_2(x) &= b_2 x^2 + b_1 x + b_0, \\ T_3(x) &= c_4 x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0, \end{aligned} \tag{1.4}$$

where b_2, c_1, c_4 positive integers and others parametres are nonnegative integers as initial conditions for tribonacci polynomials.

Theorem 1.2. *[7]The Binet formula for generalized tribonacci polynomials defined by (1.3) with initial conditions (1.4) is*

$$T_n(x) = C_{1,T}\alpha_T^{n-1} + C_{2,T}\beta_T^{n-1} + C_{3,T}\gamma_T^{n-1} \tag{1.5}$$

where n is positive integer,

$$\begin{aligned} C_{1,T} &= \frac{T_3(x) - (\gamma_T + \beta_T)T_2(x) + \gamma_T\beta_T T_1(x)}{(\alpha_T - \gamma_T)(\alpha_T - \beta_T)}, \\ C_{2,T} &= \frac{T_3(x) - (\gamma_T + \alpha_T)T_2(x) + \gamma_T\alpha_T T_1(x)}{(\beta_T - \gamma_T)(\beta_T - \alpha_T)}, \\ C_{3,T} &= \frac{T_3(x) - (\alpha_T + \beta_T)T_2(x) + \alpha_T\beta_T T_1(x)}{(\gamma_T - \alpha_T)(\gamma_T - \beta_T)} \end{aligned}$$

and $\alpha_T, \beta_T, \gamma_T$ are different solutions of characteristic equation $y^3 - x^2y^2 - xy - 1 = 0$ of (1.3).

$$\begin{aligned} \alpha_T &= \frac{x^2}{3} - \frac{2^{1/3}(-3x - x^4)}{3\delta_T} + \frac{\delta_T}{3.2^{1/3}}, \\ \beta_T &= \frac{x^2}{3} + \frac{(1 + i\sqrt{3})(-3x - x^4)}{3.2^{2/3}\delta_T} - \frac{(1 - i\sqrt{3})\delta_T}{6.2^{1/3}}, \\ \gamma_T &= \frac{x^2}{3} + \frac{(1 - i\sqrt{3})(-3x - x^4)}{3.2^{2/3}\delta_T} - \frac{(1 + i\sqrt{3})\delta_T}{6.2^{1/3}} \end{aligned} \tag{1.6}$$

with

$$\delta_T = \sqrt[3]{27 + 9x^3 + 2x^6 + 3\sqrt{3}\sqrt{27 + 14x^3 + 3x^6}}. \tag{1.7}$$

In [7], tricobsthal polynomials are defined by recurrence formula

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x) + x^2 J_{n-3}(x) \quad \text{for} \quad n \geq 4,$$

with initial conditions:

$$J_1(x) = 1, \;\; J_2(x) = 1 \text{ and } J_3(x) = x + 1.$$

The choice of initial conditions is according to property $J_n(1) = t_n(1) = T_n$ is $n - th$ tribonacci number, by analogy to Jacobsthal and Fibonacci polynomials ([5], [6]). Analogously they can define generalized tricobsthal polynomials:

Definition 1.3. *[7]Generalized tricobsthal polynomials are defined by recurrence relation*

$$\mathbf{J}_n(x) = \mathbf{J}_{n-1}(x) + x\mathbf{J}_{n-2}(x) + x^2\mathbf{J}_{n-3}(x) \quad \text{for} \quad n \geq 4, \tag{1.8}$$

with initial condition:

$$\begin{aligned} \mathbf{J}_1(x) &= a \\ \mathbf{J}_2(x) &= b \\ \mathbf{J}_3(x) &= c_1 x + c_0 \end{aligned} \tag{1.9}$$

where parameters c_1 is positive integers and a, b, c_0 are non-negative integers.

Theorem 1.4. *[7]The Binet formula for generalized tricobsthal polynomials defined by (1.8) with initial conditions (1.9) is*

$$\mathbf{J}_n(x) = C_{1,\mathbf{J}}\alpha_{\mathbf{J}}^{n-1} + C_{2,\mathbf{J}}\beta_{\mathbf{J}}^{n-1} + C_{3,\mathbf{J}}\gamma_{\mathbf{J}}^{n-1}, \tag{1.10}$$

where n is positive integer, $x \neq 0$ and

$$\begin{aligned} C_{1,\mathbf{J}} &= \frac{\mathbf{J}_3(x) - (\gamma_{\mathbf{J}} + \beta_{\mathbf{J}})\mathbf{J}_2(x) + \gamma_{\mathbf{J}}\beta_{\mathbf{J}}\mathbf{J}_1(x)}{(\alpha_{\mathbf{J}} - \gamma_{\mathbf{J}})(\alpha_{\mathbf{J}} - \beta_{\mathbf{J}})}, \\ C_{2,\mathbf{J}} &= \frac{\mathbf{J}_3(x) - (\gamma_{\mathbf{J}} + \alpha_{\mathbf{J}})\mathbf{J}_2(x) + \gamma_{\mathbf{J}}\alpha_{\mathbf{J}}\mathbf{J}_1(x)}{(\beta_{\mathbf{J}} - \gamma_{\mathbf{J}})(\beta_{\mathbf{J}} - \alpha_{\mathbf{J}})}, \\ C_{3,\mathbf{J}} &= \frac{\mathbf{J}_3(x) - (\alpha_{\mathbf{J}} + \beta_{\mathbf{J}})\mathbf{J}_2(x) + \alpha_{\mathbf{J}}\beta_{\mathbf{J}}\mathbf{J}_1(x)}{(\gamma_{\mathbf{J}} - \alpha_{\mathbf{J}})(\gamma_{\mathbf{J}} - \beta_{\mathbf{J}})} \end{aligned}$$

and $\alpha_{\mathbf{J}}$, $\beta_{\mathbf{J}}$, $\gamma_{\mathbf{J}}$ are different solutions of characteristic equation $y^3 - y^2 - xy - x^2 = 0$ of (1.8).

$$\begin{aligned} \alpha_{\mathbf{J}} &= \frac{8(3x+1)}{3\sqrt[3]{4}\delta_{\mathbf{J}}} + \frac{\delta_{\mathbf{J}}}{3\sqrt[3]{2}} + \frac{1}{3}, \\ \beta_{\mathbf{J}} &= \frac{-(1+i\sqrt{3})(3x+1)}{3\sqrt[3]{4}\delta_{\mathbf{J}}} - \frac{(1-i\sqrt{3})\delta_{\mathbf{J}}}{6.\sqrt[3]{2}} + \frac{1}{3}, \\ \gamma_{\mathbf{J}} &= \frac{-(1-i\sqrt{3})(3x+1)}{3.\sqrt[3]{4}\delta_{\mathbf{J}}} - \frac{(1+i\sqrt{3})\delta_{\mathbf{J}}}{6.\sqrt[3]{2}} + \frac{1}{3} \end{aligned} \tag{1.11}$$

and

$$\delta_{\mathbf{J}} = \sqrt[3]{27x^2 + 3\sqrt{3}\sqrt{27x^4 + 14x^3 + 3x^2} + 9x + 2}.$$

Theorem 1.5. *[7] Generating function for generalized tribonacci polynomials is given by formula*

$$\mathcal{G}_T(y) = \frac{T_1(x) + y(T_2(x) - x^2T_1(x)) + y^2(T_3(x) - x^2T_2(x) - xT_1(x))}{1 - yx^2 - y^2x - y^3} \tag{1.12}$$

and for generalized tricobsthal polynomials by

$$\mathcal{G}_{\mathbf{J}}(y) = \frac{\mathbf{J}_1(x) + y(\mathbf{J}_2(x) - \mathbf{J}_1(x)) + y^2(\mathbf{J}_3(x) - \mathbf{J}_2(x) - x\mathbf{J}_1(x))}{1 - y - xy^2 - x^2y^3}. \tag{1.13}$$

Definition 1.6. *Generalized tribonacci polynomials and generalized tricobsthal polynomials are defined for generating function by respectively:*

$$\sum_{n=0}^{\infty} T_{n+1}(x)t^n = \mathcal{G}_T(t) \tag{1.14}$$

$$\sum_{n=0}^{\infty} \mathbf{J}_{n+1}(x)t^n = \mathcal{G}_{\mathbf{J}}(t) \tag{1.15}$$

where $\mathcal{G}_T(t)$ *in (1.12) and* $\mathcal{G}_{\mathbf{J}}(t)$ *in (1.13) .*

Note that for generalized tribonacci polynomials and generalized tricobsthal polynomials are

$$\alpha_T + \beta_T + \gamma_T = x^2 \tag{1.16}$$

$$\alpha_T \beta_T \gamma_T = 1 \tag{1.17}$$

$$\alpha_{\mathbf{J}} + \beta_{\mathbf{J}} + \gamma_{\mathbf{J}} = \mathcal{T}$$

$$\alpha_{\mathbf{J}} \beta_{\mathbf{J}} \gamma_{\mathbf{J}} = \mathcal{K}$$

with

$$\begin{aligned} \mathcal{T} &= 1 + \frac{2(3x+1)}{\sqrt[3]{4}\delta_{\mathbf{J}}} \\ \mathcal{K} &= \frac{1}{864\delta_{\mathbf{J}}}\left(\sqrt[3]{4}\delta_{\mathbf{J}}^2 + 2\delta_{\mathbf{J}} + 8\sqrt[3]{2}(1+3x)\right) \\ &\times \left(4\delta_{\mathbf{J}} + 2\sqrt[3]{2}i(i+\sqrt{3})(1+3x) - \delta_{\mathbf{J}}^2\sqrt[3]{4}(1+i\sqrt{3})\right) \\ &\times \left(4\delta_{\mathbf{J}} - 2\sqrt[3]{2}i(-i+\sqrt{3})(1+3x) - \delta_{\mathbf{J}}^2\sqrt[3]{4}(1-i\sqrt{3})\right). \end{aligned}$$

2. Bilinear and Bilateral Generating Functions

In this section we will consider the families of bilinear and bilateral generating functions for generalized tribonacci polynomials $T_n(x)$ and generalized tricobsthal polynomials $\mathbf{J}_n(x)$ which are generated by (1.14), (1.15) and given explicitly by (1.12), (1.13) using the similar method considered in [1], [2], [3], [4], [8].

Using the polynomials mentioned above, we derived the following results:

Theorem 2.1. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1, ..., y_r)$ *of* r *complex variables* $y_1, ..., y_r$ $(r \in \mathbb{N})$ *and of complex order* μ*, let*

$$\Lambda_{\mu,\psi}(y_1, ..., y_r; t) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r)t^k$$

where $a_k \neq 0$, $\mu, \psi \in \mathbb{C}$ *and*

$$\theta_{n,p,\mu,\psi}(x; y_1, ..., y_s; \xi) := \sum_{k=0}^{[n/p]} a_k T_{n+1-pk}(x)\Omega_{\mu+\psi k}(y_1, ..., y_r)\xi^k.$$

Then, for $n, p \in \mathbb{N}$; *we have*

$$\sum_{n=0}^{\infty} \theta_{n,p,\mu,\psi}(x; y_1, ..., y_s; \frac{\eta}{t^p})t^n = \mathcal{G}_T(t)\Lambda_{\mu,\psi}(y_1, ..., y_r; \eta). \tag{2.1}$$

Proof. For convenience, let S denote the first member of the assertion (2.1) of Theorem 2.1. Then,

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k T_{n+1-pk}(x) \Omega_{\mu+\psi k}(y_1, ..., y_r) \frac{\eta^k}{t^{pk}} t^n.$$

Replacing n by $n + pk$ and then using relation (1.14) we may write

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} T_{n+1}(x) a_k \, \Omega_{\mu+\psi k}(y_1, ..., y_r) \eta^k t^n \\ &= \left(\sum_{n=0}^{\infty} T_{n+1}(x) t^n \right) \left(\sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r) \eta^k \right) \\ &= \mathcal{G}_{\mathbf{T}}(t) \Lambda_{\mu,\psi}(y_1, ..., y_r; \eta) \end{aligned}$$

which completes the proof. ■

By using a similar idea, we also get the next result immediately.

Theorem 2.2. *Let*

$$\Theta_{n,p}^{\mu,\psi}(x; y_1, ..., y_r; \xi) := \sum_{k=0}^{[n/p]} a_k \mathbf{J}_{n-pk+1}(x) \Omega_{\mu+\psi k}(y_1, ..., y_r) \xi^k. \tag{2.2}$$

If

$$\Lambda_{\mu,\psi}(y_1, ..., y_r; \zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r) \zeta^k$$

then, for every nonnegative integer μ, *we have*

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi} \left(x; y_1, ..., y_r; \frac{\eta}{t^p} \right) t^n = \mathcal{G}_{\mathbf{J}}(t) \Lambda_{\mu,\psi}(y_1, ..., y_r; \eta). \tag{2.3}$$

Proof. If we denote the left-hand side of (2.3) by T and use (2.2), then we obtain

$$T = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k \mathbf{J}_{n-pk+1}(x) \Omega_{\mu+\psi k}(y_1, ..., y_r) \Omega_{\mu+\psi k}(y_1, ..., y_r) \eta^k t^{n-pk}.$$

Replacing n by $n + pk$,

$$\begin{aligned} T &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k \, \mathbf{J}_{n+1}(x) \Omega_{\mu+\psi k}(y_1, ..., y_r) \eta^k t^n \\ &= \sum_{n=0}^{\infty} \mathbf{J}_{n+1}(x) t^n \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r) \eta^k \\ &= \mathcal{G}_{\mathbf{J}}(t) \Lambda_{\mu,\psi}(y_1, ..., y_r; \eta) \end{aligned}$$

which completes the proof. ■

We derive generating functions for the $(m + n) - th$ order of generalized tribonacci polynomials and generalized tricobsthal polynomials for $m \geq 2$.

Theorem 2.3. *The following generating functions holds true for generalized tribonacci polynomials and generalized tricobsthal polynomials defined by (1.3) and (1.8) respectively:*

$$g_{T,m}(x,t) = \frac{T_m(x) + t\left(T_{m+1}(x) - x^2 T_m(x)\right) + t^2 T_{m-1}(x)}{1 - tx^2 - xt^2 - t^3}, \quad m \geq 2 \tag{2.4}$$

$$g_{\mathbf{J},m}(x,t) = \frac{\mathbf{J}_m(x) + t\left(\mathbf{J}_{m+1}(x) - \mathcal{T}\mathbf{J}_m(x)\right) + t^2 \mathcal{K}\mathbf{J}_{m-1}(x)}{1 - tx^2 - xt^2 - t^3}, \quad m \geq 2 \tag{2.5}$$

where

$$\sum_{n=0}^{\infty} T_{n+m}(x)t^n = g_{T,m}(x,t), \tag{2.6}$$

$$\sum_{n=0}^{\infty} \mathbf{J}_{n+m}(x)t^n = g_{\mathbf{J},m}(x,t). \tag{2.7}$$

Proof. From Binet formulas for generalized tribonacci polynomials and equation (1.16) and (1.17), we obtained

$$\begin{aligned}
\sum_{n=0}^{\infty} T_{n+m}(x)t^n &= \sum_{n=0}^{\infty}\left(C_{1,T}\alpha_T^{n+m-1} + C_{2,T}\beta_T^{n+m-1} + C_{3,T}\gamma_T^{n+m-1}\right)t^n \\
&= \left(\alpha_T^{m-1}C_{1,T}\sum_{n=0}^{\infty}\alpha_T^n t^n\right) + \left(\beta_T^{m-1}C_{2,T}\sum_{n=0}^{\infty}\beta_T^n t^n\right) \\
&\quad + \left(\gamma_T^{m-1}C_{3,T}\sum_{n=0}^{\infty}\gamma_T^n t^n\right) \\
&= \frac{\alpha_T^{m-1}C_{1,T}}{1-\alpha_T t} + \frac{\alpha\beta_T^{m-1}C_{2,T}}{1-\beta_T t} + \frac{\gamma_T^{m-1}C_{3,T}}{1-\gamma_T t} \\
&= \frac{\left\{\begin{array}{c} (C_{1,T}\alpha_T^{m-1} + C_{2,T}\beta_T^{m-1} + C_{3,T}\gamma_T^{m-1}) \\ -t(C_{1,T}\alpha_T^{m-1}(x^2-\alpha_T) + C_{2,T}\beta_T^{m-1}(x^2-\beta_T) + C_{3,T}\gamma_T^{m-1}(x^2-\gamma_T)) \\ +t^2(C_{1,T}\alpha_T^{m-1}\beta_T\gamma_T + C_{2,T}\beta_T^{m-1}\alpha_T\gamma_T + C_{3,T}\gamma_T^{m-1}\alpha_T\beta_T) \end{array}\right\}}{\left\{\begin{array}{c} 1 - t(\alpha_T+\beta_T+\gamma_T) + t^2(\alpha_T\beta_T + \alpha_T\gamma_T + \beta_T\gamma_T) \\ -t^3(\alpha_T\beta_T\gamma_T) \end{array}\right\}} \\
&= \frac{T_m(x) + t\left(T_{m+1}(x) - x^2T_m(x)\right) + t^2T_{m-1}(x)}{1 - tx^2 - xt^2 - t^3}.
\end{aligned}$$

The other cases for generalized tricobsthal polynomials can be done similarly. ■

For $T_{n+m}(x)$ and $\mathbf{J}_{n+m}(x)$, similar theorems will be found.

Theorem 2.4. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,...,y_r)$ of r complex variables $y_1,...,y_r$ $(r \in \mathbb{N})$ and of complex order μ, let*

$$\Lambda_{m,\mu,\psi}(x; y_1,...,y_r; t) := \sum_{k=0}^{\infty} a_k T_{m+pk}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)t^k$$

where $a_k \neq 0$, $\mu, \psi \in \mathbb{C}$ and

$$\theta_{\mu,\psi}(y_1,...,y_r;\xi) := \sum_{k=0}^{[n/p]} a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k.$$

Then, for $n,m\in\mathbb{N}$; *we have*

$$\sum_{n=0}^{\infty}T_{n+m}(x)\theta_{\mu,\psi}(y_1,...,y_r;z)t^n=\Lambda_{m,\mu,\psi}(x,t;y_1,...,y_r;zt^p). \tag{2.8}$$

Proof. For convenience, let H denote the first member of the assertion (2.8) of Theorem 2.4. Then,

$$H=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_kT_{n+m}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)z^kt^n.$$

Replacing n by $n+pk$ and then using relation (1.14) we may write

$$\begin{aligned}H&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}T_{n+m+pk}(x)a_k\,\Omega_{\mu+\psi k}(y_1,...,y_r)z^kt^{n+pk}\\&=\left(\sum_{k=0}^{\infty}a_k\left(\sum_{n=0}^{\infty}T_{n+m+pk}(x)t^n\right)\Omega_{\mu+\psi k}(y_1,...,y_r)(zt^p)^k\right)\\&=\sum_{k=0}^{\infty}a_kg_{T,m+pk}(x,t)\Omega_{\mu+\psi k}(y_1,...,y_r)(zt^p)^k\\&=\Lambda_{m,\mu,\psi}(x,t;y_1,...,y_r;zt^p).\end{aligned}$$

which completes the proof. ■

Theorem 2.5. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,...,y_r)$ *of* r *complex variables* $y_1,...,y_r$ $(r\in\mathbb{N})$ *and of complex order* μ, *let*

$$\Lambda_{m,\mu,\psi}(x;y_1,...,y_r;t):=\sum_{k=0}^{\infty}a_k\mathbf{J}_{m+pk}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)t^k$$

where $a_k\neq 0$, $\mu,\psi\in\mathbb{C}$ *and*

$$\theta_{\mu,\psi}(y_1,...,y_r;\xi):=\sum_{k=0}^{[n/p]}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k.$$

Then, for $n,m\in\mathbb{N}$; *we have*

$$\sum_{n=0}^{\infty}\mathbf{J}_{n+m}(x)\theta_{\mu,\psi}(y_1,...,y_r;z)t^n=\Lambda_{m,\mu,\psi}(x,t;y_1,...,y_r;zt^p). \tag{2.9}$$

Proof. For convenience, let S denote the first member of the assertion (2.9) of Theorem 2.5. Then,

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k\mathbf{J}_{n+m}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)z^kt^n.$$

Replacing n by $n+pk$ and then using relation (2.4) we may write

$$\begin{aligned}S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathbf{J}_{n+m+pk}(x)a_k\,\Omega_{\mu+\psi k}(y_1,...,y_r)z^k t^{n+pk}\\ &= \left(\sum_{k=0}^{\infty}a_k\left(\sum_{n=0}^{\infty}\mathbf{J}_{n+m+pk}(x)t^n\right)\Omega_{\mu+\psi k}(y_1,...,y_r)(zt^p)^k\right)\\ &= \sum_{k=0}^{\infty}a_k g_{\mathbf{J},m+pk}(x,t)\Omega_{\mu+\psi k}(y_1,...,y_r)(zt^p)^k\\ &= \Lambda_{m,\mu,\psi}(x,t;y_1,...,y_r;zt^p).\end{aligned}$$

which completes the proof. ■

3. Special Cases

We formulate the sum of the first n terms of generalized tribonacci polynomials and generalized tricobsthal polynomials respectively.

Theorem 3.1. *The sum of the first $n-$terms of generalized tribonacci polynomials and generalized tricobsthal polynomials are given by*

$$\begin{aligned}\sum_{j=1}^{n}T_j(x) &= \frac{\left\{\begin{array}{c}T_{n+3}(x)+(1-x^2)T_{n+2}(x)+(1-x^2-x)T_{n+1}(x)\\ -(1-x^2-x)T_1(x)+(x^2-1)T_2(x)-T_3(x)\end{array}\right\}}{x^2+x},\\ \sum_{j=0}^{n}\mathbf{J}_j(x) &= \frac{\mathbf{J}_{n+3}(x)-x\mathbf{J}_{n+1}(x)-\mathbf{J}_3(x)+x\mathbf{J}_1(x)}{x^2+x}\end{aligned}$$

respectively.

Proof. Note that, applying $T_n(x)=x^2T_{n-1}(x)+xT_{n-2}(x)+T_{n-3}(x)$, we deduce that

$$\begin{aligned}&n=4\Rightarrow T_4(x)=x^2T_3(x)+xT_2(x)+T_1(x)\\ &n=5\Rightarrow T_5(x)=x^2T_4(x)+xT_3(x)+T_2(x)\\ &\cdots\\ &n=n+2\Rightarrow T_{n+2}(x)=x^2T_{n+1}(x)+xT_n(x)+T_{n-1}(x)\\ &n=n+3\Rightarrow T_{n+3}(x)=x^2T_{n+2}(x)+xT_{n+1}(x)+T_n(x).\end{aligned} \tag{3.1}$$

If we sum of both sides of (3.1), then we obtain

$$\begin{aligned}T_4(x)+T_5(x)+\cdots+T_{n+3}(x) &= xT_2(x)\\ &\quad+\left[(x^2+x)\sum_{j=3}^{n+1}T_j(x)\right]+x^2T_{n+2}(x)\\ &\quad+\sum_{j=1}^{n}T_j(x).\end{aligned} \tag{3.2}$$

If we make necessary regulations, (3.2) becomes

$$(x^2+x)\sum_{j=1}^{n}T_j(x)=\left\{\begin{array}{c}(1-x^2)T_{n+2}(x)+T_{n+3}(x)-(x^2+x)T_3(x)\\ -xT_2(x)+(1-x^2-x)T_{n+1}(x)\\ -(1-x^2-x)(T_1(x)+T_1(x)+T_1(x))\end{array}\right\}.$$

Therefore

$$\sum_{j=1}^{n}T_j(x)=\frac{\left\{\begin{array}{c}T_{n+3}(x)+(1-x^2)T_{n+2}(x)+(1-x^2-x)T_{n+1}(x)\\ -(1-x^2-x)T_1(x)+(x^2-1)T_2(x)-T_3(x)\end{array}\right\}}{x^2+x}$$

as we claimed. The other cases for generalized tricobsthal polynomials can be done similarly. ■

Theorem 3.2. *The exponential generating function of generalized tribonacci polynomials and generalized tricobsthal polynomials are given by*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{T_n(x)}{n!}t^n &= \frac{\beta_T\gamma_T C_{1,T}e^{\alpha_T t}+\alpha_T\gamma_T C_{2,T}e^{\beta_T t}+\alpha_T\beta_T C_{3,T}e^{\gamma_T t}}{\alpha_T\beta_T\gamma_T},\\ \sum_{n=0}^{\infty}\frac{\mathbf{J}_n(x)}{n!}t^n &= \frac{\beta_{\mathbf{J}}\gamma_{\mathbf{J}} C_{1,\mathbf{J}}e^{\alpha_{\mathbf{J}} t}+\alpha_{\mathbf{J}}\gamma_{\mathbf{J}} C_{2,\mathbf{J}}e^{\beta_{\mathbf{J}} t}+\alpha_{\mathbf{J}}\beta_{\mathbf{J}} C_{3,\mathbf{J}}e^{\gamma_{\mathbf{J}} t}}{\alpha_{\mathbf{J}}\beta_{\mathbf{J}}\gamma_{\mathbf{J}}}\end{aligned}$$

respectively.

Proof. Assuming that the exponential generating function of the generalized tribonacci polynomials, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{T_n(x)}{n!}t^n &= \sum_{n=0}^{\infty}\left(C_{1,T}\alpha_T^{n-1}+C_{2,T}\beta_T^{n-1}+C_{3,T}\gamma_T^{n-1}\right)\frac{t^n}{n!}\\ &= \frac{C_{1,T}}{\alpha_T}\sum_{n=0}^{\infty}\frac{(\alpha_T t)^n}{n!}+\frac{C_{2,T}}{\beta_T}\sum_{n=0}^{\infty}\frac{(\beta_T t)^n}{n!}+\frac{C_{3,T}}{\gamma_T}\sum_{n=0}^{\infty}\frac{(\gamma_T t)^n}{n!}\\ &= \frac{\beta_T\gamma_T C_{1,T}e^{\alpha_T t}+\alpha_T\gamma_T C_{2,T}e^{\beta_T t}+\alpha_T\beta_T C_{3,T}e^{\gamma_T t}}{\alpha_T\beta_T\gamma_T}.\end{aligned}$$

The other cases for generalized tricobsthal polynomials can be done similarly. ■

We can give many applications of our teorems obtained in the previous section with help of appropriate choices of the multivariable functions $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $k\in\mathbb{N}_0$, $r\in\mathbb{N}$, is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems.

If we set

$$s=1 \text{ and } \Omega_{\mu+\psi k}(y)=g_{\mu+\psi k}^{(s)}(\lambda,y)$$

in Theorem 2.1. Recall that, by $g_n^{(s)}(\lambda,x)$ we denote the generalized Cesáro polynomials (see, e.g. [3]) generated by

$$\sum_{n=0}^{\infty}g_n^{(s)}(\lambda,x)t^n=(1-t)^{-s-1}(1-xt)^{-\lambda} \tag{3.3}$$

where $|t|<\min\left\{1,|x|^{-1}\right\}$. Then, from Teorem 2.1, we get a family of the bilateral generating functions for the generalized Cesáro polynomials and the generalized tribonacci polynomials.

Corollary 3.3. *If*

$$\Lambda_{\mu,\psi}(\lambda,y;\zeta):=\sum_{k=0}^{\infty}a_k g_{\mu+\psi k}^{(s)}(\lambda,y)\zeta^k$$
$$(a_k\neq 0\,,\ \mu,\psi\in\mathbb{C})$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k T_{n+1-pk}(x)g_{\mu+\psi k}^{(s)}(\lambda,y)\eta^k t^{n-pk}=\mathcal{G}_T(t)\Lambda_{\mu,\psi}(\lambda,y;\eta)$$

Remark 3.4. *Using the generating relation (1.14) for generalized tribonacci polynomials and $a_k=1$, $\mu=0$, $\psi=1$ in Corollary 3.3, we find that*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k T_{n+1-pk}(x)g_{k}^{(s)}(\lambda,y)\eta^k t^{n-pk}=\mathcal{G}_T(t)(1-\eta)^{-s-1}(1-y\eta)^{-\lambda}.$$

We first set

$$\Omega_{\mu+\psi k}(y_1,...,y_r\,)=\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)$$

in Theorem 2.2, where the multivariable polynomials $\Phi_{\mu+\psi k}^{(\alpha)}(x_1,...,x_r)$ [1], generated by

$$\sum_{n=0}^{\infty}\Phi_{n}^{(\alpha)}(x_1,...,x_r)z^n=(1-x_1z)^{-\alpha}\,e^{(x_2+...+x_r)z} \tag{3.4}$$

where $|z|<|x_1|^{-1}$.

The following results which provides a class of bilateral generating functions for generalized tribonacci polynomials and the family of multivariable polynomials given explicitly by (3.4).

Corollary 3.5. *If*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta):=\sum_{k=0}^{\infty}a_k\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)\zeta^k$$
$$(a_k\neq 0\,,\ \mu,\psi\in\mathbb{C})$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k\mathbf{J}_{n+1-pk}(x)\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)\eta^k t^{n-pk}=\mathcal{G}_{\mathbf{J}}(t)\Lambda_{\mu,\psi}(y_1,...,y_r;\eta) \tag{3.5}$$

provided that each member of (3.5) exists.

Remark 3.6. *Using the generating relation (3.4) for the multivariable polynomials and getting $a_k=1$, $\mu=0$, $\psi=1$ in Corollary 3.1, we find that*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}\mathbf{J}_{n+1-pk}(x)\Phi_{k}^{(\alpha)}(y_1,...,y_r)\eta^k t^{n-pk}=\mathcal{G}_{\mathbf{J}}(t)\,(1-y_1\eta)^{-\alpha}\,e^{(y_2+...+y_r)\eta},$$

$$\left(|\eta|<\left\{|y_1|^{-1}\right\}\right).$$

If we set $s = 1$

$$\Omega_{\mu+\psi k}(y) = \mathbf{J}_{\mu+\psi k-1}(y)$$

in Theorem 2.2.Then, from Teorem 2.2, we get a family of the bilinear generating functions for generalized tricobsthal polynomials given explicitly by (1.8).

Corollary 3.7. *If*

$$\Lambda_{\mu,\psi}(y;\zeta) : = \sum_{k=0}^{\infty} a_k \mathbf{J}_{\mu+\psi k-1}(y)\zeta^k$$
$$(a_k \neq 0\,,\ \mu, \psi \in \mathbb{C})$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k \mathbf{J}_{n+1-pk}(x)\mathbf{J}_{\mu+\psi k-1}(y)\eta^k t^{n-pk} = \mathcal{G}(t)\Lambda_{\mu,\psi}(y;\eta) \tag{3.6}$$

provided that each member of (3.6) exists.

Remark 3.8. *Using the generating relation (1.15) for generalized tricobsthal polynomials and getting $a_k = 1$, $\mu = 0, \psi = 1$ in Corollary 3.2, we find that*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} \mathbf{J}_{n+1-pk}(x)\mathbf{J}_{k-1}(y)\eta^k t^{n-pk} = \mathcal{G}_{\mathbf{J}}(t)\mathbf{g}_{\mathbf{J}}(\eta)$$

If we set

$$\Omega_{\mu+\psi k}(y_1,...,y_r) = h_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)$$

in Theorem 2.4. Recall that, by $h_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)$ we denote the multivariable Lagrange-Hermite polynomials [8] generated by

$$\sum_{n=0}^{\infty} h_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)t^n = \prod_{j=1}^{r}\left\{\left(1 - x_j t^j\right)^{-\alpha_j}\right\} \tag{3.7}$$

where $|t| < \min\left\{|x_1|^{-1},...,|x_r|^{-1/r}\right\}$. Then, from Teorem 2.4, we obtain the following result which is aclass of bilateral generating functions for the multivariable Lagrange-Hermite polynomials and generalized tribonacci polynomials.

Corollary 3.9. *If*

$$\Lambda_{m,\mu,\psi}(x;y_1,...,y_r;t) : = \sum_{k=0}^{\infty} a_k T_{m+pk}(x) h_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)t^k$$
$$(a_k \neq 0\,,\ \mu, \psi \in \mathbb{C})$$

and

$$\theta_{\mu,\psi}(y_1,...,y_r;\xi) := \sum_{k=0}^{[n/p]} a_k h_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)\xi^k$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k T_{n+m}(x) h_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)z^k t^n = \Lambda_{m,\mu,\psi}(x.t;y_1,...,y_r;zt^p).$$

If we set

$$\Omega_{\mu+\psi k}(y_1,...,y_r)=g_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)$$

in Theorem 2.5. Recall that, by $g_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)$ we denote the Chan-Chyan-Srivastava polynomials [9] generated by

$$\sum_{n=0}^{\infty} g_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)t^n=\prod_{j=1}^{r}\left\{(1-x_jt)^{-\alpha_j}\right\} \tag{3.8}$$

where $|t|<\min\left\{|x_1|^{-1},...,|x_r|^{-1}\right\}$. Then, from Teorem 2.5, we obtain the following result which is aclass of bilateral generating functions for the Chan-Chyan-Srivastava polynomials and generalized tricobsthal polynomials.

Corollary 3.10. *If*

$$\Lambda_{m,\mu,\psi}(x;y_1,...,y_r;t):=\sum_{k=0}^{\infty} a_k\mathbf{J}_{m+pk}(x)g_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)t^k$$
$$(a_k\neq 0\,,\ \mu,\psi\in\mathbb{C})$$

and

$$\theta_{\mu,\psi}(y_1,...,y_r;\xi):=\sum_{k=0}^{[n/p]} a_k g_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)\xi^k$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k\mathbf{J}_{n+m}(x)g_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)z^kt^n=\Lambda_{m,\mu,\psi}(x.t;y_1,...,y_r;zt^p).$$

Notice that, for every suitable choice of the coefficients a_k $(k\in\mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $r\in\mathbb{N}$, are expressed as an appropriate product of several simpler relatively functions, the assertions of Theorem 2.1, 2.2, 2.4 and Theorem 2.5 can be applied to yield many different families of multilinear and multilateral generating functions for generalized tribonacci polynomials and generalized tricobsthal polynomials.

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