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The planar and outerplanar indices of Cayley graphs of finite groups

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Abstract. In our study, we consider the Cayley graph of finite groups and its iterated line graphs. We present a complete characterization of finite groups with planar and outerplanar indices.

AMS Subject Classifications: 05C25, 05C76, 20F65.

Keywords: Cayley graph, Finite group, Planar index, Outerplanar index.

Contents

1. Introduction and Background

Let S generates group G. We define the Cayley graph $\overrightarrow{\text{Cay}}(G, S)$ of generators S on G as follows. The vertices of $\overrightarrow{\text{Cay}}(G, S)$ are the elements of G, and there is an arc from g to gs whenever $g \in G$ and $s \in S$. The Cayley graph $\text{Cay}(G, S)$ of S on G is obtained by replacing each arc in $\text{Cay}(G, S)$ with an (undirected) edge. One can identify $\text{Cay}(G, S)$ with $\overrightarrow{\text{Cay}}(G, S \cup S^{-1})$, where $S^{-1} = \{s^{-1}; s \in S\}$.

Cayley graphs of groups enjoy a rich research history and they are a classic point of interaction of graph theory and algebra. The original definition of the Cayley graph of a group was introduced by Cayley in 1878 [1] to explain the concept of abstract groups described by a set of generators. In the last 50 years, the theory of Cayley graphs have grown into a substantial branch in algebraic graph theory. We refer the reader to [3, 6, 7, 10, 14], for more details.

It is interesting to find graphs that can be drawn respecting certain geometric or topological criteria. This work is done for some Cayley graphs on some algebraic structures. Also, there are some characterizations for these algebraic structures which their Cayley graphs can be drawn in a plane. For example see [9], and [11, 12].

A group is called planar if it admits a generating system such that the resulting Cayley graph is planar, that is, it admits a plane drawing. In 1896, Maschke characterized planar finite groups, that is groups which admit a generating system such that the resulting Cayley graph is planar.

Theorem 1.1. *[12, Maschke's Theorem] The groups and minimal generating systems in Table 1 are exactly those pairs having a planar Cayley graph.*

Group	Minimal generating systems
$\mathbb{Z}_n$	1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(1,0), (0,1)$
$\mathbb{Z}_2 \times \mathbb{Z}_n$	$(1,0), (0,1)$
D_3	$(123), (12), (23)$
D_4	$(1234), (13)$
D_n	$(12), (13)$ $(12\ldots n), (12)$
$\mathbb{Z}_2 \times D_n$	$(1,e), (0,(12)), (0,(13))$
A_4	$(123), (12)(34)$ $(123), (234)$ $(123), (234), (13)(24)$
$\mathbb{Z}_2 \times A_4$	$(0,(123)), (1,(12)(34))$
S_4	$(123), (34)$ $(12), (23), (34)$ $(12), (1234)$ $(123), (1234)$ $(1234), (123), (34)$
$\mathbb{Z}_2 \times S_4$	$(1,(12)), (0,(23)), (0,(34))$
A_5	$(124), (23)(45)$ $(12345), (23)(45)$ $(12345), (124)$ $(12345), (124), (23)(45)$
$\mathbb{Z}_2 \times A_5$	$(1,(12)(35)), (1,(24)(35)), (1,(23)(45))$

Table 1: The planar groups and their minimal generating systems giving planar Cayley graphs

In this paper, we will focus on embeddability of the Cayley graph and its iterated line graphs into a plane. Given a graph G, its line graph $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G. Also, we denote the kth iterated line graph of G by $L^k(G)$ and define it as follows: $L^k(G) = L(L^{k-1}(G))$ for $k \geqslant 1$ and $L^0(G) = G$ and $L^1(G) = L(G)$ is the line graph of G. We define the planar index of G, denoted by $\xi(G)$, as the smallest k such that $L^k(G)$ is non-planar. If $L^k(G)$ is planar for all $k \geqslant 0$, we define $\xi(G) = \infty$. Further, the outerplanar index of G is defined as the smallest k such that $L^k(G)$ is non-outerplanar. We denote the outerplanar index of G by $\zeta(G)$. As well as, if $L^k(G)$ is outerplanar for all $k \geqslant 0$, we define $\zeta(G) = \infty$.

This paper is organized as follows. At first, we deal with planar index of the Cayley graph of a finite group. In addition, we classify all finite groups which admit outerplanar Cayley graph. Also, we study the outerplanar index of the Cayley graph when G is a finite group.

In order to make this paper easier to follow, let recall some standard definitions and notation of group theory and graph theory we use in this paper. Let n be a positive integer. The group of integers modulo n is denoted by $\mathbb{Z}_n = \{0, \ldots, n-1\}$. Also, the notation D_n stands for the dihedral group. The elements of D_n are the symmetries of the n-gon with the vertices $1, \ldots, n$ and so $|D_n| = 2n$. Further, A_n and S_n are the alternating group and the symmetric group on n points, respectively. The identity element is denoted by e for all groups G except for $\mathbb{Z}_n$, where we use 0.

Now let us summarize some notations, concepts of graph theory which will be needed in the subsequent sections. For basic definitions on graphs, one may refer to [2]. Let G be a graph. Then the degree of a vertex v, denoted by $\deg(v)$, is the number of edges of G incident to v. Also, an r-regular graph is a graph where every vertex has the degree r. The maximum degree of G, denoted by $\Delta(G)$, is the maximum degree of its vertices. The

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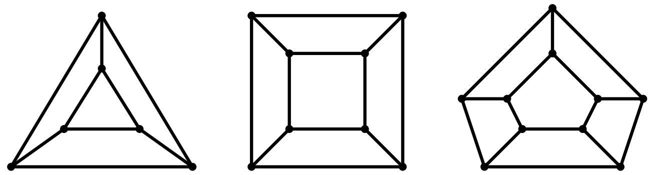

Figure 1: n-prisms

graph G is connected if for every pair of vertices, there is a path in the graph between those vertices, otherwise G is said to be disconnected. A connected component of a disconnected graph is a maximal connected subgraph of the graph. A cut vertex v is a vertex that when we removed it (with its boundary edges) from G creates more connected components than previously in G. We use the notations K_n and C_n for complete graphs and cycles on n vertices, respectively. The Cartesian product $G \times H$ of graphs G and H is a graph such that

(i) the vertex set of the graph $G \times H$ is the Cartesian product $V(G) \times V(H)$; and

(ii) any two vertices (u, u') and (v, v') are adjacent in $G \times H$ if and only if either $u = v$and u' is adjacent with v' in H, or $u' = v'$ and u is adjacent with v in G.

An n-prism graph is a simple graph which can be constructed as the Cartesian product of the cycle C_n with K_2. In Figure 1, 3-prism, 4-prism and 5-prism are drawn.

2. The planar index of Cay(G, S)

This section consists on classifying all finite groups with respect to planar index of their Cayley graphs. At first, we determine when $L(\text{Cay}(G, S))$ is planar. Sedláček [13], characterized graphs whose their line graph is planar. He showed that the line graph of a graph G is planar if and only if G is planar, $\Delta(G) \leqslant 4$, and every vertex of degree 4 in G is a cut-vertex. Using Sedláček's characterization, in the following lemma, we characterizes all Cayley graphs whose their line graph is planar.

Lemma 2.1. *The groups and minimal generating systems in Table 2 are exactly those pairs which $L(Cay(G, S))$ is planar.*

Proof. By using Sedláček's characterization, if $L(\text{Cay}(G, S))$ is planar, then G is planar. So, we must only check the planar groups. By Maschke's Theorem, Theorem 1.1, we have the following cases:

Case 1. $G \cong \mathbb{Z}_n$ with $S = \{1\}$ and $G \cong D_n$ with $S = \{(12), (13)\}$. In both cases, the graph Cay(G, S) is 2-reqular graph which implies that these graphs are cycles. So, $L(\text{Cay}(G, S))$ is planar.

Case 2. $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with $S = \{(1, 0), (0, 1)\}$. In this case the Cayley graph Cay(G, S) is a cycle with 4 vertices and so $L(\text{Cay}(G, S))$ is planar.

Case 3. $G \cong \mathbb{Z}_2 \times \mathbb{Z}_n$ with $S = \{(1, 0), (0, 1)\}$ and $G \cong D_n$ with $S = \{(12 \dots n), (12)\}$ where $n \geqslant 3$. In both these cases, it is not hard to see that the graph Cay(G, S) is a n-prism graph. Since n-prisms are 3-regular graphs we have that $L(\text{Cay}(G, S))$ is planar.

Case 4. $G \cong D_3$ with $S = \{(123), (12), (23)\}$. Since $|S \cup S^{-1}| = 4$, the graph Cay(G, S) is a 4-reqular graph. This graph is drawn in Figure 2. By this figure, we see that the graph Cay(G, S) has a vertex of degree 4 which is not a cut vertex. Hence $L(\text{Cay}(G, S))$ is not planar.

Case 5. $G \cong D_4$ with $S = \{(1234), (13)\}$. The graph Cay(G, S) is a 4-prism graph. So it is a 3-regular graph and we have that $L(\text{Cay}(G, S))$ is planar.

Group	Minimal generating systems
$\mathbb{Z}_n$	1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(1,0), (0,1)$
$\mathbb{Z}_2 \times \mathbb{Z}_n$ where $n \geqslant 3$	$(1,0), (0,1)$
D_3	$(123), (12), (23)$
D_4	$(1234), (13)$
D_n	$(12), (13)$ $(12\dots n), (12)$
$\mathbb{Z}_2 \times D_n$	$(1,e), (0,(12)), (0,(13))$
A_4	$(123), (12)(34),$
$\mathbb{Z}_2 \times A_4$	$(0,(123)), (1,(12)(34))$
S_4	$(123), (34)$ $(12), (23), (34)$ $(12), (1234)$
$\mathbb{Z}_2 \times S_4$	$(1,(12)), (0,(23), (0,(34))$
A_5	$(124), (23)(45)$ $(12345), (23)(45)$
$\mathbb{Z}_2 \times A_5$	$(1,(12)(35)), (1,(24)(35)), (1,(23)(45))$

Table 2: The groups and their minimal generating systems which the line of their cayley graphs are planar

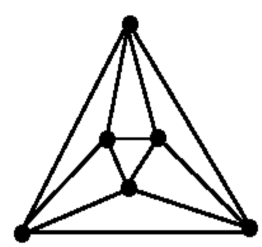

Figure 2: Cay$(D_3, \{(123), (12), (23)\})$

Case 6. $G \cong \mathbb{Z}_2 \times D_n$ and $S = \{(1,e), (0,(12)), (0,(13))\}$. The Cayley graph Cay$(G,S)$ is a $2n$-prism. Therefore this graph is a 3-regular graph which implies that $L($Cay$(G,S))$ is planar.

Case 7. $G \cong A_4$ and $S = \{(123), (12)(34)\}$. Since $|S \cup S^{-1}| = 3$, the Cayley graph Cay(G,S) is a 3-regular graph (Figure 3) and so $L($Cay$(G,S))$ is planar.

Case 8. $G \cong A_4$ and $S = \{(123), (234)\}$. By Figure 4, Cayley graph Cay(G,S) is a 4-regular graph and it has a vertex of degree 4 which is not a cut vertex. So $L($Cay$(G,S))$ is not planar.

Case 9. $G \cong A_4$ and $S = \{(123), (234), (13)(24)\}$. Since $|S \cup S^{-1}| = 5$, Cayley graph Cay(G,S) is a 5-regular graph and so $L($Cay$(G,S))$ is not planar.

Case 10. $G \cong \mathbb{Z}_2 \times A_4$ with $S = \{(0,(123)), (1,(12)(34))\}$ and $G \cong S_4$ with $S = \{(123), (34)\}$. Since $|S \cup S^{-1}| = 3$, in both cases the Cayley graph Cay(G,S) is a 3-regular graph. It is not hard to see that

$$\text{Cay}(\mathbb{Z}_2 \times A_4, \{(0,(123)), (1,(12)(34))\}) \cong \text{Cay}(S_4, \{(123), (34)\}).$$

The graph is pictured in Figure 5. Hence we can conclude that $L($Cay$(G,S))$ is planar.

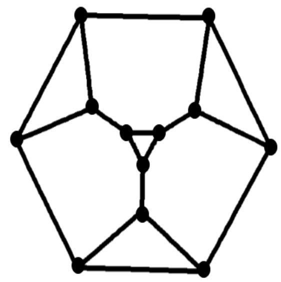

Figure 3: Cay$(A_4, \{(123), (12)(34)\})$

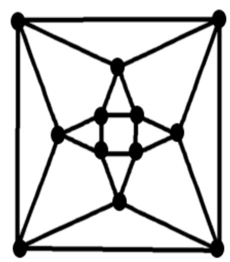

Figure 4: Cay$(A_4, \{(123), (234)\})$

Case 11. $G \cong S_4$ with $S = \{(12), (23), (34)\}$ or $S = \{(12), (1234)\}$. It is easy to see that the Cayley graph Cay(G, S) is a 3-regular graph and they are isomorphic to Figure 6. Hence $L($Cay$(G, S))$ is planar.

Case 12. $G \cong S_4$ with $S = \{(123), (1234)\}$. It is easy to see that the Cayley graph Cay(G, S) is a 4-regular graph and none of the vertices is a cut vertex. So $L($Cay$(G, S))$ is not planar.

Case 13. $G \cong S_4$ with $S = \{(1234), (123), (34)\}$. Since $|S \cup S^{-1}| = 5$, the Cayley graph Cay(G, S) is a 5-regular graph. Therefore $L($Cay$(G, S))$ is not planar.

Case 14. $G \cong \mathbb{Z}_2 \times S_4$ with $S = \{(1, (12)), (0, (23)), (0, (34))\}$. Since $|S \cup S^{-1}| = 3$, the Cayley graph Cay(G, S) is a 3-regular graph which is pictured in Figure 7. Therefore $L($Cay$(G, S))$ is planar.

Case 15. $G \cong A_5$ with $S = \{(124), (23)(45)\}$. Since $|S \cup S^{-1}| = 3$, the Cayley graph Cay(G, S) is a 3-regular graph (Figure 8). Therefore $L($Cay$(G, S))$ is planar.

Case 16. $G \cong A_5$ with $S = \{(12345), (23)(45)\}$. Since $|S \cup S^{-1}| = 3$, the Cayley graph Cay(G, S) is a 3-regular graph which is drawn in Figure 9. Hence $L($Cay$(G, S))$ is planar.

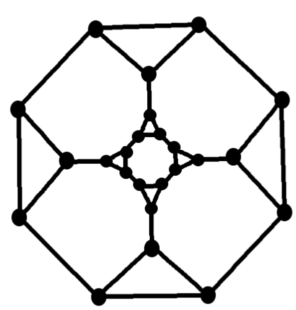

Figure 5: Cay$(\mathbb{Z}_2 \times A_4, \{(0, (123)), (1, (12)(34))\})$

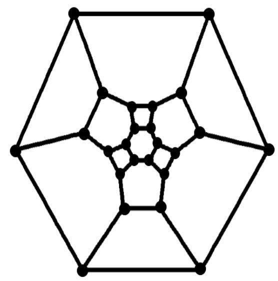

Figure 6: Cay$(S_4, \{(12),(23),(34)\}) \cong$ Cay$(S_4, \{(12),(1234)\})$

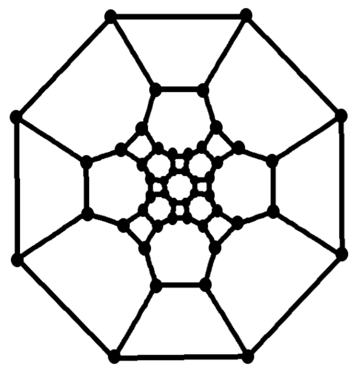

Figure 7: Cay$(\mathbb{Z}_2 \times S_4, \{(1,(12)),(0,(23)),(0,(34))\})$

Case 17. $G \cong A_5$ with $S = \{(12345),(124)\}$. Since $|S \cup S^{-1}| = 4$, the Cayley graph Cay(G,S) is a 4-regular graph. It is easy to see that none of the vertices of this graph is a cut vertex. Therefore $L($Cay$(G,S))$ is not planar.

Case 18. $G \cong A_5$ with $S = \{(12345),(124),(23)(45)\}$. Since $|S \cup S^{-1}| = 5$, the Cayley graph Cay(G,S) is a 5-regular graph, which implies that $L($Cay$(G,S))$ is not planar.

Case 19. $G \cong \mathbb{Z}_2 \times A_5$ with $S = \{(1,(12)(35)),(1,(24)(35)),(1,(23)(45))\}$. Since $|S \cup S^{-1}| = 3$, the Cayley graph Cay(G,S) is a 3-regular graph(Figure 10), which implies that $L($Cay$(G,S))$ is planar.

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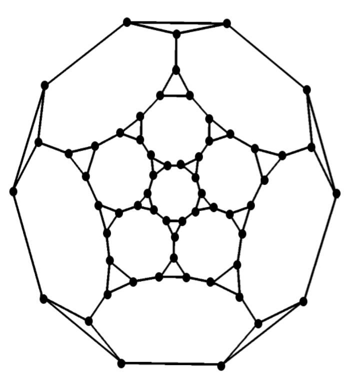

Figure 8: Cay$(A_5, \{(124),(23)(45)\})$

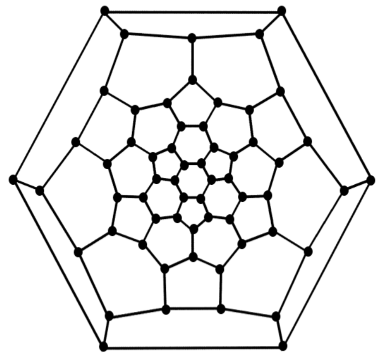

Figure 9: Cay$(A_5, \{(12345), (23)(45)\})$

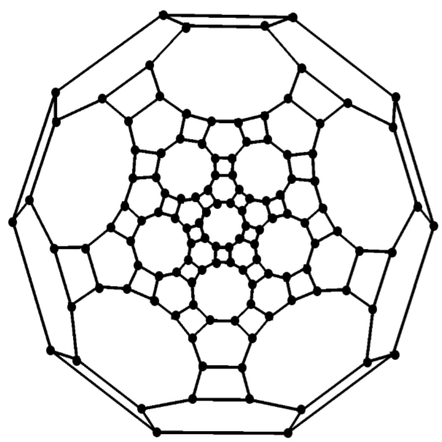

Figure 10: Cay$(\mathbb{Z}_2 \times A_5, \{(1, (12)(35)), (1, (24)(35)), (1, (23)(45))\})$

In the rest of this section, we deal with planar index of Cay(G, S). It was shown in [13] that if G is non-planar, then $L(G)$ is also non-planar. Also, if H is a subgraph of G, in [4, Lemma 4], it was shown that $\xi(G) \leqslant \xi(H)$, and hence the planar index of a graph is the minimum of the planar indices of its connected components. Further, in [4], the authors gave a full characterization of connected graphs with respect to their planar index.

Theorem 2.2. *[4, Theorem 10] Let G be a connected graph. Then:*

(i) $\xi(G) = 0$ *if and only if G is non-planar.*

(ii) $\xi(G) = \infty$ *if and only if G is either a path, a cycle, or* $K_{1,3}$.

(iii) $\xi(G) = 1$ *if and only if G is planar and either* $\Delta(G) \geqslant 5$ *or G has a vertex of degree* 4 *which is not a cut-vertex.*

(iv) $\xi(G) = 2$ *if and only if* $L(G)$ *is planar and G contains one of the graphs* H_i *in Figure 11 as a subgraph.*

(v) $\xi(G) = 4$ *if and only if G is one of the graphs* X_k *or* Y_k *(Figure 11) for some* $k \geqslant 2$.

(vi) $\xi(G) = 3$ *otherwise.*

In the next theorem we classify the Cayley graphs Cay(G, S) of finite groups with respect to their planar index.

Theorem 2.3. *Let G be a finite group. Then:*

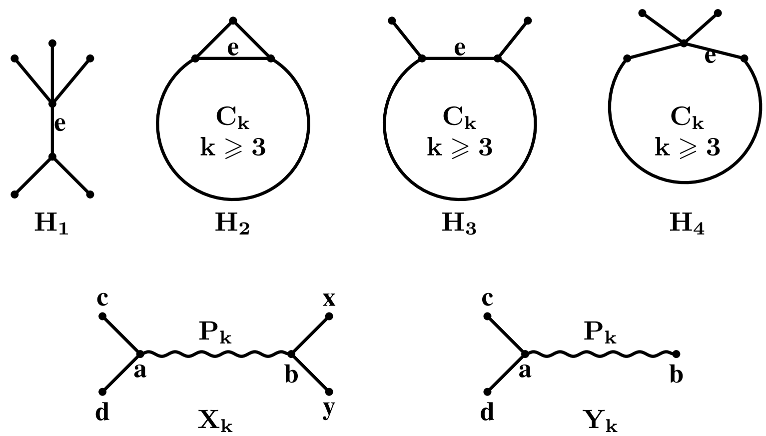

Figure 11: Figures of Theorem 2.2

Group	Minimal generating systems
$\mathbb{Z}_n$	1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(1,0),(0,1)$
D_n	$(12),(13)$

Table 3: The groups and their minimal generating systems which $\xi(\text{Cay}(G,S)) = \infty$.

(i) $\xi(Cay(G,S)) = 0$ *if and only if* $Cay(G,S)$ *is non-planar.*

(ii) $\xi(Cay(G,S)) = \infty$ *if and only if* G *and* S *are as in Table 3.*

(iii) $\xi(Cay(G,S)) = 1$ *if and only if* G *and* S *are as in Table 4.*

(iv) $\xi(Cay(G,S)) = 2$ *if and only if* G *and* S *are as in Table 5.*

Proof. We know $\xi(\text{Cay}(G,S)) = 0$ if $\text{Cay}(G,S)$ is non-planar. Thus we may assume that $\text{Cay}(G,S)$ is planar. So, by Maschke's Theorem, we must consider the groups and minimal generating sets which were stated in Table 1. By comparing Tables 1 and 2, we can conclude that $\xi(\text{Cay}(G,S)) = 1$ for the groups and minimal generating sets of Table 4. Now, by Lemma 2.1 and Table 2, we have the following cases:

Case 1. Let $G \cong \mathbb{Z}_n$ with $S = \{1\}$, $G \cong D_n$ with $S = \{(12),(13)\}$ and $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with $S = \{(1,0),(0,1)\}$. Since the Cayley graphs of these groups and their generating sets are cycles, we can conclude that $\xi(\text{Cay}(G,S)) = \infty$.

Case 2. Let $G \cong \mathbb{Z}_2 \times \mathbb{Z}_n$ with $S = \{(1,0),(0,1)\}$, $G \cong D_n$ with $S = \{(12\ldots n),(12)\}$ and $G \cong \mathbb{Z}_2 \times D_n$ and

Group	Minimal generating systems
A_4	$(123),(234)$
	$(123),(234),(13)(24)$
S_4	$(123),(1234)$
	$(1234),(123),(34)$
A_5	$(12345),(124)$
	$(12345),(124),(23)(45)$

Table 4: The groups and their minimal generating systems which $\xi(\text{Cay}(G,S)) = 1$.

Group	Minimal generating system
$\mathbb{Z}_2 \times \mathbb{Z}_n$ where $n \geqslant 3$	$(1, 0), (0, 1)$
D_3	$(123), (12), (23)$
D_4	$(1234), (13)$
D_n	$(12 \ldots n), (12)$
$\mathbb{Z}_2 \times D_n$	$(1, e), (0, (12)), (0, (13))$
A_4	$(123), (12)(34)$
$\mathbb{Z}_2 \times A_4$	$(0, (123)), (1, (12)(34))$
S_4	$(123), (34)$ $(12), (23), (34)$
$\mathbb{Z}_2 \times S_4$	$(1, (12)), (0, (23), (0, (34))$
A_5	$(124), (23)(45)$ $(12345), (23)(45)$ $(12345), (124)$
$\mathbb{Z}_2 \times A_5$	$(1, (12)(35)), (1, (24)(35)), (1, (23)(45))$

Table 5: The groups and their minimal generating systems which $\xi(\text{Cay}(G, S)) = 2$.

$S = \{(1, e), (0, (12)), (0, (13))\}$. In these cases, the graph Cay(G, S) is a prism and so they have H_3 as a subgraph. Hence $\xi(L(\text{Cay}(G, S)) = 2$.

Case 3. Assume that $G \cong D_3$ and $S = \{(123), (12), (23)\}$ be the minimal generating set of it. By Figure 2, the Cayley graph Cay(G, S) has H_4 as a subgraph which implies that $\xi(L(\text{Cay}(G, S)) = 2$.

Case 4. Suppose that $G \cong A_4$ and $S = \{(123), (12)(34)\}$. By Figure 3, the Cayley graph of this group has H_3 as a subgraph. Hence $\xi(L(\text{Cay}(G, S)) = 2$.

Case 5. $G \cong A_4$ and $S = \{(123), (234)\}$. By Figure 4, the Cayley graph of this group has H_2 as a subgraph. Hence $\xi(L(\text{Cay}(G, S)) = 2$.

Case 6. $G \cong \mathbb{Z}_2 \times A_4$ with $S = \{(0, (123)), (1, (12)(34))\}$ and $G \cong S_4$ with $S = \{(123), (34)\}$. By Figure 5, the Cayley graph of this group has H_2 as a subgraph. Hence $\xi(L(\text{Cay}(G, S)) = 2$.

Case 7. $G \cong S_4$ with $S = \{(12), (23), (34)\}$ or $S = \{(12), (1234)\}$. It is easy to see that the Cayley graph Cay(G, S) has H_3 as a subgraph (Figure 6). Hence $\xi(L(\text{Cay}(G, S)) = 2$.

Case 8. $G \cong \mathbb{Z}_2 \times S_4$ with $S = \{(1, (12)), (0, (23)), (0, (34))\}$. Since the Cayley graph Cay(G, S) has a subgraph isomorphic to H_3 (Figure 7). Therefore $\xi(\text{Cay}(G, S)) = 2$.

Case 9. $G \cong A_5$ with $S = \{(124), (23)(45)\}$. By Figure 8, it is easy to see that the Cayley graph Cay(G, S) has a subgraph which is isomorphic to H_3. Thus $\xi(\text{Cay}(G, S)) = 2$.

Case 10. $G \cong A_5$ with $S = \{(12345), (23)(45)\}$. It is easy to see that the Cayley graph Cay(G, S) has H_3 as a subgraph (Figure 9). Hence $\xi(\text{Cay}(G, S)) = 2$.

Case 11. $G \cong \mathbb{Z}_2 \times A_5$ with $S = \{(1, (12)(35)), (1, (24)(35)), (1, (23)(45))\}$. By Figure 10, the Cayley graph Cay(G, S) has a subgraph isomorphic to H_3 which implies that $\xi(\text{Cay}(G, S)) = 2$

■

Group	Minimal generating systems
$\mathbb{Z}_n$	1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(1,0), (0,1)$
D_n	$(12), (13)$

Table 6: The groups and their minimal generating systems giving outerplanar Cayley graphs.

3. Outerplanar index of Cayley graphs

In this section, we study the outerplanar index of the Cayley graphs of finite groups. An undirected graph is an outerplanar graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization of outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of the complete graph K_4 or the complete bipartite graph $K_{2,3}$. At first, we deal with the investigation of when Cayley graphs of finite groups are outerplanar.

Lemma 3.1. *The groups and minimal generating systems in Table 6 are exactly those pairs having an outerplanar Cayley graph.*

Proof. It is well known that every outerplanar graph is planar. So, we must only check planar groups. By Maschke's Theorem, we must consider the groups and minimal generating sets which were stated in Table 1. If $G \cong \mathbb{Z}_n$ with $S = \{1\}$, $G \cong D_n$ with $S = \{(12), (13)\}$ and $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with $S = \{(1,0), (0,1)\}$, since the Cayley graphs of these groups and their generating sets are cycles, we can conclude Cay(G, S) is outerplanar. For other groups and their minimal generating sets of Table 1, it is easy to see that their Cayley graphs have a subgraph homeomorphic to $K_{2,3}$ which implies that Cay(G, S) is not outerplanar. ■

Recall that the outerplanar index of a graph G, which is denoted by $\zeta(G)$, is the smallest integer k such that the kth iterated line graph of G is non-outerplanar. If $L^k(G)$ is outerplanar for all $k \geqslant 0$, we define $\zeta(G) = \infty$. In [5], the authors gave a full characterization of all graphs with respect to their outerplanar index which is stated in the following theorem.

Theorem 3.2. *Let G be a connected graph. Then:*

(i) $\zeta(G) = 0$ *if and only if G is non-outerplanar.*

(ii) $\zeta(G) = \infty$ *if and only if G is a path, a cycle, or* $K_{1,3}$.

(iii) $\zeta(G) = 1$ *if and only if G is planar and G has a subgraph homeomorphic to* $K_{1,4}$ *or* $K_1 + P_3$ *in Figure 12.*

(iv) $\zeta(G) = 2$ *if and only if* $L(G)$ *is planar and G has a subgraph isomorphic to one of the graphs* G_2 *and* G_3 *in Figure 12.*

(v) $\zeta(G) = 3$ *if and only if* $G \in I(d_1, d_2, \ldots, d_t)$ *where* $d_i \geqslant 2$ *for* $i = 2, \ldots, t-1$, *and* $d_1 \geqslant 1$ *(Figure 12).*

Theorem 3.3. *Let G be a finite group. Then:*

(i) $\zeta(G) = \infty$ *if and only if G and S are as in Table 6.*

(ii) $\zeta(Cay(G, S)) = 0$ *otherwise.*

Proof. It follows from Lemma 3.1 and Theorem 3.2. ■

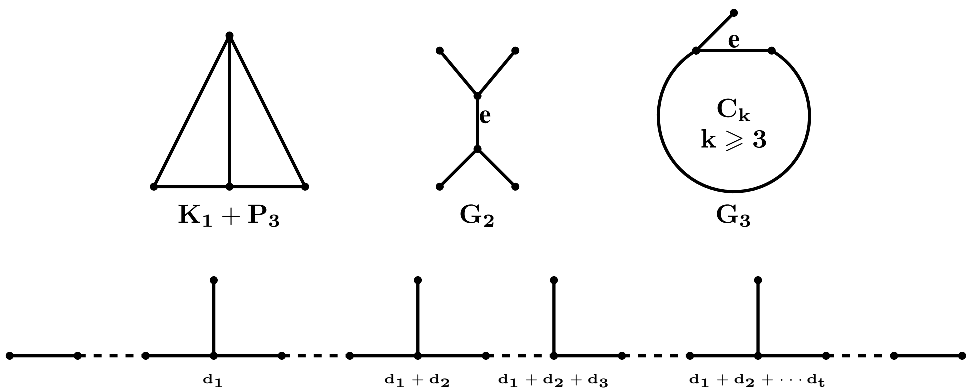

Figure 12: Figures of Theorem 3.2

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Generalized mixed higher order functional equation in various Banach spaces

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Abstract. In this article, the we establish the generalized Ulam-Hyers stability of a generalized mixed $n^{th}(n+1)^{th}$ Order Functional Equation in various Banach Spaces.
AMS Subject Classifications: 39B52, 32B72, 32B82.

Keywords: Mixed functional equations, Generalized Ulam - Hyers stability, Banach space, Modular Space, Fuzzy Banach space, Random Banach Space.

Contents

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1. Introduction

A inspiring and renowned talk presented by Ulam [50] in 1940, encouraged the study of stability problems for various functional equations. He gave a wide range of talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems.

The first confident answer to celebrated Ulam's question concerning the problem of stability of functional equations was given by Hyers [17] for the case of additive mappings in Banach spaces. In development of time, the theorem conveyed by Hyers was generalized by Aoki [3], Rassias [39, 40], Gavruta [14] for additive mappings and Ravi [42] for quadratic mappings.

The general solution and generalized Ulam - Hyers stability of several types of functional equations in various normed spaces were discussed by many authors one can see [1, 9, 11, 19, 20, 41] and references there in.

The simplest functional equations are

$$f(-x) = -f(x); \qquad and \qquad g(-x) = g(x) \tag{1.1}$$

which are the well known odd and even functions.

Inspiring by the overhead idea, Arunkumar et. al., [4] introduced and established the general solution and generalized Ulam - Hyers stability of the simple additive-quadratic and simple cubic-quartic functional equations

$$f(2x) = 3f(x) + f(-x); \qquad and \qquad g(2x) = 12g(x) + 4g(-x); \tag{1.2}$$

having solutions

$$f(x) = ax + bx^2 \qquad and \qquad g(x) = cx^3 + dx^4. \tag{1.3}$$

Also, the generalized Ulam - Hyers stability of the functional equations (1.2) in Quasi-Beta Banach space, Intuitionistic fuzzy Banach space applying direct and fixed point methods were discussed in [5].

Infact, the generalized version of (1.2) was introduced and examined the generalized Ulam - Hyers stability of single variable generalized additive-quadratic and generalized cubic-quartic functional equations of the form

$$\phi(\lambda w) = \frac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) + \frac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right); \tag{1.4}$$

$$\psi(\mu w) = \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) + \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right) \tag{1.5}$$

having solutions

$$\phi(w) = aw + bw^2 \qquad and \qquad \phi(w) = cw^3 + dw^4, \tag{1.6}$$

was investigated by Arunkumar et. al., [6] .

Motivated from overhead ideas in this article, we establish the generalized Ulam-Hyers stability of a Generalized Mixed $n^{th}(n+1)^{th}$ Order Functional Equation

$$\mathcal{N}_{n;n+1}(\mathcal{T}v) = \frac{\mathcal{T}^n}{2}\left(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\right) + \frac{\mathcal{T}^{n+1}}{2}\left(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\right) \tag{1.7}$$

having solutions

$$\mathcal{N}_{n;n+1}(v) = av^n + bv^{n+1} \tag{1.8}$$

with $n \neq 0$ is an **odd positive integer** and $\mathcal{T} \geq 2$ in various Banach Spaces via Hyers Method.

The solution of the functional equation (1.7) are as follows. Assume A_1 and A_2 are vector spaces. Applying oddness and evenness of $\mathcal{N}_{n;n+1}$ the following lemmas are trivial.

Lemma 1.1. *An odd function $\mathcal{N}_{n;n+1} : A_1 \to A_2$ satisfying (1.7) and if we define*

$$\mathcal{N}_{n;n+1} = \mathcal{N}_n \tag{1.9}$$

then $\mathcal{N}_n$ is an n^{th} order function.

Lemma 1.2. *An even function $\mathcal{N}_{n;n+1} : A_1 \to A_2$ satisfying (1.7) and if we define*

$$\mathcal{N}_{n;n+1} = \mathcal{N}_{n+1} \tag{1.10}$$

then $\mathcal{N}_{n+1}$ is an $(n+1)^{th}$ order function.

2. Stability In Banach Space of (1.7)

In this section, we investigate the generalized Ulam - Hyers stability of the functional equations (1.7) in Banach space. To prove stability results, let us take $\mathcal{R}_1$ be an normed space and $\mathcal{R}_2$ be an Banach space.

2.1. Stability Results: Odd Case

Theorem 2.1. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an odd function fulfilling the inequality*

$$\begin{aligned}\Bigg\| \mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\Bigg\| \le \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1,\end{aligned} \tag{2.1}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0,\infty)$ with the condition

$$\lim_{m\to\infty} \frac{\mathcal{M}(\mathcal{T}^{nmt}v)}{\mathcal{T}^{nmt}} = 0; \quad \forall\, v \in \mathcal{R}_1. \tag{2.2}$$

Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\|\Gamma_n(v) - \mathcal{N}_n(v)\| \le \frac{1}{\mathcal{T}^n} \sum_{r=\frac{1-t}{2}}^{\infty} \frac{\mathcal{M}(\mathcal{T}^{nrt}v)}{\mathcal{T}^{nrt}}; \quad \forall\, v \in \mathcal{R}_1; \tag{2.3}$$

with $t = \pm 1$. The mapping $\Gamma_n(v)$ is defined by

$$\Gamma_n(v) = \lim_{m\to\infty} \frac{\mathcal{N}_n(\mathcal{T}^{nmt}v)}{\mathcal{T}^{nmt}}; \quad \forall\, v \in \mathcal{R}_1. \tag{2.4}$$

Proof. Applying oddness of $\mathcal{N}_{n;n+1}$ in (2.1) and by (1.9), we observe that

$$\|\mathcal{N}_n(\mathcal{T}v) - \mathcal{T}^n \mathcal{N}_n(v)\| \le \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1. \tag{2.5}$$

The overhead inequality can be rewritten as

$$\left\| \frac{\mathcal{N}_n(\mathcal{T}v)}{\mathcal{T}^n} - \mathcal{N}_n(v) \right\| \le \frac{\mathcal{M}(v)}{\mathcal{T}^n}; \quad \forall\, v \in \mathcal{R}_1. \tag{2.6}$$

Changing v by $\mathcal{T}v$ and multiplying by $\dfrac{1}{\mathcal{T}^n}$ in (2.6), we notice that

$$\left\| \frac{\mathcal{N}_n(\mathcal{T}^2v)}{\mathcal{T}^{2n}} - \frac{\mathcal{N}_n(\mathcal{T}v)}{\mathcal{T}^n} \right\| \le \frac{\mathcal{M}(\mathcal{T}v)}{\mathcal{T}^{2n}}; \quad \forall\, v \in \mathcal{R}_1. \tag{2.7}$$

From (2.6) and (2.7), we obtain that

$$\left\|\frac{\mathcal{N}_n(\mathcal{T}^2 v)}{\mathcal{T}^{2n}}-\mathcal{N}_n(v)\right\| \leq \frac{1}{\mathcal{T}^n}\left(\mathcal{M}(v)+\frac{\mathcal{M}(\mathcal{T}v)}{\mathcal{T}^n}\right); \quad \forall\, v \,\in\, \mathcal{R}_1. \tag{2.8}$$

Generalizing for a positive integer m, we acquire that

$$\left\|\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}-\mathcal{N}_n(v)\right\| \leq \frac{1}{\mathcal{T}^n}\sum_{r=0}^{m-1}\frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{nr}}; \quad \forall\, v \,\in\, \mathcal{R}_1. \tag{2.9}$$

Thus $\left\{\dfrac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}\right\}$ is a Cauchy sequence and it converges to a point $\Gamma_n(v) \in \mathcal{R}_2$.

Indeed, replacing v by $\mathcal{T}^\kappa w$ and divided by $\mathcal{T}^{n\kappa}$ in (2.9), we achieve that

$$\begin{aligned}\left\|\frac{\mathcal{N}_n(\mathcal{T}^{m+\kappa} v)}{\mathcal{T}^{nm+n\kappa}}-\frac{\mathcal{N}_n(\mathcal{T}^\kappa v)}{\mathcal{T}^{n\kappa}}\right\| &= \frac{1}{\mathcal{T}^{n\kappa}}\left\|\frac{\mathcal{N}_n(\mathcal{T}^m\cdot\mathcal{T}^\kappa v)}{\mathcal{T}^{nm}}-\mathcal{N}_n(\mathcal{T}^\kappa v)\right\|\\ &\leq \frac{1}{\mathcal{T}^n}\sum_{r=0}^{m-1}\frac{\mathcal{M}(\mathcal{T}^{r+\kappa} v)}{\mathcal{T}^{nr+n\kappa}}\\ &\to 0 \quad as \quad \kappa \quad \to \quad \infty \end{aligned}\tag{2.10}$$

for all $v \in \mathcal{R}_1$. Thus, we define mapping $\Gamma_n(v) : \mathcal{R}_1 \to\ \mathcal{R}_2$ such that

$$\Gamma_n(v) = \lim_{m\to\infty}\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}; \quad \forall\, v \,\in\, \mathcal{R}_1.$$

Letting limit $m \to \infty$ in (2.9) and applying the definition of $\Gamma_n(v)$, we arrive that

$$\left\|\lim_{m\to\infty}\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}-\mathcal{N}_n(v)\right\| \leq \frac{1}{\mathcal{T}^n}\sum_{r=0}^{\infty}\frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{nr}} \Rightarrow\ \|\Gamma_n(v)-\mathcal{N}_n(v)\| \leq \frac{1}{\mathcal{T}^n}\sum_{r=0}^{\infty}\frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{nr}}$$

for all $v \in \mathcal{R}_1$. Thus (2.3) holds for $t = 1$. Now, to show that $\Gamma_n(v)$ satisfies (1.7), changing v by $\mathcal{T}^m v$ and divided by $\mathcal{T}^{nm}$ in (2.1), we observe that

$$\begin{aligned}\frac{1}{\mathcal{T}^{nm}}\Big\|\mathcal{N}_{n;n+1}(\mathcal{T}^m\cdot\mathcal{T}v)-\frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v)-\mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)&\\ -\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v)+\mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\Big\| &\leq \frac{1}{\mathcal{T}^{nm}}\mathcal{M}(\mathcal{T}^m v)\end{aligned}$$

for all $v \in \mathcal{R}_1$. Approaching $m \to \infty$ and applying the definition of $\Gamma_n(v)$ and (2.2) in the overhead inequality, we identify that

$$\Gamma_n(\mathcal{T}v) = \frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v)-\Gamma_n(-v)\Big)+\frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v)+\Gamma_n(-v)\Big); \quad \forall\, v \,\in\, \mathcal{R}_1.$$

Hence $\Gamma_n(v)$ satisfies the functional equation (1.7) for all $v \in \mathcal{R}_1$. In order to prove the existence of $\Gamma_n(v)$ is unique, assume $\Gamma_B(v)$ be another n^{th} order mapping satisfying (1.7) and (2.3). Now,

$$\begin{aligned}\|\Gamma_n(v)-\Gamma_B(v)\| &= \frac{1}{\mathcal{T}^{n\kappa}}\|\Gamma_n(\mathcal{T}^\kappa v)-\Gamma_B(\mathcal{T}^\kappa v)\|\\ &= \frac{1}{\mathcal{T}^{n\kappa}}\|\Gamma_n(\mathcal{T}^\kappa v)-\mathcal{N}_n(\mathcal{T}^\kappa v)+\mathcal{N}_n(\mathcal{T}^\kappa v)-\Gamma_B(\mathcal{T}^\kappa v)\|\\ &\leq \frac{1}{\mathcal{T}^{n\kappa}}\{\|\Gamma_n(\mathcal{T}^\kappa v)-\mathcal{N}_n(\mathcal{T}^\kappa v)\|+\|\Gamma_B(\mathcal{T}^\kappa v)-\mathcal{N}_n(\mathcal{T}^\kappa v)\|\}\\ &\leq \frac{2}{\mathcal{T}^n}\sum_{r=0}^{\infty}\frac{\mathcal{M}(\mathcal{T}^{r+\kappa} v)}{\mathcal{T}^{nr+n\kappa}}\\ &\to 0 \quad as \quad \kappa \quad \to \quad \infty\end{aligned}$$

for all $v \in \mathcal{R}_1$. This proves that $\Gamma_n(v) = \Gamma_B(v)$ for all $v \in \mathcal{R}_1$. Thus $\Gamma_n(v)$ is unique. Hence the theorem holds for $t = 1$.
Further, replacing v by $\frac{v}{\mathcal{T}}$ in (2.5), we find that

$$\left\| \mathcal{N}_n(v) - \mathcal{T}^n \mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right) \right\| \leq \mathcal{M}\left(\frac{v}{\mathcal{T}}\right) \tag{2.11}$$

for all $v \in \mathcal{R}_1$. Again replacing v by $\frac{v}{\mathcal{T}}$ and multiply by $\mathcal{T}^n$ in (2.11), we notice that

$$\left\| \mathcal{T}^n \mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right) - \mathcal{T}^{2n} \mathcal{N}_n\left(\frac{v}{\mathcal{T}^2}\right) \right\| \leq \mathcal{T}^n \mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{2.12}$$

for all $v \in \mathcal{R}_1$. applying triangle inequality on (2.11) and (2.12), we obtain that

$$\left\| \mathcal{N}_n(v) - \mathcal{T}^{2n} \mathcal{N}_n\left(\frac{v}{\mathcal{T}^2}\right) \right\| \leq \mathcal{M}\left(\frac{v}{\mathcal{T}}\right) + \mathcal{T}^n \mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{2.13}$$

for all $v \in \mathcal{R}_1$. Generalizing for a positive integer m, we acquire that

$$\left\| \mathcal{N}_n(v) - \mathcal{T}^{nm} \mathcal{N}_n\left(\frac{v}{\mathcal{T}^m}\right) \right\| \leq \sum_{r=1}^{m-1} \mathcal{T}^{nr-n} \mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right) = \frac{1}{\mathcal{T}^n} \sum_{r=1}^{m-1} \mathcal{T}^{nr} \mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right) \tag{2.14}$$

for all $v \in \mathcal{R}_1$. The rest of the proof is similar ideas to that of case $t = 1$. Thus the theorem is true for $t = -1$. Hence the proof is complete. ■

The following corollary is the immediate consequence of Theorem 2.1 concerning the stabilities of (1.7).

Corollary 2.2. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an odd function fulfilling the inequality*

$$\begin{aligned} \Bigg\| \mathcal{N}_{n;n+1}(\mathcal{T}v) &- \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ &- \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big) \Bigg\| \leq \begin{cases} s; \\ s||v||^\mu;\ \mu \neq n \end{cases} \end{aligned} \tag{2.15}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\|\Gamma_n(v) - \mathcal{N}(v)\| \leq \begin{cases} \dfrac{s}{|\mathcal{T}^n - 1|}; \\ \dfrac{s||v||^\mu}{|\mathcal{T}^n - \mathcal{T}^\mu|}; \end{cases} \tag{2.16}$$

for all $v \in \mathcal{R}_1$.

2.2. Stability Results: Even Case

The proof of the following theorem and corollary is similar clues that of Theorem 2.1 and Corollary 2.2 with the help of (1.10). Hence the details of the proof are omitted.

Theorem 2.3. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an even function satisfies the inequality*

$$\begin{aligned} \Bigg\| \mathcal{N}_{n;n+1}(\mathcal{T}v) &- \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ &- \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big) \Bigg\| \leq \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1, \end{aligned} \tag{2.17}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0, \infty)$ with the condition

$$\lim_{m\to\infty} \frac{\mathcal{M}(\mathcal{T}^{mt}v)}{\mathcal{T}^{2nmt}} = 0; \quad \forall\, v \in \mathcal{R}_1. \tag{2.18}$$

Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\|\Gamma_{n+1}(v) - \mathcal{N}_{n+1}(v)\| \le \frac{1}{\mathcal{T}^{2n}} \sum_{r=\frac{1-t}{2}}^{\infty} \frac{\mathcal{M}(\mathcal{T}^{rt}v)}{\mathcal{T}^{2nrt}}; \quad \forall\, v \in \mathcal{R}_1 \tag{2.19}$$

with $t = \pm 1$. The mapping $\Gamma_{n+1}(v)$ is defined by

$$\Gamma_{n+1}(v) = \lim_{m\to\infty} \frac{\mathcal{M}(\mathcal{T}^{mt}v)}{\mathcal{T}^{2nmt}}; \quad \forall\, v \in \mathcal{R}_1. \tag{2.20}$$

Corollary 2.4. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an even function fulfilling the inequality*

$$\begin{aligned}\Bigg\| \mathcal{N}_{n;n+1}(\mathcal{T}v) &- \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ &- \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\Bigg\| \le \begin{cases} s; \\ s||v||^{\mu};\ \mu \neq 2n \end{cases}\end{aligned} \tag{2.21}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\|\Gamma_{n+1}(v) - \mathcal{N}_{n+1}(v)\| \le \begin{cases} \dfrac{s}{|\mathcal{T}^{2n} - 1|}; \\ \dfrac{s||v||^{\mu}}{|\mathcal{T}^{2n} - \mathcal{T}^{\mu}|}; \end{cases} \tag{2.22}$$

for all $v \in \mathcal{R}_1$.

2.3. Stability Results: Odd-Even Case

Theorem 2.5. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be a function satisfies the inequality*

$$\begin{aligned}\Bigg\| \mathcal{N}_{n;n+1}(\mathcal{T}v) &- \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ &- \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\Bigg\| \le \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1,\end{aligned} \tag{2.23}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0, \infty)$ satisfying the conditions (2.2) and (2.18) for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\begin{aligned}&\|\mathcal{N}_{n;n+1}(v) - \Gamma_n(v) - \Gamma_{n+1}(v)\| \\ &\le \frac{1}{2}\left\{ \frac{1}{\mathcal{T}^n} \sum_{r=\frac{1-t}{2}}^{\infty} \frac{\mathcal{M}(\mathcal{T}^{rt}v)}{\mathcal{T}^{rnt}} + \frac{\mathcal{M}(-\mathcal{T}^{rt}v)}{\mathcal{T}^{rnt}} + \frac{1}{\mathcal{T}^{2n}} \sum_{r=\frac{1-t}{2}}^{\infty} \frac{\mathcal{M}(\mathcal{T}^{rt}v)}{\mathcal{T}^{2rnt}} + \frac{\mathcal{M}(-\mathcal{T}^{rt}v)}{\mathcal{T}^{2rnt}} \right\}\end{aligned} \tag{2.24}$$

for all $v \in \mathcal{R}_1$ with $t = \pm 1$. The mappings $\Gamma_n(v)$ and $\Gamma_{n+1}(v)$ are respectively defined in (2.4) and (2.20) for all $v \in \mathcal{R}_1$.

Proof. Suppose define a function $\mathcal{N}_{odd}(v)$ by

$$\mathcal{N}_{odd}(v) = \frac{\mathcal{N}_n(v) - \mathcal{N}_n(-v)}{2}; \quad \forall \ v \ \in \ \mathcal{R}_1. \tag{2.25}$$

Then it is easy to verify from (2.25) that

$$\mathcal{N}_{odd}(0) = 0 \qquad and \qquad \mathcal{N}_{odd}(-v) = -\mathcal{N}_{odd}(v); \quad \forall \ v \ \in \ \mathcal{R}_1.$$

By Theorem 2.1 and (2.25), we notice that

$$\|\Gamma_n(v) - \mathcal{N}_{odd}(v)\| \leq \frac{1}{2\mathcal{T}^n} \sum_{r=\frac{1-t}{2}}^{\infty} \frac{\mathcal{M}(\mathcal{T}^{rt}v)}{\mathcal{T}^{rnt}} + \frac{\mathcal{M}(-\mathcal{T}^{rt}v)}{\mathcal{T}^{rnt}}; \quad \forall \ v \ \in \ \mathcal{R}_1. \tag{2.26}$$

Again define a function $\mathcal{N}_{even}(v)$ by

$$\mathcal{N}_{even}(v) = \frac{\mathcal{N}_{n+1}(v) + \mathcal{N}_{n+1}(-v)}{2}; \quad \forall \ v \ \in \ \mathcal{R}_1. \tag{2.27}$$

Then it is easy to verify from (2.27) that

$$\mathcal{N}_{even}(0) = 0 \qquad and \qquad \mathcal{N}_{even}(-v) = \mathcal{N}_{even}(v); \quad \forall \ v \ \in \ \mathcal{R}_1.$$

By Theorem 2.3 and (2.27), we notice that

$$\|\Gamma_{n+1}(v) - \mathcal{N}_{even}(v)\| \leq \frac{1}{2\mathcal{T}^{2n}} \sum_{r=\frac{1-t}{2}}^{\infty} \frac{\mathcal{M}(\mathcal{T}^{rt}v)}{\mathcal{T}^{2rnt}} + \frac{\mathcal{M}(-\mathcal{T}^{rt}v)}{\mathcal{T}^{2rnt}}; \ \forall \ v \ \in \ \mathcal{R}_1. \tag{2.28}$$

Define a function $\mathcal{N}_{n;n+1}$ by

$$\mathcal{N}_{n;n+1}(v) = \mathcal{N}_{odd}(v) + \mathcal{N}_{even}(v); \quad \forall \ v \ \in \ \mathcal{R}_1. \tag{2.29}$$

Now, it follows from (2.26), (2.28) and (2.29), we achieve our desired result. ■

Corollary 2.6. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be a function fulfilling the inequality*

$$\begin{aligned} \Bigg\| \mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ -\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\Bigg\| \leq \begin{cases} s; \\ s||v||^{\mu}; \ \mu \neq n; 2n \end{cases} \end{aligned} \tag{2.30}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\|\mathcal{N}_{n;n+1}(v) - \Gamma_n(v) - \Gamma_{n+1}(v)\| \leq \begin{cases} s\left(\dfrac{1}{|\mathcal{T}^n - 1|} + \dfrac{1}{|\mathcal{T}^{2n} - 1|}\right); \\ s||v||^{\mu}\left(\dfrac{1}{|\mathcal{T}^n - \mathcal{T}^{\mu}|} + \dfrac{1}{|\mathcal{T}^{2n} - \mathcal{T}^{\mu}|}\right); \end{cases} \tag{2.31}$$

for all $v \in \mathcal{R}_1$.

3. Stability In Modular Space of (1.7)

In this section, we investigate the generalized Ulam - Hyers stability of the functional equation (1.7) in Modular space. To prove stability results, let us take $\mathcal{R}_1$ be an linear space and $\mathcal{R}_{2\rho}$ be an $\rho-$ complete convex modular space.

3.1. Basic Concepts on Modular Spaces

Now, we introduce to adopt the usual terminologies, notations, definitions and properties of the theory of modular spaces given in [2, 22, 23, 25, 27–29, 34, 37, 38, 45, 49, 52, 55].

Definition 3.1. *Let X be a linear space over a field $K(R$ or $C)$. We say that a generalized functional $\rho : X \to [0,\infty]$ is a modular if for any $x, y \in X$,*

(MS1) $\rho(x) = 0$ *if and only if* $x = 0$;

(MS2) $\rho(\alpha x) = \rho(x)$ *for all scalar* α *with* $|\alpha| = 1$;

(MS3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ *for all scalar* $\alpha, \beta \geq 0$ *with* $\alpha + \beta = 1$.

(MS4) If (MS3) is replaced by $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ *for all scalar* $\alpha, \beta \geq 0$ *with* $\alpha + \beta = 1$*, then the functional* ρ *is called a convex modular.*

Definition 3.2. *A modular ρ defines the following vector space:*

$$X_\rho = \{x \in X : \rho(\lambda x) \ \to \ 0 \ as \ \lambda \ \to 0\},$$

and we say that X_ρ is a modular space.

Definition 3.3. *Let X_ρ be a modular space and let $\{x_n\}$ be a sequence in X_ρ then $\{x_n\}$ is $\rho-$convergent to a point $x \in X_\rho$and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.*

Definition 3.4. *Let X_ρ be a modular space and let $\{x_n\}$ be a sequence in X_ρ then $\{x_n\}$ is called $\rho-$Cauchy if for any $\epsilon > 0$one has $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in N$.*

Definition 3.5. *Let X_ρ be a modular space and let $\{x_n\}$ be a sequence in X_ρ. A subset $K \subseteq X_\rho$ is called $\rho-$complete if any $\rho-$Cauchy sequence is $\rho-$convergent to a point in K.*

Definition 3.6. *A modular space ρ has the Fatou property if and only if $\rho(x) \leq \liminf_{n \to \infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is $\rho-$convergent to x in modular space X_ρ .*

Definition 3.7. *A modularρ is said to satisfy the Δ_2-condition if there exists $k > 0$ such that $\rho(\mathcal{T}^n x) \leq k\rho(x)$ for all $x \in X_\rho$.*

Remark 3.8. *Suppose that ρ is convex and satisfies Δ_2-condition with Δ_2- constant $k > 0$. If $k < \mathcal{T}^n$, then $\rho(x) \leq k\rho(x) \leq \frac{k}{\mathcal{T}^n}\rho(x)$, which implies $\rho = 0$. Therefore, we must have the Δ_2- constant $k \geq \mathcal{T}^n$ if ρ is convex modular.*

3.2. Stability Results: Odd Case : Without Applying Δ_2 Condition

Theorem 3.9. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an odd function fulfilling the inequality*

$$\rho\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1, \tag{3.1}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0, \infty)$ *with the condition*

$$\lim_{m\to\infty} \frac{\mathcal{M}(\mathcal{T}^{nm}v)}{\mathcal{T}^{nm}} = 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.2}$$

Then there exists one and only n^{th} *order mapping* $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ *satisfying the functional equation (1.7) and*

$$\rho\ (\Gamma_n(v) - \mathcal{N}_n(v)) \leq \frac{1}{\mathcal{T}^n} \sum_{r=0}^{\infty} \frac{\mathcal{M}(\mathcal{T}^{nr}v)}{\mathcal{T}^{nr}}; \quad \forall\, v \in \mathcal{R}_1. \tag{3.3}$$

The mapping $\Gamma_n(v)$ *is defined by*

$$\lim_{m\to\infty} \rho \left(\frac{\mathcal{N}_n(\mathcal{T}^{nm}v)}{\mathcal{T}^{nm}} - \Gamma_n(v) \right) \to 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.4}$$

Proof. Using oddness of $\mathcal{N}_{n;n+1}$ in (3.1) and by (1.9), we observe that

$$\rho\ (\mathcal{N}_n(\mathcal{T}v) - \mathcal{T}^n \mathcal{N}_n(v)) \leq \mathcal{M}(v) \tag{3.5}$$

for all $v \in \mathcal{R}_1$. Without applying the Δ_2- condition it follows from (3.5), generalizing for a positive integer m, we acquire that

$$\begin{aligned} \rho\ \left(\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \mathcal{N}_n(v) \right) &= \rho\ \left(\sum_{r=0}^{m-1} \frac{1}{\mathcal{T}^{n(r+1)}} \left[\mathcal{T}^{nr} \mathcal{N}_n(\mathcal{T}^r v) - \mathcal{N}_n(\mathcal{T}^{r+1}v) \right] \right) \\ &\leq \sum_{r=0}^{m-1} \frac{1}{\mathcal{T}^{n(r+1)}} \rho\ \left(\mathcal{T}^{nr} \mathcal{N}_n(\mathcal{T}^r v) - \mathcal{N}_n(\mathcal{T}^{r+1}v) \right) \\ &\leq \frac{1}{\mathcal{T}^n} \sum_{r=0}^{m-1} \frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{nr}} \end{aligned} \tag{3.6}$$

for all $v \in \mathcal{R}_1$. Thus $\left\{ \frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} \right\}$ is a $\rho-$ Cauchy sequence in $\mathcal{R}_{2\rho}$ and $\mathcal{R}_{2\rho}$ is $\rho-$complete there exists a $\rho-$ limit function $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ given by

$$\lim_{m\to\infty} \rho \left(\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \Gamma_n(v) \right) \to 0; \quad \forall\, v \in \mathcal{R}_1.$$

Indeed, replacing v by $\mathcal{T}^\kappa w$ and divided by $\mathcal{T}^{n\kappa}$ in (3.6), we achieve that

$$\begin{aligned} \rho\ \left(\frac{\mathcal{N}_n(\mathcal{T}^{m+\kappa}v)}{\mathcal{T}^{nm+n\kappa}} - \frac{\mathcal{N}_n(\mathcal{T}^\kappa v)}{\mathcal{T}^{n\kappa}} \right) &= \frac{1}{\mathcal{T}^{n\kappa}} \rho\ \left(\frac{\mathcal{N}_n(\mathcal{T}^m \cdot \mathcal{T}^\kappa v)}{\mathcal{T}^{nm}} - \mathcal{N}_n(\mathcal{T}^\kappa v) \right) \\ &\leq \frac{1}{\mathcal{T}^n} \sum_{r=0}^{m-1} \frac{\mathcal{M}(\mathcal{T}^{r+\kappa}v)}{\mathcal{T}^{nr+n\kappa}} \\ &\to 0 \quad as \quad \kappa \quad \to \quad \infty \end{aligned} \tag{3.7}$$

for all $v \in \mathcal{R}_1$. Thus, we define mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ such that

$$\Gamma_n(v) = \lim_{m\to\infty} \frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}$$

for all $v \in \mathcal{R}_1$. It follows from the Fatou property that the inequality

$$\rho\ \left(\Gamma_n(v) - \mathcal{N}_n(v)\right) \le \liminf_{m \to \infty} \rho\ \left(\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \mathcal{N}_n(v)\right) \le \frac{1}{\mathcal{T}^n} \sum_{r=0}^{\infty} \frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{nr}}$$

for all $v \in \mathcal{R}_1$. Thus, we see (3.3) holds. Now, to show that $\Gamma_n(v)$ satisfies (1.7), changing v by $\mathcal{T}^m v$ and divided by $\mathcal{T}^{nm}$ in (3.1), we observe that

$$\begin{aligned}\rho\ \Bigg(\frac{1}{\mathcal{T}^{nm}} \Big\{\mathcal{N}_{n;n+1}(\mathcal{T}^m \cdot \mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) - \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big) \\ -\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) + \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\Big\}\Bigg) \le \frac{1}{\mathcal{T}^{nm}} \mathcal{M}(\mathcal{T}^m v)\end{aligned}$$

for all $v \in \mathcal{R}_1$. By convexity of ρ that

$$\begin{aligned}&\rho\left(\frac{1}{4}\Gamma_n(\mathcal{T}v) - \frac{1}{4}\,\frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) - \frac{1}{4}\,\frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big)\right) \\ &\le \frac{1}{4}\rho\left(\Gamma_n(\mathcal{T}v) - \frac{1}{\mathcal{T}^{nm}}\,\mathcal{N}_{n;n+1}(\mathcal{T}^m \cdot \mathcal{T}v)\right) \\ &\quad + \frac{1}{4}\rho\left(-\frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) + \frac{1}{\mathcal{T}^{nm}}\frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) - \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\right) \\ &\quad + \frac{1}{4}\rho\left(-\frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big) + \frac{1}{\mathcal{T}^{nm}}\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) + \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\right) \\ &\quad + \frac{1}{4}\rho\ \Bigg(\frac{1}{\mathcal{T}^{nm}}\Big\{\mathcal{N}_{n;n+1}(\mathcal{T}^m \cdot \mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) - \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big) \\ &\qquad\qquad -\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) + \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\Big\}\Bigg)\end{aligned}$$

for all $v \in \mathcal{R}_1$. Approaching $m \to \infty$, we notice that

$$\rho\left(\frac{1}{4}\Gamma_n(\mathcal{T}v) - \frac{1}{4}\,\frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) - \frac{1}{4}\,\frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big)\right) = 0$$

for all $v \in \mathcal{R}_1$. Hence $\Gamma_n(v)$ satisfies the functional equation (1.7) for all $v \in \mathcal{R}_1$. In order to prove the existence of $\Gamma_n(v)$ is unique, assume $\Gamma_B(v)$ be another n^{th} order mapping satisfying (1.7) and (3.3). Now,

$$\begin{aligned}\rho\ \left(\frac{1}{2}\Gamma_n(v) - \frac{1}{2}\Gamma_B(v)\right) &\le \frac{1}{2}\rho\ \left(\frac{1}{\mathcal{T}^{n\kappa}}\Gamma_n(\mathcal{T}^\kappa v) - \frac{1}{\mathcal{T}^{n\kappa}}\Gamma_B(\mathcal{T}^\kappa v)\right) \\ &\le \frac{1}{2}\frac{1}{\mathcal{T}^{n\kappa}}\rho\ \left(\Gamma_n(\mathcal{T}^\kappa v) - \mathcal{N}_n(\mathcal{T}^\kappa v) + \mathcal{N}_n(\mathcal{T}^\kappa v) - \Gamma_B(\mathcal{T}^\kappa v)\right) \\ &\le \frac{1}{2}\frac{1}{\mathcal{T}^{n\kappa}}\left\{\rho\ \left(\Gamma_n(\mathcal{T}^\kappa v) - \mathcal{N}_n(\mathcal{T}^\kappa v)\right) + \rho\ \left(\Gamma_B(\mathcal{T}^\kappa v) - \mathcal{N}_n(\mathcal{T}^\kappa v)\right)\right\} \\ &\le \frac{1}{2}\frac{2}{\mathcal{T}^n}\sum_{r=0}^{\infty}\frac{\mathcal{M}(\mathcal{T}^{r+\kappa} v)}{\mathcal{T}^{nr+n\kappa}} \\ &\to 0 \quad as \quad \kappa \quad \to \quad \infty\end{aligned}$$

for all $v \in \mathcal{R}_1$. This proves that $\Gamma_n(v) = \Gamma_B(v)$ for all $v \in \mathcal{R}_1$. Thus $\Gamma_n(v)$ is unique. This completes the proof of the theorem. ■

The following corollary is the immediate consequence of Theorem 3.9 concerning the stabilities of (1.7).

Corollary 3.10. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an odd function fulfilling the inequality*

$$\rho\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \begin{cases} s; \\ s||v||^{\mu};\ \mu < n \end{cases} \tag{3.8}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\rho\ (\Gamma_n(v) - \mathcal{N}(v)) \leq \begin{cases} \dfrac{s}{(\mathcal{T}^n - 1)}; \\ \dfrac{s||v||^{\mu}}{(\mathcal{T}^n - \mathcal{T}^{\mu})}; \end{cases} \tag{3.9}$$

for all $v \in \mathcal{R}_1$.

3.3. Stability Results: Even Case : Without Applying Δ_2 Condition

The proof of the following theorem and corollary is similar clues that of Theorem 3.9 and Corollary 3.10 with the help of (1.10). Hence the details of the proof are omitted.

Theorem 3.11. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an even function satisfies the inequality*

$$\rho\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1, \tag{3.10}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0, \infty)$ with the condition

$$\lim_{m\to\infty} \frac{\mathcal{M}(\mathcal{T}^m v)}{\mathcal{T}^{2nm}} = 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.11}$$

Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\rho\ (\Gamma_{n+1}(v) - \mathcal{N}_{n+1}(v)) \leq \frac{1}{\mathcal{T}^{2n}} \sum_{r=0}^{\infty} \frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{2nr}}; \quad \forall\, v \in \mathcal{R}_1. \tag{3.12}$$

The mapping $\Gamma_{n+1}(v)$ is defined by

$$\rho\left(\lim_{m\to\infty} \frac{\mathcal{M}(\mathcal{T}^m v)}{\mathcal{T}^{2nm}} - \Gamma_{n+1}(v)\right) \to 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.13}$$

Corollary 3.12. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an even function fulfilling the inequality*

$$\rho\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \begin{cases} s; \\ s||v||^{\mu};\ \mu < 2n \end{cases} \tag{3.14}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\rho\ (\Gamma_{n+1}(v) - \mathcal{N}_{n+1}(v)) \leq \begin{cases} \dfrac{s}{(\mathcal{T}^{2n}-1)|}; \\ \dfrac{s||v||^{\mu}}{(\mathcal{T}^{2n}-\mathcal{T}^{\mu})}; \end{cases} \tag{3.15}$$

for all $v \in \mathcal{R}_1$.

3.4. Stability Results: Odd- Even Case: Without Applying Δ_2 Condition

Theorem 3.13. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be a function satisfies the inequality*

$$\rho\ \left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1, \tag{3.16}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0,\infty)$ satisfying the conditions (3.2) and (3.11) for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\rho\ (\mathcal{N}_{n;n+1}(v) - \Gamma_n(v) - \Gamma_{n+1}(v)) \leq \frac{1}{2}\left\{\frac{1}{\mathcal{T}^n}\sum_{r=0}^{\infty}\frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{rn}} + \frac{\mathcal{M}(-\mathcal{T}^r v)}{\mathcal{T}^{rn}} + \frac{1}{\mathcal{T}^{2n}}\sum_{r=0}^{\infty}\frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{2rn}} + \frac{\mathcal{M}(-\mathcal{T}^r v)}{\mathcal{T}^{2rn}}\right\} \tag{3.17}$$

for all $v \in \mathcal{R}_1$ with $t = \pm 1$. The mappings $\Gamma_n(v)$ and $\Gamma_{n+1}(v)$ are respectively defined in (3.4) and (3.13) for all $v \in \mathcal{R}_1$.

Proof. The proof is similar lines to that of Theorem 2.5. ■

Corollary 3.14. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be a function fulfilling the inequality*

$$\rho\ \left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \begin{cases} s; \\ s||v||^{\mu};\ \mu < n; 2n \end{cases} \tag{3.18}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\rho\ (\mathcal{N}_{n;n+1}(v) - \Gamma_n(v) - \Gamma_{n+1}(v)) \leq \begin{cases} s\left(\dfrac{1}{(\mathcal{T}^n-1)} + \dfrac{1}{(\mathcal{T}^{2n}-1)}\right); \\ s||v||^{\mu}\left(\dfrac{1}{(\mathcal{T}^n-\mathcal{T}^{\mu})} + \dfrac{1}{(\mathcal{T}^{2n}-\mathcal{T}^{\mu})}\right); \end{cases} \tag{3.19}$$

for all $v \in \mathcal{R}_1$.

3.5. Stability Results: Odd Case: Applying Δ_2 Condition

Theorem 3.15. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ be an odd function fulfilling the inequality*

$$\rho\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1, \tag{3.20}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0,\infty)$ with the condition

$$\lim_{m\to\infty}\left(\frac{k^2}{\mathcal{T}^n}\right)^m \mathcal{M}\left(\frac{v}{\mathcal{T}^m}\right) = 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.21}$$

Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ satisfying the functional equation (1.7) and

$$\rho\ (\Gamma_n(v) - \mathcal{N}_n(v)) \leq \frac{1}{k}\sum_{r=1}^{\infty}\left(\frac{k^2}{\mathcal{T}^n}\right)^r \mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right); \quad \forall\, v \in \mathcal{R}_1. \tag{3.22}$$

The mapping $\Gamma_n(v)$ is defined by

$$\lim_{m\to\infty}\rho\left(\frac{\mathcal{N}_n(\mathcal{T}^{nm}v)}{\mathcal{T}^{nm}} - \Gamma_n(v)\right) \to 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.23}$$

Proof. Applying oddness of $\mathcal{N}_{n;n+1}$ in (3.20) and by (1.9), we observe that

$$\rho\ (\mathcal{N}_n(\mathcal{T}v) - \mathcal{T}^n\mathcal{N}_n(v)) \leq \mathcal{M}(v) \tag{3.24}$$

for all $v \in \mathcal{R}_1$. Further, replacing v by $\frac{v}{\mathcal{T}}$ in (3.24), we find that

$$\rho\ \left(\mathcal{N}_n(v) - \mathcal{T}^n\mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right)\right) \leq \mathcal{M}\left(\frac{v}{\mathcal{T}}\right) \tag{3.25}$$

for all $v \in \mathcal{R}_1$. Applying the Δ_2 condition it follows from (3.25) and the convexity of the modular ρ that,

$$\rho\ \left(\mathcal{N}_n(v) - \mathcal{T}^n\mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right)\right) \leq \frac{k}{\mathcal{T}^n}\,\mathcal{M}\left(\frac{v}{\mathcal{T}}\right) \tag{3.26}$$

for all $v \in \mathcal{R}_1$. Again, replacing v by $\frac{v}{\mathcal{T}}$ in (3.26), we notice that

$$\rho\ \left(\mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right) - \mathcal{T}^n\mathcal{N}_n\left(\frac{v}{\mathcal{T}^2}\right)\right) \leq \frac{k}{\mathcal{T}^n}\,\mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{3.27}$$

for all $v \in \mathcal{R}_1$. Applying the Δ_2 condition it follows from (3.27) and the convexity of the modular ρ that,

$$\rho\ \left(\mathcal{T}^n\mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right) - \mathcal{T}^{2n}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^2}\right)\right) \leq \frac{k^3}{\mathcal{T}^{2n}}\,\mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{3.28}$$

for all $v \in \mathcal{R}_1$. From (3.26) and (3.28), we obtain that

$$\rho\ \left(\mathcal{N}_n(v) - \mathcal{T}^{2n}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^2}\right)\right) \leq \frac{k}{\mathcal{T}^n}\,\mathcal{M}\left(\frac{v}{\mathcal{T}}\right) + \frac{k^3}{\mathcal{T}^{2n}}\,\mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{3.29}$$

for all $v \in \mathcal{R}_1$. Generalizing for a positive integer m, we acquire that

$$\rho\left(\mathcal{N}_n(v) - \mathcal{T}^{nm}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^m}\right)\right) \le \frac{1}{k}\sum_{r=1}^{m}\left(\frac{k^2}{\mathcal{T}^n}\right)^r \mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right) \tag{3.30}$$

for all $v \in \mathcal{R}_1$. Thus $\left\{\mathcal{T}^{nm}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^m}\right)\right\}$ is a $\rho-$ Cauchy sequence in $\mathcal{R}_{2\rho}$ and $\mathcal{R}_{2\rho}$ is $\rho-$complete there exists a $\rho-$ limit function $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ given by

$$\lim_{m\to\infty}\rho\left(\mathcal{T}^{nm}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^m}\right) - \Gamma_n(v)\right) \to 0; \quad \forall\, v \in \mathcal{R}_1.$$

Indeed, replacing v by $\mathcal{T}^{\kappa}w$ and divided by $\mathcal{T}^{n\kappa}$ in (3.29), we achieve that

$$\begin{aligned}
\rho\left(\mathcal{T}^{n\kappa}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^\kappa}\right) - \mathcal{T}^{nm}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^m}\right)\right) &\le k^{\kappa}\,\rho\left(\mathcal{N}_n\left(\frac{v}{\mathcal{T}^\kappa}\right) - \mathcal{T}^{nm-n\kappa}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^m}\right)\right)\\
&\le k^{\kappa-1}\sum_{r=1}^{m-\kappa}\left(\frac{k^2}{\mathcal{T}^n}\right)^r \mathcal{M}\left(\frac{v}{\mathcal{T}^{r+\kappa}}\right)\\
&= k^{\kappa-1}\left(\frac{\mathcal{T}^n}{k^2}\right)^{\kappa}\sum_{r=\kappa+1}^{m}\left(\frac{k^2}{\mathcal{T}^n}\right)^r \mathcal{M}\left(\frac{v}{\mathcal{T}^{r+\kappa}}\right)\\
&\to 0 \quad as \quad \kappa \to \infty
\end{aligned} \tag{3.31}$$

for all $v \in \mathcal{R}_1$. It follows from (3.30) and the Fatou property that

$$\rho\left(\Gamma_n(v) - \mathcal{N}_n(v)\right) \le \liminf_{m\to\infty}\rho\left(\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \mathcal{N}_n(v)\right) \le \frac{1}{k}\sum_{r=1}^{\infty}\left(\frac{k^2}{\mathcal{T}^n}\right)^r \mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right)$$

for all $v \in \mathcal{R}_1$. Thus, we see that (3.22) holds. The rest of the proof is similar to that of Theorem 3.9. ■

The following corollary is the immediate consequence of Theorem 3.15 concerning the stabilities of (1.7).

Corollary 3.16. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ be an odd function fulfilling the inequality*

$$\begin{aligned}
\rho\Bigg(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big)&\\
-\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\Bigg) &\le \begin{cases} s; \\ s\|v\|^{\mu};\ \mu > \log_2\frac{k^2}{\mathcal{T}^n}\end{cases}
\end{aligned} \tag{3.32}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ satisfying the functional equation (1.7) and

$$\rho\left(\Gamma_n(v) - \mathcal{N}(v)\right) \le \begin{cases} \dfrac{sk}{\mathcal{T}^n - k^2}; \\ \dfrac{sk\|v\|^{\mu}}{\mathcal{T}^{n+\mu} - k^2}; \end{cases} \tag{3.33}$$

for all $v \in \mathcal{R}_1$.

3.6. Stability Results: Even Case : Applying Δ_2 Condition

Theorem 3.17. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ be an even function satisfies the inequality*

$$\rho\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1, \tag{3.34}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0, \infty)$ with the condition

$$\lim_{m\to\infty}\left(\frac{k^2}{\mathcal{T}^{2n}}\right)^m \mathcal{M}\left(\frac{v}{\mathcal{T}^m}\right) = 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.35}$$

Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ satisfying the functional equation (1.7) and

$$\rho\ (\Gamma_n(v) - \mathcal{N}_n(v)) \leq \frac{1}{k^2}\sum_{r=1}^{\infty}\left(\frac{k^3}{\mathcal{T}^{2n}}\right)^r \mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right); \quad \forall\, v \in \mathcal{R}_1.; \quad \forall\, v \in \mathcal{R}_1. \tag{3.36}$$

The mapping $\Gamma_{n+1}(v)$ is defined by

$$\lim_{m\to\infty}\rho\left(\frac{\mathcal{N}_n(\mathcal{T}^{2nm}v)}{\mathcal{T}^{nm}} - \Gamma_n(v)\right) \to 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.37}$$

Proof. Applying even of $\mathcal{N}_{n;n+1}$ in (3.34) and by (1.10), we observe that

$$\rho\ \big(\mathcal{N}_{n+1}(\mathcal{T}v) - \mathcal{T}^{2n}\mathcal{N}_{n+1}(v)\big) \leq \mathcal{M}(v) \tag{3.38}$$

for all $v \in \mathcal{R}_1$. Further, replacing v by $\dfrac{v}{\mathcal{T}}$ in (3.38), we find that

$$\rho\ \left(\mathcal{N}_{n+1}(v) - \mathcal{T}^{2n}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}}\right)\right) \leq \mathcal{M}\left(\frac{v}{\mathcal{T}}\right) \tag{3.39}$$

for all $v \in \mathcal{R}_1$. Applying the Δ_2 condition it follows from (3.39) and the convexity of the modular ρ that,

$$\rho\ \left(\mathcal{N}_{n+1}(v) - \mathcal{T}^{2n}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}}\right)\right) \leq \frac{k}{\mathcal{T}^{2n}}\,\mathcal{M}\left(\frac{v}{\mathcal{T}}\right) \tag{3.40}$$

for all $v \in \mathcal{R}_1$. Again, replacing v by $\dfrac{v}{\mathcal{T}}$ in (3.40), we notice that

$$\rho\ \left(\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}}\right) - \mathcal{T}^{2n}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}^2}\right)\right) \leq \frac{k}{\mathcal{T}^{2n}}\,\mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{3.41}$$

for all $v \in \mathcal{R}_1$. Applying the Δ_2 condition it follows from (3.41) and the convexity of the modular ρ that,

$$\rho\ \left(\mathcal{T}^{2n}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}}\right) - \mathcal{T}^{4n}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}^2}\right)\right) \leq \frac{k^4}{\mathcal{T}^{4n}}\,\mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{3.42}$$

for all $v \in \mathcal{R}_1$. From (3.40) and (3.42), we obtain that

$$\rho\ \left(\mathcal{N}_{n+1}(v) - \mathcal{T}^{2n}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}^2}\right)\right) \leq \frac{k}{\mathcal{T}^{2n}}\,\mathcal{M}\left(\frac{v}{\mathcal{T}}\right) + \frac{k^4}{\mathcal{T}^{4n}}\,\mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{3.43}$$

for all $v \in \mathcal{R}_1$. Generalizing for a positive integer m, we acquire that

$$\rho\ \left(\mathcal{N}_{n+1}(v) - \mathcal{T}^{2nm}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}^m}\right)\right) \leq \frac{1}{k^2}\sum_{r=1}^{m}\left(\frac{k^3}{\mathcal{T}^{2n}}\right)^r \mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right) \tag{3.44}$$

for all $v \in \mathcal{R}_1$. Thus $\left\{\mathcal{T}^{2nm}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}^m}\right)\right\}$ is a $\rho-$ Cauchy sequence in $\mathcal{R}_{2\rho}$ and $\mathcal{R}_{2\rho}$ is $\rho-$complete there exists a $\rho-$ limit function $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ given by

$$\lim_{m\to\infty} \rho\left(\mathcal{T}^{2nm}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}^m}\right) - \Gamma_{n+1}(v)\right) \to 0; \quad \forall\, v \in \mathcal{R}_1.$$

The rest of the proof is similar to that of Theorem 3.15. ■

The following corollary is the immediate consequence of Theorem 3.17 concerning the stabilities of (1.7).

Corollary 3.18. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ be an even function fulfilling the inequality*

$$\rho\ \left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big)\right.$$
$$\left. - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \begin{cases} s; \\ s||v||^{\mu};\ \mu > \log_2 \frac{k^3}{\mathcal{T}^{2n}} \end{cases} \tag{3.45}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ satisfying the functional equation (1.7) and

$$\rho\ (\Gamma_{n+1}(v) - \mathcal{N}_{n+1}(v)) \leq \begin{cases} \dfrac{sk}{\mathcal{T}^{2n} - k^3}; \\ \dfrac{sk||v||^{\mu}}{\mathcal{T}^{2n+\mu} - k^3}; \end{cases} \tag{3.46}$$

for all $v \in \mathcal{R}_1$.

3.7. Stability Results: Odd-Even Case: Applying Δ_2 Condition

Theorem 3.19. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ be a function satisfies the inequality*

$$\rho\ \left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big)\right.$$
$$\left. - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1, \tag{3.47}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0,\infty)$ satisfying the conditions (3.21) and (3.35) for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ satisfying the functional equation (1.7) and

$$\rho\ (\mathcal{N}_{n;n+1}(v) - \Gamma_n(v) - \Gamma_{n+1}(v)\|$$
$$\leq \frac{1}{2}\left\{\frac{1}{k}\sum_{r=1}^{\infty}\left(\frac{k^2}{\mathcal{T}^n}\right)^r\left[\mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right) + \mathcal{M}\left(\frac{-v}{\mathcal{T}^r}\right)\right] + \frac{1}{k^2}\sum_{r=1}^{\infty}\left(\frac{k^3}{\mathcal{T}^{2n}}\right)^r\left[\mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right) + \mathcal{M}\left(\frac{-v}{\mathcal{T}^r}\right)\right]\right\} \tag{3.48}$$

for all $v \in \mathcal{R}_1$. The mappings $\Gamma_n(v)$ and $\Gamma_{n+1}(v)$ are respectively defined in (3.23) and (3.37) for all $v \in \mathcal{R}_1$.

Proof. The proof is similar lines to that of Theorem 2.5. ■

Corollary 3.20. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ be a function fulfilling the inequality*

$$\rho\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big)\right.$$
$$\left. - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \begin{cases} s; \\ s||v||^{\mu};\ \mu > \log_2 \frac{k^2}{\mathcal{T}^n}; \log_2 \frac{k^3}{\mathcal{T}^{2n}} \end{cases} \tag{3.49}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ satisfying the functional equation (1.7) and

$$\rho\ (\mathcal{N}_{n;n+1}(v) - \Gamma_n(v) - \Gamma_{n+1}(v)) \leq \begin{cases} sk\left(\dfrac{1}{\mathcal{T}^n - k^2} + \dfrac{1}{\mathcal{T}^{2n} - k^2}\right); \\ sk||v||^{\mu}\left(\dfrac{1}{\mathcal{T}^{n+\mu} - k^2} + \dfrac{1}{\mathcal{T}^{2n+\mu} - k^3}\right); \end{cases} \tag{3.50}$$

for all $v \in \mathcal{R}_1$.

4. Stability In Fuzzy Banach Space of (1.7)

In this section, we investigate the generalized Ulam - Hyers stability of the functional equation (1.7) in Fuzzy Banach space. To prove stability results, let us take $\mathcal{R}_3, (\mathcal{R}_1, F)$ and $(\mathcal{R}_2, F')$ are linear space, fuzzy normed space and fuzzy Banach space.

4.1. Definitions on Fuzzy Banach Spaces

In this section, we present the definitions and notations on fuzzy normed spaces given in [7, 30–33].

Definition 4.1. *Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*
$(FNS1)$ $N(x,c) = 0$ *for* $c \leq 0$;
$(FNS2)$ $x = 0$ *if and only if* $N(x,c) = 1$ *for all* $c > 0$;
$(FNS3)$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;
$(FNS4)$ $N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;
$(FNS5)$ $N(x,\cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $lim_{t\to\infty} N(x,t) = 1$;
$(FNS6)$ *for* $x \neq 0, N(x,\cdot)$ *is (upper semi) continuous on* $\mathbb{R}$.
The pair (X,N) is called a fuzzy normed linear space. One may regard $N(X,t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t'.

Example 4.2. *Let $(X, ||\cdot||)$ be a normed linear space. Then*

$$N(x,t) = \begin{cases} \dfrac{t}{t + \|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 4.3. *Let (X,N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim\limits_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim\limits_{n\to\infty} x_n = x$.*

Definition 4.4. *A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we obtain that $N(x_{n+p} - x_n, t) > 1 - \epsilon$.*

Definition 4.5. *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

4.2. Stability Results: Odd Case

Theorem 4.6. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an odd mapping fullfilling the inequality*

$$F\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big), c\right) \geq F'\left(\mathcal{M}(v), c\right) \tag{4.1}$$

where $\mathcal{M} : \mathcal{R}_1 \to \mathcal{R}_3$ with the conditions

$$\lim_{m\to\infty} F'\left(\mathcal{M}\left(\mathcal{T}^{nmt}v\right), \mathcal{T}^{nmt}c\right) = 1; \tag{4.2}$$

$$F'\left(\mathcal{M}\left(\mathcal{T}^{t}v\right), c\right) \geq F'\left(O^t\mathcal{M}(v), c\right); \tag{4.3}$$

with $0 < \left(\frac{c}{\mathcal{T}^n}\right)^t < 1$. Then there exists a unique n^{th} order mapping $\Gamma_n : \mathcal{R}_1 \to \mathcal{R}_2$ which satisfies (1.7) and

$$F\left(\mathcal{N}_n(v) - \Gamma_n(v), c\right) \geq F'\left(\mathcal{M}(v),\ c\left|\mathcal{T}^n - O\right|\right). \tag{4.4}$$

The mapping $\Gamma_n(v)$ is defined by

$$\lim_{m\to\infty} F\left(\Gamma_n(v) - \frac{\mathcal{N}_n(\mathcal{T}^{nmt}v)}{\mathcal{T}^{nmt}}, c\right) = 1 \tag{4.5}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$ with $t \pm 1$.

Proof. Applying oddness of $\mathcal{N}_{n;n+1}$ in (4.1) and by (1.9), we observe that

$$F\left(\mathcal{N}_n(\mathcal{T}v) - \mathcal{T}^n\,\mathcal{N}_n(v), c\right) \geq F'\left(\mathcal{M}(v), c\right); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{4.6}$$

Applying $(FNS3)$ in (4.6), we obtain that

$$F\left(\frac{1}{\mathcal{T}^n}\,\mathcal{N}_n(\mathcal{T}v) - \mathcal{N}_n(v), \frac{c}{\mathcal{T}^n}\right) \geq F'\left(\mathcal{M}(v), c\right); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{4.7}$$

Replacing v by $\mathcal{T}^m v$ in (4.7) and applying (4.3), $(FNS3)$, we find that

$$\begin{aligned} F\left(\frac{1}{\mathcal{T}^n}\,\mathcal{N}_n(\mathcal{T}^{m+1}v) - \mathcal{N}_n(\mathcal{T}^m v), \frac{c}{\mathcal{T}^n}\right) &\geq F'\left(\mathcal{M}\left(\mathcal{T}^m v\right), c\right) \\ &\geq F'\left(O^m\mathcal{M}(v), c\right) \\ &= F'\left(\mathcal{M}(v), \frac{c}{O^m}\right); \forall\ v \in \mathcal{R}_1;\ \ c > 0. \end{aligned} \tag{4.8}$$

With the help of $(FNS3)$ it follows from (4.8), that

$$F\left(\frac{1}{\mathcal{T}^{n+mn}}\,\mathcal{N}_n(\mathcal{T}^{m+1}v) - \frac{1}{\mathcal{T}^{mn}}\,\mathcal{N}_n(\mathcal{T}^m v), \frac{c}{\mathcal{T}^n \cdot \mathcal{T}^{mn}}\right) \geq F'\left(\mathcal{M}(v), \frac{c}{O^m}\right); \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{4.9}$$

Changing c by $O^m c$ in (4.9), we achieve that

$$F\left(\frac{1}{\mathcal{T}^{n+mn}}\,\mathcal{N}_n(\mathcal{T}^{m+1}v)-\frac{1}{\mathcal{T}^{mn}}\,\mathcal{N}_n(\mathcal{T}^{m}v),\frac{c}{\mathcal{T}^n}\cdot\left[\frac{O}{\mathcal{T}^n}\right]^m\right)\geq F'\left(\mathcal{M}\left(v\right),c\right);\forall\;\;v\in\mathcal{R}_1;\;\;c>0. \quad (4.10)$$

It is easy to see that

$$\frac{1}{\mathcal{T}^{mn}}\,\mathcal{N}_n(\mathcal{T}^{m}v)-\mathcal{N}_n(v)=\sum_{r=0}^{m-1}\left[\frac{1}{\mathcal{T}^{n+rn}}\,\mathcal{N}_n(\mathcal{T}^{r+1}v)-\frac{1}{\mathcal{T}^{rn}}\,\mathcal{N}_n(\mathcal{T}^{r}v)\right];\quad\forall\;\;v\in\mathcal{R}_1. \quad (4.11)$$

for all $v\in\mathcal{R}_1$. From equations (4.10) and (4.11), we obtain that

$$\begin{aligned}
&F\left(\frac{1}{\mathcal{T}^{mn}}\,\mathcal{N}_n(\mathcal{T}^{m}v)-\mathcal{N}_n(v),\frac{c}{\mathcal{T}^n}\cdot\sum_{r=0}^{m-1}\left[\frac{O}{\mathcal{T}^n}\right]^r\right)\\
&\geq \min F\left(\sum_{r=0}^{m-1}\left[\frac{1}{\mathcal{T}^{n+rn}}\,\mathcal{N}_n(\mathcal{T}^{r+1}v)-\frac{1}{\mathcal{T}^{rn}}\,\mathcal{N}_n(\mathcal{T}^{r}v)\right],\frac{c}{\mathcal{T}^n}\cdot\sum_{r=0}^{m-1}\left[\frac{O}{\mathcal{T}^n}\right]^r\right)\\
&\geq \min\bigcup_{r=0}^{m-1}\left\{F\left(\left[\frac{1}{\mathcal{T}^{n+rn}}\,\mathcal{N}_n(\mathcal{T}^{r+1}v)-\frac{1}{\mathcal{T}^{rn}}\,\mathcal{N}_n(\mathcal{T}^{r}v)\right],\frac{c}{\mathcal{T}^n}\cdot\left[\frac{O}{\mathcal{T}^n}\right]^r\right)\right\}\\
&\geq \min\bigcup_{r=0}^{m-1}\left\{F'\Big(\mathcal{M}\left(v\right),c\Big)\right\}=F'\left(\mathcal{M}\left(v\right),c\right);\forall\;\;v\in\mathcal{R}_1;\;\;c>0. \quad (4.12)
\end{aligned}$$

Replacing v by $\mathcal{T}^\kappa v$ in (4.12) and applying (4.3), $(FNS3)$, we find that

$$\begin{aligned}
&F\left(\frac{1}{\mathcal{T}^{mn+\kappa n}}\,\mathcal{N}_n(\mathcal{T}^{m+\kappa}v)-\frac{1}{\mathcal{T}^{\kappa n}}\mathcal{N}_n(\mathcal{T}^{\kappa}v),\frac{c}{\mathcal{T}^n\cdot\mathcal{T}^{\kappa n}}\sum_{r=0}^{m-1}\left[\frac{O}{\mathcal{T}^n}\right]^r\right)\\
&\geq F'\left(\mathcal{M}\left(\mathcal{T}^\kappa v\right),c\right)\geq F'\left(O^\kappa\mathcal{M}\left(v\right),c\right)=F'\left(\mathcal{M}\left(v\right),\frac{c}{O^\kappa}\right);\forall\;\;v\in\mathcal{R}_1;\;\;c>0. \quad (4.13)
\end{aligned}$$

Changing c by $O^\kappa c$ in (4.13), we achieve that

$$F\left(\frac{1}{\mathcal{T}^{mn+\kappa n}}\,\mathcal{N}_n(\mathcal{T}^{m+\kappa}v)-\frac{1}{\mathcal{T}^{\kappa n}}\mathcal{N}_n(\mathcal{T}^{\kappa}v),\frac{c}{\mathcal{T}^n}\cdot\sum_{r=0}^{m-1}\left[\frac{O}{\mathcal{T}^{n+\kappa}}\right]^r\right)\geq F'\left(\mathcal{M}\left(v\right),c\right);\forall\;\;v\in\mathcal{R}_1;\;\;c>0. \quad (4.14)$$

for all $\kappa>m\geq 0$. It follows from (4.14), we see that

$$F\left(\frac{1}{\mathcal{T}^{mn+\kappa n}}\,\mathcal{N}_n(\mathcal{T}^{m+\kappa}v)-\frac{1}{\mathcal{T}^{\kappa n}}\mathcal{N}_n(\mathcal{T}^{\kappa}v),c\right)\geq F'\left(\mathcal{M}\left(v\right),\frac{c}{\frac{1}{\mathcal{T}^n}\sum_{r=0}^{m-1}\left[\frac{O}{\mathcal{T}^{n+\kappa}}\right]^r}\right);\forall\;\;v\in\mathcal{R}_1;\;\;c>0. \quad (4.15)$$

Since $0<t<\mathcal{T}^n$ and $\sum_{r=0}^{m}\left(\frac{c}{\mathcal{T}^n}\right)^r<\infty$, the Cauchy criterion for convergence and $(FNS5)$ implies that $\left\{\frac{1}{\mathcal{T}^{mn}}\,\mathcal{N}_n(\mathcal{T}^{m}v)\right\}$ is a Cauchy sequence in $(\mathcal{R}_2,N')$ and it is complete, this sequence converges to some point $\Gamma_n\in\mathcal{R}_2$. So one can define the mapping $\Gamma_n:\mathcal{R}_1\rightarrow\;\mathcal{R}_2$ by

$$\lim_{m\to\infty}F\left(\Gamma_n(v)-\frac{1}{\mathcal{T}^{mn}}\,\mathcal{N}_n(\mathcal{T}^{m}v),c\right)=1 \quad (4.16)$$

for all $v \in \mathcal{R}_1$ and all $s > 0$. Letting $\kappa = 0$ and $m \to \infty$ in (4.15), we achieve that

$$F\ (\Gamma_n(v) - \mathcal{N}_n(v), c) \geq F'\left(\mathcal{M}\left(v\right), c \cdot (\mathcal{T}^n - O)\right); \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{4.17}$$

To prove Γ_n satisfies the (1.7), replacing v by $\mathcal{T}^m v$ in (4.3), we arrive that

$$F\ \left(\frac{1}{\mathcal{T}^{nm}} \cdot \mathcal{N}_{n;n+1}(\mathcal{T}\,\mathcal{T}^m v) - \frac{1}{\mathcal{T}^{nm}} \cdot \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) - \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\right.$$
$$\left. - \frac{1}{\mathcal{T}^{nm}} \cdot \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) + \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big), c\ \right) \geq F'\ \left(\mathcal{M}\left(\mathcal{T}^m v\right), \mathcal{T}^{nm} c\ \right); \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{4.18}$$

Now,

$$\begin{aligned} &F\ \left(\Gamma_n(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big), c\ \right) \\ &\geq min\left\{F\ \left(\Gamma_n(\mathcal{T}v) - \frac{1}{\mathcal{T}^{nm}} \cdot \mathcal{N}_{n;n+1}(\mathcal{T}\,\mathcal{T}^m v), \frac{c}{4}\right),\right. \\ &\quad F\ \left(-\frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) + \frac{1}{\mathcal{T}^{nm}} \cdot \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) - \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big), \frac{c}{4}\right), \\ &\quad F\ \left(-\frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big) + \frac{1}{\mathcal{T}^{nm}} \cdot \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) + \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big), \frac{c}{4}\right), \\ &\quad F\ \left(\frac{1}{\mathcal{T}^{nm}} \cdot \mathcal{N}_{n;n+1}(\mathcal{T}\,\mathcal{T}^m v) - \frac{1}{\mathcal{T}^{nm}} \cdot \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) - \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\right. \\ &\qquad\left.\left. - \frac{1}{\mathcal{T}^{nm}} \cdot \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) + \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big), c\ \right)\right\} \end{aligned} \tag{4.19}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Applying (4.16), (4.18), $(FNS5)$ in (4.19), we observe that

$$\begin{aligned} &F\ \left(\Gamma_n(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big), c\ \right) \\ &\qquad \geq \min\left\{1, 1, 1, F'\ \left(\mathcal{M}\left(\mathcal{T}^m v\right), \mathcal{T}^{nm} c\ \right)\right\}; \forall\ v \in \mathcal{R}_1;\ \ c > 0. \end{aligned} \tag{4.20}$$

Approaching m tends to infinity in (4.20) and applying (4.3), we achieve that

$$F\ \left(\Gamma_n(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big), c\ \right) = 1 \tag{4.21}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Applying $(FNS2)$ in (4.21) we identify that

$$\Gamma_n(\mathcal{T}v) = \frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) + \frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big); \forall\ v \in \mathcal{R}_1;\ \ c > 0.$$

for all $v \in \mathcal{R}_1$. Hence Γ_n satisfies the functional equation (1.7). To prove $\Gamma_n(v)$ is unique, let $\Gamma_n'(v)$ be another additive functional equation satisfying (1.7) and (4.5). So,

$$\begin{aligned} N(\Gamma_n(v) - \Gamma_n'(v), s) &= F\ \left(\frac{\Gamma_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \frac{\Gamma_n'(\mathcal{T}^m v)}{\mathcal{T}^{nm}}, c\right) \\ &\geq \min\left\{F\ \left(\frac{\Gamma_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \frac{\Gamma_n'(\mathcal{T}^m v)}{\mathcal{T}^{nm}}, \frac{c}{2}\right), F\ \left(\frac{\Gamma_n'(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \frac{\Gamma_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}, \frac{c}{2}\right)\right\} \\ &\geq F'\ \left(\mathcal{M}\left(\mathcal{T}^m v\right), \frac{c\left(\mathcal{T}^n - O\right)\mathcal{T}^{nm}}{2}\right) \\ &= F'\ \left(\mathcal{M}\left(v\right), \frac{c\left(\mathcal{T}^n - O\right)\mathcal{T}^{nm}}{2\,O^m}\right) \end{aligned}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Since $\lim_{m\to\infty} \frac{c\,(\mathcal{T}^n - O)\mathcal{T}^{nm}}{2\,O^m} = \infty$, it follows that $\lim_{m\to\infty} F'\left(\mathcal{M}(v), \frac{c\,(\mathcal{T}^n-0)\,\mathcal{T}^{nm}}{2\,O^m}\right) = 1$ for all $v \in \mathcal{R}_1$ and all $c > 0$. Thus

$$N(\Gamma_n(v) - \Gamma'_n(v), s) = 1$$

for all $v \in \mathcal{R}_1$ and all $c > 0$, hence $\Gamma_n(v) = \Gamma'_n(v)$. Therefore $\Gamma_n(v)$ is unique. Hence for $t = 1$ the theorem holds.

Replacing v by $\dfrac{v}{\mathcal{T}}$ in (4.6), we notice that

$$F\left(\mathcal{N}_n(v) - \mathcal{T}^n\,\mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right), c\right) \geq F'\left(\mathcal{M}\left(\frac{v}{\mathcal{T}}\right), c\right); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{4.22}$$

The rest of the proof is similar lines to that of case $t = 1$ Hence the theorem holds for the case $t = -1$. This completes the proof of the theorem. ■

The following corollary is the immediate consequence of Theorem 4.6 concerning the stabilities of (1.7).

Corollary 4.7. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an odd function fulfilling the inequality*

$$\begin{aligned} F\ \Bigg(&\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ &-\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big),\ c\Bigg) \geq \begin{cases} F'\ (s, c\,); \\ F'\ (s||v||^{\mu}, c\,); \end{cases} \end{aligned} \tag{4.23}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$F\ (\Gamma_n(v) - \mathcal{N}_n(v), c) \geq \begin{cases} F'\,(s, c\cdot|\mathcal{T}^n - 1|)\,, \\ F'\,(s\,||v||^{\mu}, c\cdot|\mathcal{T}^n - \mathcal{T}^{\mu}|)\,,\ \mu \neq n; \end{cases} \tag{4.24}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$.

4.3. Stability Results: Even Case

The proof of the following theorem and corollary is similar clues that of Theorem 4.6 and Corollary 4.7 with the help of (1.10). Hence the details of the proof are omitted.

Theorem 4.8. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an even mapping fullfilling the inequality*

$$\begin{aligned} F\ \Bigg(&\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ &-\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big), c\Bigg) \geq F'\ (\mathcal{M}\,(v)\,, c\,) \end{aligned} \tag{4.25}$$

where $\mathcal{M} : \mathcal{R}_1 \to \mathcal{R}_3$ with the conditions

$$\lim_{m\to\infty} F'\ \left(\mathcal{M}\left(\mathcal{T}^{2nmt}v\right), \mathcal{T}^{2nmt}c\,\right) = 1; \tag{4.26}$$

$$F'\ \left(\mathcal{M}\left(\mathcal{T}^t v\right), c\,\right) \geq F'\ \left(O^t\mathcal{M}\,(v)\,, c\,\right); \tag{4.27}$$

with $0 < \left(\dfrac{c}{\mathcal{T}^{2n}}\right)^t < 1$. Then there exists a unique $(n+1)^{th}$ order mapping $\Gamma_{n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ which satisfies (1.7) and

$$F\ (\mathcal{N}_n(v) - \Gamma_{n+1}(v), c) \geq F'\ \left(\mathcal{M}\,(v)\,,\ c\,|\mathcal{T}^{2n} - O|\right). \tag{4.28}$$

The mapping $\Gamma_{n+1}(v)$ is defined by

$$\lim_{m\to\infty} F\left(\Gamma_{n+1}(v) - \frac{\mathcal{N}_n(\mathcal{T}^{2nmt}v)}{\mathcal{T}^{2nmt}}, c\right) = 1 \tag{4.29}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$ with $t \pm 1$.

Corollary 4.9. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an even function fulfilling the inequality*

$$F\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big),\ c\right) \geq \begin{cases} F'\ (s, c\,); \\ F'\ (s||v||^{\mu}, c\,); \end{cases} \tag{4.30}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$F\ (\Gamma_{n+1}(v) - \mathcal{N}_n(v), c) \geq \begin{cases} F'\left(s, c\cdot|\mathcal{T}^{2n} - 1|\right), \\ F'\left(\ s\ ||v||^{\mu}, c\cdot|\mathcal{T}^{2n} - \mathcal{T}^{\mu}|\right),\ \mu \neq 2n; \end{cases} \tag{4.31}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$.

4.4. Stability Results: Odd-Even Case

Theorem 4.10. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be a function satisfies the inequality*

$$F\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big), c\right) \geq F'\ (\mathcal{M}\,(v)\,, c\,) \tag{4.32}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0, \infty)$ satisfying the conditions (4.2), (4.3), (4.26) and (4.27) with $0 < \left(\frac{c}{\mathcal{T}^{2n}}\right)^t < \left(\frac{c}{\mathcal{T}^{n}}\right)^t < 1$. Then there exists one and only n^{th} order mapping $\Gamma_n : \mathcal{R}_1 \to \mathcal{R}_2$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ which satisfies (1.7) and

$$F\ (\mathcal{N}_n(v) - \Gamma_n(v) - \Gamma_{n+1}(v), 2c) \geq \min\left\{F'\ \left(\Big(\mathcal{M}\,(v) + \mathcal{M}\,(-v)\Big),\ 2c\left(|\mathcal{T}^n - O| + |\mathcal{T}^{2n} - O|\right)\right)\right\} \tag{4.33}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$ with $t \pm 1$. The mappings $\Gamma_n(v)$ and $\Gamma_{n+1}(v)$ are respectively defined in (4.5) and (4.29).

Proof. The proof is similar lines to that of Theorem 2.5. ■

Corollary 4.11. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be a function fulfilling the inequality*

$$F\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big),\ c\right) \geq \begin{cases} F'\ (s, c\,); \\ F'\ (s||v||^{\mu}, c\,); \end{cases} \tag{4.34}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Then there exists one and only n^{th} order mapping $\Gamma_n : \mathcal{R}_1 \to \mathcal{R}_2$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ which satisfies (1.7) and

$$F\ (\Gamma_{n+1}(v) - \mathcal{N}_n(v), 2c) \geq \begin{cases} F'\left(s, c\Big(|\mathcal{T}^n - 1| + |\mathcal{T}^{2n} - 1|\Big)\right), \\ F'\left(s, c\Big(|\mathcal{T}^n - \mathcal{T}^\mu| + |\mathcal{T}^{2n} - \mathcal{T}^\mu|\Big)\right),\ \mu \neq n, 2n; \end{cases} \tag{4.35}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$.

5. Stability In Random Banach Space of (1.7)

In this section, we investigate the generalized Ulam - Hyers stability of the functional equations (1.7) in Fuzzy Banach space. To prove stability results, let us take $\mathcal{R}_1$ and $(\mathcal{R}_2, \eta, c)$ are linear space and Random Banach space.

5.1. Definitions on Random Banach Spaces

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces as in [46, 47].

From now on, D^+ is the space of distribution functions, that is, the space of all mappings

$$F : R \cup \{-\infty, \infty\} \to [0,1],$$

such that F is left continuous and nondecreasing on $R, F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of D^+ consisting of all functions $F \in D^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is,

$$l^- f(x) = \lim_{t \to x^-} f(t).$$

The space D^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for D^+ in this order is the distribution function ϵ_0 given by

$$\epsilon_0(t) = \begin{cases} 0,\ if\ t \leq 0, \\ 1,\ if\ t > 0. \end{cases} \tag{5.1}$$

Definition 5.1. *A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular norm (briefly, a continuous $t-$norm) if T satisfies the following conditions:*

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(a,1) = a$ for all $a \in [0,1]$;

(d) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Typical examples of continuous $t-$norms are $T_P(a,b) = ab, T_M(a,b) = min(a,b)$ and $T_L(a,b) = max(a+b-1, 0)$ (the Lukasiewicz $t-$norm). Recall (see [15, 16]) that if T is a $t-$norm and x_n is a given sequence of numbers in $[0,1]$, then $T_{i=1}^n x_{n+i}$ is defined recurrently by

$$T_{i=1}^1 x_i = x_1 \qquad and \qquad T_{i=1}^n x_i = T\left(T_{i=1}^{n-1} x_i, x_n\right) \qquad for \qquad n \geq 2.$$

$T_{i=n}^\infty x_i$ is defined as $T_{i=1}^\infty x_{n+i}$. It is known [16] that, for the Lukasiewicz $t-$norm, the following implication holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^\infty x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty \tag{5.2}$$

Definition 5.2. *A random normed space (briefly, RN-space) is a triple (X, η, T), where X is a vector space, T is a continuous $t-$norm and η is a mapping from X into D^+ satisfying the following conditions:*

(RBS1) $\eta_x(t) = \varepsilon_0(t)$ *for all* $t > 0$ *if and only if* $x = 0$*;*

(RBS2) $\eta_{\alpha\, x}(t) = \eta_x(t/|\alpha|)$ *for all* $x \in X$*, and* $\alpha \in \mathbb{R}$ *with* $\alpha \neq 0$*;*

(RBS3) $\eta_{x+y}(t+s) \geq T\left(\eta_x(t), \eta_y(s)\right)$ *for all* $x, y \in X$ *and* $t, s \geq 0$*.*

Example 5.3. *Every normed spaces $(X, ||\cdot||)$ defines a random normed space (X, η, T_M), where*

$$\eta_x(t) = \frac{t}{t + ||x||}$$

and T_M is the minimum $t-$norm. This space is called the induced random normed space.

Definition 5.4. *Let (X, η, T) be a RN-space.*

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\eta_{x_n - x}(\varepsilon) > 1 - \lambda$ for all $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\eta_{x_n - x_m}(\varepsilon) > 1 - \lambda$ for all $n \geq m \geq N$.

(3) A RN-space (X, η, T) is said to be complete if every Cauchy sequence in X is convergent to a point in X.

Theorem 5.5. *If (X, η, T) is a RN-space and $\{x_n\}$ is a sequence in X such that $x_n \to x$, then $\lim_{n\to\infty} \eta_{x_n}(t) = \eta_x(t)$ almost everywhere.*

To prove stability results, let us take

$$\mathcal{N}^{\mathcal{T}}_{n;n+1}(v) = \mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)$$

5.2. Stability Results: Odd Case

Theorem 5.6. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an odd function fulfilling the inequality*

$$\eta_{\mathcal{N}^{\mathcal{T}}_{n;n+1}(v)}(\, c\,) \geq \eta'_v(\, c\,); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{5.3}$$

for which there exist a function $\eta' : \mathcal{R}_1 \to D^+$ with the condition

$$\lim_{m\to\infty} T^{\infty}_{r=0}\eta'_{\mathcal{T}^{(m+r)t}v}\left(\mathcal{T}^{(m+r+1)t}c\right) = 1 = \lim_{m\to\infty} \eta'_{\mathcal{T}^{mt}v}\left(\mathcal{T}^{mt}c\,\right); \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{5.4}$$

Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\eta_{\Gamma_n(v) - \mathcal{N}_n(v)}(\, c\,) \geq T^{\infty}_{r=0}\eta'_{\mathcal{T}^{rt}v}(\mathcal{T}^{(rn+n)t}\, c\,); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{5.5}$$

with $t = \pm 1$. The mapping $\Gamma_n(v)$ is defined by

$$\eta_{\Gamma_n(v)}(\, c\,) = \lim_{m\to\infty} \eta_{\frac{\mathcal{N}_n(\mathcal{T}^{mt}v)}{\mathcal{T}^{nmt}}}(\, c\,); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{5.6}$$

Proof. Applying oddness of $\mathcal{N}_{n;n+1}$ in (5.3) and by (1.9), we observe that

$$\eta_{\mathcal{N}_n(\mathcal{T}v)-\mathcal{T}^n\mathcal{N}_n(v)}(\ c\ ) \geq \eta'_v(\ c\ );\quad \forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.7}$$

Applying $(RBS2)$ in (5.7), we obtain that

$$\eta_{\frac{\mathcal{N}_n(\mathcal{T}v)}{\mathcal{T}^n}-\mathcal{N}_n(v)}\left(\frac{c}{\mathcal{T}^n}\right) \geq \eta'_v(\ c\ );\quad \forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.8}$$

Changing c by $\mathcal{T}^n\ c$ in (5.8), we notice that

$$\eta_{\frac{\mathcal{N}_n(\mathcal{T}v)}{\mathcal{T}^n}-\mathcal{N}_n(v)}(c) \geq \eta'_v(\mathcal{T}^n\ c\ );\quad \forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.9}$$

Replacing v by $\mathcal{T}^m v$ in (5.9), we see that

$$\eta_{\frac{\mathcal{N}_n(\mathcal{T}\ \mathcal{T}^m v)}{\mathcal{T}^n}-\mathcal{N}_n(\mathcal{T}^m v)}(c) \geq \eta'_{\mathcal{T}^m v}(\mathcal{T}^n\ c\ );\quad \forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.10}$$

Applying $(RBS2)$ in (5.10), we achieve that

$$\eta_{\frac{\mathcal{N}_n(\mathcal{T}^{m+1}v)}{\mathcal{T}^{nm+n}}-\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}}\left(\frac{c}{\mathcal{T}^{nm}}\right) \geq \eta'_{\mathcal{T}^m v}(\mathcal{T}^n\ c\ );\quad \forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.11}$$

Changing c by $\mathcal{T}^m\ c$ in (5.11), we obtain that

$$\eta_{\frac{\mathcal{N}_n(\mathcal{T}^{m+1}v)}{\mathcal{T}^{nm+n}}-\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}}(\ c\ ) \geq \eta'_{\mathcal{T}^m v}(\mathcal{T}^{nm+n}\ c\ );\quad \forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.12}$$

It is easy to see that

$$\frac{1}{\mathcal{T}^{mn}}\,\mathcal{N}_n(\mathcal{T}^m v)-\mathcal{N}_n(v)=\sum_{r=0}^{m-1}\left[\frac{1}{\mathcal{T}^{n+rn}}\,\mathcal{N}_n(\mathcal{T}^{r+1}v)-\frac{1}{\mathcal{T}^{rn}}\,\mathcal{N}_n(\mathcal{T}^r v)\right];\quad \forall\ v\in\mathcal{R}_1. \tag{5.13}$$

From equations (5.12) and (5.13) and $(RBS3)$, we observe that

$$\begin{aligned}\eta_{\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}-\mathcal{N}_n(v)}(\ c\ ) &= \eta_{\sum\limits_{r=0}^{m-1}\left[\frac{1}{\mathcal{T}^{n+rn}}\ \mathcal{N}_n(\mathcal{T}^{r+1}v)-\frac{1}{\mathcal{T}^{rn}}\ \mathcal{N}_n(\mathcal{T}^r v)\right]}(\ c\ )\\ &\geq T_{r=0}^{m-1}\eta_{\left[\frac{1}{\mathcal{T}^{n+rn}}\ \mathcal{N}_n(\mathcal{T}^{r+1}v)-\frac{1}{\mathcal{T}^{rn}}\ \mathcal{N}_n(\mathcal{T}^r v)\right]}(\mathcal{T}^{rn+n}\ c\ )\\ &\geq T_{r=0}^{m-1}\eta'_{\mathcal{T}^r v}(\mathcal{T}^{rn+n}\ c\ );\quad \forall\ v\in\mathcal{R}_1;\ \ c>0.\end{aligned} \tag{5.14}$$

In order to prove the convergence of the sequence $\left\{\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^m}\right\}$, replacing v by $\mathcal{T}^\kappa v$ in (5.14) and applying $(RBS2)$, (5.4), we arrive

$$\begin{aligned}\eta_{\frac{\mathcal{N}_n(\mathcal{T}^{m+\kappa}v)}{\mathcal{T}^{nm+n\kappa}}-\frac{\mathcal{N}_n(\mathcal{T}^\kappa v)}{\mathcal{T}^{n\kappa}}}(\ c\ ) &\geq T_{r=0}^{m-1}\eta'_{\mathcal{T}^{r+\kappa}v}(\mathcal{T}^{nm+nr+n}\ c\ )\\ &\to 1 \quad as \quad m\to\infty;\quad \forall\ v\in\mathcal{R}_1;\ \ c>0.\end{aligned}$$

Thus $\left\{\frac{1}{\mathcal{T}^{nm}}\,\mathcal{N}_n(\mathcal{T}^m v)\right\}$ is a Cauchy sequence $\mathcal{R}_2$ and it is complete, this sequence converges to some point $\Gamma_n\in\mathcal{R}_2$. So one can define the mapping $\Gamma_n:\mathcal{R}_1\to\ \mathcal{R}_2$ by

$$\eta_{\Gamma_n(v)}(\ c\ )=\lim_{m\to\infty}\eta_{\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}}(\ c\ );\quad \forall\ v\in\mathcal{R}_1;\ \ c>0.$$

Letting $m\to\infty$ in (5.14), we identify that (5.5) holds for $t=1$, for all $v\in\mathcal{R}_1$ and all $c>0$. To prove that Γ_n satisfies (1.7), replacing v by $\mathcal{T}^m v$ in (5.3), we find that

$$\eta_{\frac{1}{\mathcal{T}^{nm}}\ \mathcal{N}^{\mathcal{T}}_{n;n+1}(\mathcal{T}^m v)}(\ c\ ) \geq \eta'_{\mathcal{T}^m v}\,(\mathcal{T}^{nm}\ c);\quad \forall\ v\in\mathcal{R}_1;\ \ c>0.$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Letting $n \to \infty$ in the overhead inequality and applying the definition of $\Gamma_n(v)$, we identify that Γ_n satisfies (1.7) for all $v \in \mathcal{R}_1$. To prove $\Gamma_n(v)$ is unique, let $\Gamma'_n(v)$ be another mapping satisfying (1.7) and (5.6). So,

$$\begin{aligned}\eta_{\Gamma_n(v)-\Gamma'_n(v)}(2\,c\,) &= \eta_{\Gamma_n(\mathcal{T}^m v)-\mathcal{N}_n(\mathcal{T}^m v)+\mathcal{N}_n(\mathcal{T}^m v)-\Gamma'_n(\mathcal{T}^m v)}(\mathcal{T}^{nm}\cdot 2\,c\,)\\ &\geq T\left(\eta_{\Gamma_n(\mathcal{T}^m v)-\mathcal{N}_n(\mathcal{T}^m v)}(\mathcal{T}^{nm}\,c\,), \eta_{\mathcal{N}_n(\mathcal{T}^m v)-\Gamma'_n(\mathcal{T}^m v)}(\mathcal{T}^{nm}\,c\,)\right)\\ &\geq T\left(T_{r=0}^{\infty}\eta'_{\mathcal{T}^r v}(\mathcal{T}^{rn+nm+n})\,c\,), T_{r=0}^{\infty}\eta'_{\mathcal{T}^r v}(\mathcal{T}^{rn+nm+n}\,c\,)\right)\\ &\to\ 1 \quad as \quad m\to\infty;\quad \forall\ v\in\mathcal{R}_1;\ \ c>0.\end{aligned}$$

Hence, Γ_n is unique. Hence for $t = 1$ the theorem holds.

Replacing v by $\dfrac{v}{\mathcal{T}}$ in (5.7), we notice that

$$\eta_{\mathcal{N}_n(v)-\mathcal{T}^n\mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right)}(\,c\,) \geq \eta'_v(\,c\,);\quad \forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.15}$$

The rest of the proof is similar lines to that of case $t = 1$ Hence the theorem holds for the case $t = -1$. This completes the proof of the theorem. ■

The following corollary is the immediate consequence of Theorem 5.6 concerning the stability of (1.7).

Corollary 5.7. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an odd function fulfilling the inequality*

$$\eta_{\mathcal{N}^{\mathcal{T}}_{n;n+1}(v)}(\,c\,) \geq \begin{cases}\eta'_s(\,c\,);\\ \eta'_{s||v||^\mu}(\,c\,); & \mu\neq n,\end{cases} \tag{5.16}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\eta_{\Gamma_n(v)-\mathcal{N}_n(v)}(\,c\,) \geq \begin{cases}\eta'_{|s|}\,(|\mathcal{T}^n-1|\ \,c\,)\\ \eta'_{s||v||^\mu}(|\mathcal{T}^n-\mathcal{T}^\mu|\ \,c\,)\end{cases} \tag{5.17}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$.

5.3. Stability Results: Even Case

The proof of the following theorem and corollary is similar clues that of Theorem 5.6 and Corollary 5.7 with the help of (1.10). Hence the details of the proof are omitted.

Theorem 5.8. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an even function fulfilling the inequality*

$$\eta_{\mathcal{N}^{\mathcal{T}}_{n;n+1}(v)}(\,c\,) \geq \eta'_v(\,c\,);\quad \forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.18}$$

for which there exist a function $\eta' : \mathcal{R}_1 \to D^+$ with the condition

$$\lim_{m\to\infty} T_{r=0}^{\infty}\eta'_{\mathcal{T}^{(m+r)t}v}\left(\mathcal{T}^{2(m+r+1)t}c\right) = 1 = \lim_{m\to\infty}\eta'_{\mathcal{T}^{mt}v}\left(\mathcal{T}^{2mt}c\right);\forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.19}$$

Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\eta_{\Gamma_{n+1}(v)-\mathcal{N}_n(v)}(\,c\,) \geq T_{r=0}^{\infty}\eta'_{\mathcal{T}^{rt}v}(\mathcal{T}^{2(rn+n)t}\,c\,);\quad \forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.20}$$

with $t = \pm 1$. The mapping $\Gamma_{n+1}(v)$ is defined by

$$\eta_{\Gamma_{n+1}(v)}(\,c\,) = \lim_{m\to\infty}\eta_{\frac{\mathcal{N}_n(\mathcal{T}^{mt}v)}{\mathcal{T}^{2nmt}}}(\,c\,);\quad \forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.21}$$

Corollary 5.9. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1}:\mathcal{R}_1\to\ \mathcal{R}_2$ be an even function fulfilling the inequality*

$$\eta_{\mathcal{N}^{\mathcal{T}}_{n;n+1}(v)}(\ c\ ) \geq \begin{cases} \eta'_s(\ c\ ); & \\ \eta'_{s||v||^\mu}(\ c\ ); & \mu\neq 2n, \end{cases} \tag{5.22}$$

for all $v\in\mathcal{R}_1$ and all $c>0$. Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v):\mathcal{R}_1\to\ \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\eta_{\Gamma_{n+1}(v)-\mathcal{N}_n(v)}(\ c\ ) \geq \begin{cases} \eta'_{|s|}(|\mathcal{T}^{2n}-1|\ c\ ) \\ \eta'_{s||v||^\mu}(|\mathcal{T}^{2n}-\mathcal{T}^\mu|c\ ) \end{cases} \tag{5.23}$$

for all $v\in\mathcal{R}_1$ and all $c>0$.

5.4. Stability Results: Odd - Even Case

Theorem 5.10. *Assume $\mathcal{N}_{n;n+1}:\mathcal{R}_1\to\ \mathcal{R}_2$ be a function fulfilling the inequality*

$$\eta_{\mathcal{N}^{\mathcal{T}}_{n;n+1}(v)}(\ c\ ) \geq \eta'_v(\ c\ ); \quad \forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.24}$$

for which there exist a function $\eta':\mathcal{R}_1\to D^+$ with the conditions (5.4) and (5.19) . Then there exists one and only n^{th} order mapping $\Gamma_n(v):\mathcal{R}_1\to\ \mathcal{R}_2$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v):\mathcal{R}_1\to\ \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\begin{aligned} &\eta_{\mathcal{N}_n(v)-\Gamma_n(v)-\Gamma_{n+1}(v)}(\ c\ ) \\ &\quad\geq T^3\Big(T^{\infty}_{r=0}\eta'_{\mathcal{T}^{rt}v}(\mathcal{T}^{(rn+n)t}\ c\ ), T^{\infty}_{r=0}\eta'_{\mathcal{T}^{rt}-v}(\mathcal{T}^{(rn+n)t}\ c\ ), \\ &\qquad\quad T^{\infty}_{r=0}\eta'_{\mathcal{T}^{2rt}v}(\mathcal{T}^{(rn+n)t}\ c\ ), T^{\infty}_{r=0}\eta'_{\mathcal{T}^{2rt}-v}(\mathcal{T}^{(rn+n)t}\ c\ )\Big); \forall\ v\in\mathcal{R}_1;\ \ c>0. \end{aligned} \tag{5.25}$$

with $t=\pm1$. The mappings $\Gamma_n(v)$ and $\Gamma_{n+1}(v)$ are respectively defined in (5.6) and (5.21) .

Corollary 5.11. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1}:\mathcal{R}_1\to\ \mathcal{R}_2$ be a function fulfilling the inequality*

$$\eta_{\mathcal{N}^{\mathcal{T}}_{n;n+1}(v)}(\ c\ ) \geq \begin{cases} \eta'_s(\ c\ ); & \\ \eta'_{s||v||^\mu}(\ c\ ); & \mu\neq n, 2n, \end{cases} \tag{5.26}$$

for all $v\in\mathcal{R}_1$ and all $c>0$. Then there exists one and only n^{th} order mapping $\Gamma_n(v):\mathcal{R}_1\to\ \mathcal{R}_2$ one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v):\mathcal{R}_1\to\ \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\eta_{\mathcal{N}_n(v)-\Gamma_n(v)-\Gamma_{n+1}(v)}(\ c\ ) \geq \begin{cases} \eta'_{|s|}\Big((|\mathcal{T}^n-1|+|\mathcal{T}^{2n}-1|)\cdot c\Big) \\ \eta'_{s||v||^\mu}\Big((|\mathcal{T}^n-\mathcal{T}^\mu|+|\mathcal{T}^{2n}-\mathcal{T}^\mu|)\cdot c\ \Big) \end{cases} \tag{5.27}$$

for all $v\in\mathcal{R}_1$ and all $c>0$.

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Theoretical and empirical comparison of two discrete statistical models of crash frequencies

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Abstract. In this paper, we compare two discrete statistical models for the evaluation of a road safety measure. We give a much simpler proof of the expression of the maximum likelihood estimator for the more complex model and we demonstrate theoretical results on the measure of divergence between the two models. The results obtained on real data suggest that both models are very competitive.

AMS Subject Classifications: 62F10, 62F30, 62H10, 62P99.

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Contents

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1. Introduction

Road accidents are very unfortunate events that often cause damage of all kinds, the heaviest of which is the loss of lives. According to a report published in 2018 by the World Health Organization (WHO), it can be estimated that more than 1.35 million people are killed annually by road accidents (about 2.5% of deaths worldwide) and more than 50 million people are injured annually because of road accidents [17].

As the science of data collection, analysis and interpretation, statistics play an important role in the study of road accidents. Over the years, a plethora of statistical models have been proposed for accident data and several states of the art of these models have been made. Among them are the papers [7] and [8] in which the authors present the different existing methodological approaches as well as the strengths and weaknesses of each approach. It should be noted that statistical models very often depend on the context (available data and objective of the study).

One of the important contributions of statistics in the field of road safety concerns the statistical evaluation of the effect of a road safety measure (reduction or increase of the maximum speed allowed on a road, installation of speed bumps, installation of signs, transformation of intersections into roundabouts, modification of road markings, etc.). After a certain period of application of a certain measure, it is necessary and logical to seek to assess its effect in order to ascertain whether or not that measure has made it possible to reduce the number of accidents.

In this paper, we consider the case where accidents are classified by level of severity. Among the most widely used statistical models in this context, before-after models with control site are in a good position since they allow cause and effect interpretation [5]. These models simultaneously consider accidents by level of severity before and after the road safety measure not only at the site that underwent the measure (often referred to as the experimental site or treated site or treatment site) but also at another site (often called the control site or comparison site) with the same characteristics as the treatment site but where the measure was not applied. This avoids attributing erroneously to the measure any underlying trend due to factors other than the measure [15].

Let n be the total number of accidents at the experimental site, $r \in \mathbb{N}^*$ be the number of accident severity levels and $\boldsymbol{X} = (X_{11}, \ldots, X_{1r}, X_{21}, \ldots, X_{2r}) \in \mathbb{R}^{2r}$ be a discrete random vector where, for all $j = 1, \ldots, r$, X_{ij} is the discrete random variable representing the number of accidents of severity level j on the experimental site during the period i ($i = 1$ represents the period before the measure and $i = 2$ represents the period after). Modelling also requires the non-random vector $\boldsymbol{z} = (z_1, \ldots, z_r)$, where for all $j = 1, \ldots, r$,

$$z_j = \frac{\text{number of crashes of severity level } j \text{ on the control site in the "after" period}}{\text{number of crashes of severity level } j \text{ on the control site in the "before" period}}. \tag{1.1}$$

The objective of this paper is to theoretically and empirically compare two very relevant statistical models proposed respectively in [10] (referred to as Model 1 in the following) and [11] (referred to as Model 2 in the following). These two models represent the probability distribution of the vector $\boldsymbol{X}$ by a multinomial distribution

$$\boldsymbol{X} \rightsquigarrow \mathcal{M}(n, \boldsymbol{\pi}(\boldsymbol{\theta}|\boldsymbol{z})), \tag{1.2}$$

where

$$\boldsymbol{\pi}(\boldsymbol{\theta}|\boldsymbol{z}) = (\pi_{11}(\boldsymbol{\theta}|\boldsymbol{z}), \ldots, \pi_{1r}(\boldsymbol{\theta}|\boldsymbol{z}), \pi_{21}(\boldsymbol{\theta}|\boldsymbol{z}), \ldots, \pi_{2r}(\boldsymbol{\theta}|\boldsymbol{z})) \tag{1.3}$$

is a vector function of an unknown parameter vector (to be estimated) $\boldsymbol{\theta} \in \mathbb{R}^{r+1}$ and $\boldsymbol{z}$ is such that, for all $i = 1, 2$ and $j = 1, \ldots, r$, $0 < \pi_{ij}(\boldsymbol{\theta}|\boldsymbol{z}) < 1$ and $\sum_{i=1}^{2}\sum_{j=1}^{r} \pi_{ij}(\boldsymbol{\theta}|\boldsymbol{z}) = 1$. The notation (1.2) means that for any realization $\boldsymbol{x} = (x_{11}, \ldots, x_{1r}, x_{21}, \ldots, x_{2r})$ of $\boldsymbol{X}$ such that $\sum_{i=1}^{2}\sum_{j=1}^{r} x_{ij} = n$, the probability function evaluated to the vector $\boldsymbol{x}$ is

$$P(\boldsymbol{x}) = \frac{n!}{\prod_{i=1}^{2}\prod_{j=1}^{r} x_{ij}!} \prod_{i=1}^{2}\prod_{j=1}^{r} (\pi_{ij}(\boldsymbol{\theta}|\boldsymbol{z}))^{x_{ij}}. \tag{1.4}$$

The difference between the two models considered in this paper lies in the definition of the function $\boldsymbol{\pi}(\boldsymbol{\theta}|\boldsymbol{z})$ and the theoretical and empirical comparison that we propose is an innovation because until now, in the literature, the two models have been treated separately.

In addition to the introductory section, this paper has other sections organized as follows. Section 2 presents the two models and the estimation of the parameter vector $\boldsymbol{\theta}$ for each of them. Section 3 presents the main theoretical results of this paper. In this section, we give a much simpler proof of the expression of the maximum likelihood estimator for Model 1 and we demonstrate theoretical results on the measure of divergence between the two models. Section 4 presents the results of the comparison on real data. Some concluding remarks are given in Section 5.

2. Models and estimation of their parameters

2.1. Presentation of the models

The details of the construction of both models can be found in [10, 11] and are not presented here. The main question behind these models is: how to calculate the average effect of the measure on the number of accidents? This average effect, denoted α in the rest of this paper, is a strictly positive real number defined as the ratio of the number of accidents observed at the experimental site in the "after" period to the number of accidents that one would have expected to observe if the measure had no effect [11]. Consideration of the different types of accident severity introduces r positive secondary parameters $\beta_1, \ldots, \beta_r$ such as $\sum_{i=1}^{r} \beta_i = 1$. In each of the models, the vector parameter $\boldsymbol{\theta}$ takes the form $\boldsymbol{\theta} = (\alpha, \beta_1, \ldots, \beta_r)$.

In Model 1 [10], the components of the vector function $\boldsymbol{\pi}(\boldsymbol{\theta}|\boldsymbol{z})$ are defined by:

$$\pi_{1j}^{(1)}(\boldsymbol{\theta}|\boldsymbol{z}) = \frac{\beta_j}{1 + \alpha \sum_{k=1}^{r} z_k \beta_k}, \quad \pi_{2j}^{(1)}(\boldsymbol{\theta}|\boldsymbol{z}) = \frac{\alpha \beta_j z_j}{1 + \alpha \sum_{k=1}^{r} z_k \beta_k}, \quad j = 1, \ldots, r, \tag{2.1}$$

whereas in Model 2 [11], they are defined by:

$$\pi_{1j}^{(2)}(\boldsymbol{\theta}|\boldsymbol{z}) = \frac{\beta_j}{1 + \alpha \sum_{k=1}^{r} z_k \beta_k}, \quad \pi_{2j}^{(2)}(\boldsymbol{\theta}|\boldsymbol{z}) = \frac{\alpha \beta_j \sum_{k=1}^{r} z_k \beta_k}{1 + \alpha \sum_{k=1}^{r} z_k \beta_k}, \quad j = 1, \ldots, r. \tag{2.2}$$

The value of the mean effect α can be interpreted by comparing it to 1 (for example, if $\alpha = 0.8 < 1$, then $1 - \alpha = 0.2 = 20\%$ and we can estimate the reduction in the number of accidents due to the measure at 20%). The parameters $\beta_1, \ldots, \beta_r$ are the respective probabilities associated with the severity levels.

2.2. Maximum likelihood estimation of parameters

Let $\boldsymbol{x} = (x_{11}, \ldots, x_{1r}, x_{21}, \ldots, x_{2r})$ be an observation of $\boldsymbol{X}$ such that $\sum_{i=1}^{2} \sum_{j=1}^{r} x_{ij} = n$. Applying the logarithm to Formula (1.4) and taking into account the definition of $\boldsymbol{\pi}(\boldsymbol{\theta}|\boldsymbol{z})$ for each of the two models, one can verify that the log-likelihoods associated with the two models are respectively defined by

$$L_1(\boldsymbol{\theta}) = C + \sum_{j=1}^{r} \left\{ x_{\bullet j} \log(\beta_j) + x_{2j} \log(\alpha) - x_{\bullet j} \log\left(1 + \alpha \sum_{k=1}^{r} z_k \beta_k\right) + x_{2j} \log z_j \right\} \tag{2.3}$$

and

$$L_2(\boldsymbol{\theta}) = C + \sum_{j=1}^{r} \left\{ x_{\bullet j} \log(\beta_j) + x_{2j} \log(\alpha) - x_{\bullet j} \log\left(1 + \alpha \sum_{k=1}^{r} z_k \beta_k\right) + x_{2j} \log\left(\sum_{k=1}^{r} z_k \beta_k\right) \right\}, \tag{2.4}$$

where

$$C = \log\left(\frac{n!}{\prod_{i=1}^{2} \prod_{j=1}^{r} x_{ij}!}\right).$$

Remark 2.1. *Let* $\mathbb{S}_{r-1} = \{(\beta_1, \ldots, \beta_r) \in (\mathbb{R}_+^*)^r, \quad \sum_{i=1}^r \beta_i = 1\}$*. For all* $\boldsymbol{\theta} \in \mathbb{R}_+^* \times \mathbb{S}_{r-1}$*, the difference between the two log-likelihoods*

$$L_2(\boldsymbol{\theta}) - L_1(\boldsymbol{\theta}) = \sum_{j=1}^r x_{2j} \left\{ \log\left(\sum_{k=1}^r z_k \beta_k \right) - \log z_j \right\}$$

does not have a constant sign (see Figure 1). In the special case $z_1 = \cdots = z_r$*, we have* $L_2(\boldsymbol{\theta}) - L_1(\boldsymbol{\theta}) = 0$.

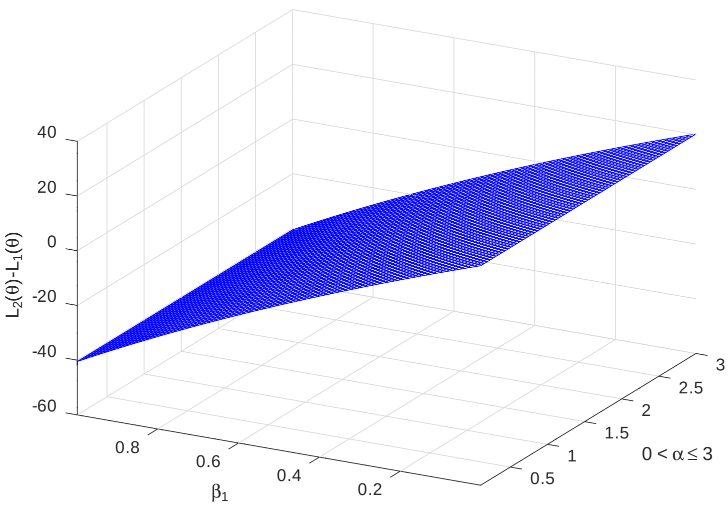

Figure 1: Example of representation of $L_2(\boldsymbol{\theta}) - L_1(\boldsymbol{\theta})$ for r =2.

The respective Maximum Likelihood Estimators (MLE) $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}_1, \ldots, \hat{\beta}_r)$ and $\tilde{\boldsymbol{\theta}} = (\tilde{\alpha}, \tilde{\beta}_1, \ldots, \tilde{\beta}_r)$ of the unknown parameter vector $\boldsymbol{\theta}$ are obtained by maximizing $L_1(\boldsymbol{\theta})$ and $L_2(\boldsymbol{\theta})$.

For Model 1, it was proved that an explicit form could not be obtained for $\boldsymbol{\theta}$ [9]. An estimation algorithm called cyclic algorithm (CA) was therefore proposed in [12]. Let $x_{1\bullet} = \sum_{j=1}^r x_{1j}$, $x_{2\bullet} = \sum_{j=1}^r x_{2j}$ and, for all $j = 1, \ldots, r$, $x_{\bullet j} = x_{1j} + x_{2j}$. The CA is written in iterative form

$$\begin{aligned} \alpha^{(k+1)} &= \frac{x_{2\bullet}}{x_{1\bullet}\left(\sum_{j=1}^r z_j \beta_j^{(k)}\right)} \\ \beta_j^{(k+1)} &= \frac{1}{1 - \dfrac{1}{n}\displaystyle\sum_{m=1}^r \frac{x_{\bullet m}\alpha^{(k+1)} z_m}{1 + \alpha^{(k+1)} z_m}} \times \frac{x_{\bullet j}}{n(1 + \alpha^{(k+1)} z_j)}, \quad j = 1, \ldots, r, \end{aligned} \tag{2.5}$$

where $\alpha^{(k+1)}$ and $\beta_j^{(k+1)}$ ($j = 1, \ldots, r$) denote the respective estimates of α and β_j after $k+1$ iterations. This algorithm begins with an initial vector $\boldsymbol{\theta}^{(0)} = (\alpha^{(0)}, \beta_1^{(0)}, \ldots, \beta_r^{(0)})$ and stops when two successive values $\boldsymbol{\theta}^{(k)} = (\alpha^{(k)}, \beta_1^{(k)}, \ldots, \beta_r^{(k)})$ and $\boldsymbol{\theta}^{(k+1)} = (\alpha^{(k+1)}, \beta_1^{(k+1)}, \ldots, \beta_r^{(k+1)})$ are such that $|L_1(\boldsymbol{\theta}^{(k+1)}) - L_1(\boldsymbol{\theta}^{(k)})| < \epsilon$ for a sufficiently small precision $\epsilon > 0$. The global convergence of the (2.5) algorithm to the MLE in the algorithmic sense (i.e. the convergence of the sequence $(\boldsymbol{\theta}^{(k)})$ to the MLE $\hat{\boldsymbol{\theta}}$ regardless of the initial vector $\boldsymbol{\theta}^{(0)}$) was demonstrated by [4] and the almost sure convergence (i.e. the strong convergence in the sense of random variables) of the EMV $\hat{\boldsymbol{\theta}}$ to the true unknown value $\boldsymbol{\theta}$ of the parameter was obtained by [3].

For Model 2, the exact analytical expression of MLE has been obtained and its almost sure convergence has been demonstrated [2] (see Lemma 2.2 below).

Lemma 2.2 (MLE in Model 2 [2])**.** *Let* $\boldsymbol{X} = (X_{11}, \ldots, X_{1r}, X_{21}, \ldots, X_{2r})$ *be a random vector with multinomial distribution defined by* (2.2) *and* $\boldsymbol{\theta} = (\alpha, \beta_1, \ldots, \beta_r)$*. The MLE* $\tilde{\boldsymbol{\theta}} = (\tilde{\alpha}, \tilde{\beta}_1, \ldots, \tilde{\beta}_r)$ *of* $\boldsymbol{\theta}$ *is defined by*

$$\begin{aligned} \tilde{\alpha} &= \frac{n \sum_{k=1}^{r} X_{2k}}{\Big(\sum_{k=1}^{r} X_{1k}\Big)\Big(\sum_{k=1}^{r} z_k (X_{1k} + X_{2k})\Big)} \\ \tilde{\beta}_j &= \frac{X_{1j} + X_{2j}}{n}, \quad j = 1, \ldots, r. \end{aligned} \tag{2.6}$$

2.3. Computation of the standard errors

Standard errors are very important in Statistics. Indeed, the MLE of α and β_j $(j = 1, \ldots, r)$ are random variables and therefore each has a standard deviation called standard error. These standard errors are given for Model 1 and Model 2 respectively by the following lemmas.

Lemma 2.3 ([14])**.** *Let* $\overline{z} = \sum_{i=1}^{r} z_i \beta_i$*,* $\gamma_n = n/(1 + \alpha \overline{z})$*,* $\tau = \gamma_n^2 \overline{z}/(n\alpha)$ *and*

$$\boldsymbol{\Gamma} = \begin{bmatrix} \tau & \frac{\gamma_n^2 z_1}{n} & \cdots & \cdots & \frac{\gamma_n^2 z_r}{n} & 0 \\ \frac{\gamma_n^2 z_1}{n} & \frac{n\gamma_n\omega_1 - \gamma_n^2\alpha^2 z_1^2}{n} & -\frac{\gamma_n^2\alpha^2 z_1 z_2}{n} & \cdots & -\frac{\gamma_n^2\alpha^2 z_1 z_r}{n} & 1 \\ \frac{\gamma_n^2 z_2}{n} & -\frac{\gamma_n^2\alpha^2 z_2 z_1}{n} & \frac{n\gamma_n\omega_2 - \gamma_n^2\alpha^2 z_2^2}{n} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -\frac{\gamma_n^2\alpha^2 z_{r-1} z_r}{n} & \vdots \\ \frac{\gamma_n^2 z_r}{n} & -\frac{\gamma_n^2\alpha^2 z_r z_1}{n} & \cdots & -\frac{\gamma_n^2\alpha^2 z_r z_{r-1}}{n} & \frac{n\gamma_n\omega_r - \gamma_n^2\alpha^2 z_r^2}{n} & 1 \\ 0 & 1 & \cdots & \cdots & 1 & 0 \end{bmatrix} \tag{2.7}$$

be a non singular matrix of order $(r+2) \times (r+2)$ *where for all* $j = 1, \ldots, r$*,* $\omega_j = (1 + \alpha z_j)/\beta_j$*. Let* $\boldsymbol{W} = (W_{i,j})_{1 \leqslant i,j \leqslant r+1}$ *be the matrix of order* $(r+1) \times (r+1)$ *composed of the first* $r+1$ *rows and* $r+1$ *columns of* $\boldsymbol{\Gamma}^{-1}$*. The approximate standard errors of* $\hat{\alpha}$ *and* $\hat{\beta}_j$ *(*$j = 1, \ldots, r$*) in Model 1 are the square roots of the diagonal elements of* $\boldsymbol{W}$*:*

$$\sigma_1(\hat{\alpha}) = \sqrt{W_{1,1}} \tag{2.8}$$

$$\sigma_1(\hat{\beta}_j) = \sqrt{W_{j+1,j+1}}, \quad j = 1, \ldots, r. \tag{2.9}$$

Lemma 2.4 ([2])**.** *Let* $\overline{z^2} = \sum_{i=1}^{r} z_i^2 \beta_i$ *and* $\gamma = 1/(1 + \alpha \overline{z})$*. The approximate standard error of* $\hat{\alpha}$ *and the exact standard error of* $\hat{\beta}_j$ *(*$j = 1, \ldots, r$*) in Model 2 are*

$$\sigma_2(\tilde{\alpha}) = \sqrt{\frac{\alpha}{n\gamma^2 \overline{z}} + \frac{\alpha^2 \overline{z^2}}{n \overline{z}^2} - \frac{\alpha^2}{n}} \tag{2.10}$$

and

$$\sigma_2(\tilde{\beta}_j) = \sqrt{\frac{\beta_j (1 - \beta_j)}{n}}, \tag{2.11}$$

respectively.

Remark 2.5. *In practice, the true values* α *and* β_j *(*$j = 1, \ldots, r$*) are unknown and replaced by their respective estimates.*

3. Main theoretical results

3.1. A much simpler proof of the expression of the MLE in Model 1

It is proven (see [10] or [13]) that the MLE of the parameter vector of Model 1 satisfies the following system of non-linear equations:

$$\begin{cases} \displaystyle\sum_{j=1}^{r}\left(x_{2j} - \frac{x_{\bullet j}\hat{\alpha}\sum_{m=1}^{r} z_m\hat{\beta}_m}{1+\hat{\alpha}\sum_{m=1}^{r} z_m\hat{\beta}_m}\right) = 0 \\ \displaystyle x_{\bullet j} - \frac{n\hat{\beta}_j(1+\hat{\alpha}z_j)}{1+\hat{\alpha}\sum_{m=1}^{r} z_m\hat{\beta}_m} = 0, \quad j = 1,\dots,r. \end{cases} \tag{3.1}$$

N'Guessan and Truffier [13] have transformed the second row of the system (3.1) into a system of r linear equations of unknowns $\beta_1, \dots, \beta_r$ whose matrix depends on the parameter α. They then proved that this matrix is invertible and then inverted it analytically using the Schur complement. This enabled them to obtain the following theorem:

Theorem 3.1 ([13])**.** *The MLE $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}_1, \dots, \hat{\beta}_r)$ of $\boldsymbol{\theta}$ in Model 1 is given by:*

$$\begin{cases} \hat{\alpha} = \dfrac{x_{2\bullet}}{x_{1\bullet}\left(\sum_{j=1}^{r} z_j\hat{\beta}_j\right)} \\ \hat{\beta}_j = \dfrac{1}{1 - \dfrac{1}{n}\displaystyle\sum_{m=1}^{r}\frac{x_{\bullet m}\hat{\alpha}z_m}{1+\hat{\alpha}z_m}} \times \dfrac{x_{\bullet j}}{n(1+\hat{\alpha}z_j)}, \quad j = 1,\dots,r. \end{cases} \tag{3.2}$$

In this paper, we give a simpler proof of the expression of the MLE in Model 1 that does not require the use of Schur complement or any other technique for analytical inversion of block-defined matrices.

Proof. Since $\sum_{j=1}^{r} x_{\bullet j} = n$, the first line of the non-linear system (3.1) is equivalent to

$$x_{2\bullet} - \frac{n\hat{\alpha}\sum_{m=1}^{r} z_m\hat{\beta}_m}{1+\hat{\alpha}\sum_{m=1}^{r} z_m\hat{\beta}_m} = 0$$

that can be rewritten as

$$x_{2\bullet} - n + \frac{n}{1+\hat{\alpha}\sum_{m=1}^{r} z_m\hat{\beta}_m} = 0$$

and then

$$\frac{n}{1+\hat{\alpha}\sum_{m=1}^{r} z_m\hat{\beta}_m} = x_{1\bullet} \tag{3.3}$$

The expression of $\hat{\alpha}$ as a function of $\hat{\beta}_j$ is a simple consequence of Equation (3.3). Replacing Equation (3.3) in the second row of the system (3.1), we get

$$x_{\bullet j} - \hat{\beta}_j(1+\hat{\alpha}z_j)x_{1\bullet} = 0, \quad j = 1,\dots,r,$$

hence

$$\hat{\beta}_j = \frac{x_{\bullet j}}{(1+\hat{\alpha}z_j)x_{1\bullet}}, \quad j = 1,\dots,r.$$

The condition $\sum_{j=1}^{r}\hat{\beta}_j = 1$ means that $\sum_{j=1}^{r}\frac{x_{\bullet j}}{(1+\hat{\alpha}z_j)} = x_{1\bullet}$, from where we have:

$$\hat{\beta}_j = \frac{1}{\displaystyle\sum_{m=1}^{r}\frac{x_{\bullet m}}{1+\hat{\alpha}z_m}} \times \frac{x_{\bullet j}}{1+\hat{\alpha}z_j}. \tag{3.4}$$

The equivalence between Equation (3.4) and the second line of (3.2) is obtained by noting that

$$
\begin{aligned}
1-\frac{1}{n}\sum_{m=1}^{r}\frac{x_{\bullet m}\hat{\alpha}z_m}{1+\hat{\alpha}z_m} &= 1-\frac{1}{n}\sum_{m=1}^{r}\left(x_{\bullet m}-\frac{x_{\bullet m}}{1+\hat{\alpha}z_m}\right)\\
&= 1-\frac{1}{n}\sum_{m=1}^{r}x_{\bullet m}+\frac{1}{n}\sum_{m=1}^{r}\frac{x_{\bullet m}}{1+\hat{\alpha}z_m}\\
&= 1-\frac{n}{n}+\frac{1}{n}\sum_{m=1}^{r}\frac{x_{\bullet m}}{1+\hat{\alpha}z_m}\\
&= \frac{1}{n}\sum_{m=1}^{r}\frac{x_{\bullet m}}{1+\hat{\alpha}z_m}.
\end{aligned}
$$

■

3.2. Measure of divergence between the two models

The notion of divergence makes it possible to quantify the "distance" between two probability distributions or to quantify the difficulty of discriminating between them. Among the most widely used divergences is the Kullback-Leibler (KL) divergence [6].

Let $m \in \mathbb{N}^*$, $\boldsymbol{p} = (p_1, \ldots, p_m) \in \mathbb{S}_{m-1}$, $\boldsymbol{q} = (q_1, \ldots, q_m) \in \mathbb{S}_{m-1}$ and

$$\mathbb{E}_n = \Big\{\boldsymbol{y} = (y_1, \ldots, y_m) \in \mathbb{N}^m, \quad y_1 + \cdots + y_m = n\Big\}.$$

Let P_n and Q_n be the probability functions associated with the multinomial distributions $\mathcal{M}(n, \boldsymbol{p})$ and $\mathcal{M}(n, \boldsymbol{q})$, respectively. That is, for any vector $\boldsymbol{y} = (y_1 \ldots, y_m) \in \mathbb{E}_n$, we have:

$$P_n(\boldsymbol{y}) = \frac{n!}{\prod_{i=1}^m y_i!}\prod_{i=1}^m p_i^{y_i} \quad \text{and} \quad Q_n(\boldsymbol{y}) = \frac{n!}{\prod_{i=1}^m y_i!}\prod_{i=1}^m q_i^{y_i}. \tag{3.5}$$

The Kullback-Leibler (KL) divergence between $\mathcal{M}(n, \boldsymbol{p})$ and $\mathcal{M}(n, \boldsymbol{q})$ is defined by

$$D_{\mathrm{KL}}\left(\mathcal{M}(n, \boldsymbol{p}) \| \mathcal{M}(n, \boldsymbol{q})\right) = \sum_{\boldsymbol{y} \in \mathbb{E}_n} P_n(\boldsymbol{y}) \log\left(\frac{P_n(\boldsymbol{y})}{Q_n(\boldsymbol{y})}\right) \tag{3.6}$$

and represents the mean information for discriminating for the distribution $\mathcal{M}(n, \boldsymbol{p})$ against $\mathcal{M}(n, \boldsymbol{q})$ when the true distribution is supposed to be $\mathcal{M}(n, \boldsymbol{p})$. The KL divergence is non-negative (greater than or equal to 0) and equals zero if and only if the two distributions are the same ($\boldsymbol{p} = \boldsymbol{q}$) but it is not symmetric [6].

We have the following result.

Theorem 3.2. *Let $m \in \mathbb{N}^*$, $\boldsymbol{p} = (p_1, \ldots, p_m) \in \mathbb{S}_{m-1}$ and $\boldsymbol{q} = (q_1, \ldots, q_m) \in \mathbb{S}_{m-1}$. The Kullback-Leibler (KL) divergence between $\mathcal{M}(n, \boldsymbol{p})$ and $\mathcal{M}(n, \boldsymbol{q})$ is*

$$D_{\mathrm{KL}}\left(\mathcal{M}(n, \boldsymbol{p}) \| \mathcal{M}(n, \boldsymbol{q})\right) = n\sum_{i=1}^m p_i \log\left(\frac{p_i}{q_i}\right). \tag{3.7}$$

Proof. We make a proof by induction. To save space, $D_{\mathrm{KL}}\left(\mathcal{M}(n, \boldsymbol{p}) \| \mathcal{M}(n, \boldsymbol{q})\right)$ will simply be denoted $d_n(\boldsymbol{p}\|\boldsymbol{q})$.

- Let $n = 1$. The elements of the set

$$\mathbb{E}_1 = \Big\{\boldsymbol{y} = (y_1, \ldots, y_m) \in \mathbb{N}^m, \quad y_1 + \cdots + y_m = 1\Big\}$$

are the m vectors $\boldsymbol{e}_{1,1} = (1, 0, 0, \ldots, 0)$, $\boldsymbol{e}_{1,2} = (0, 1, 0, \ldots, 0)$, ..., $\boldsymbol{e}_{1,m} = (0, 0, \ldots, 0, 1)$ such that, for all $i = 1, \ldots, m$, $P_1(\boldsymbol{e}_{1,i}) = p_i$ and $Q_1(\boldsymbol{e}_{1,i}) = q_i$. Thus, according to the formula (3.6),

$$d_1(\boldsymbol{p}\|\boldsymbol{q}) = \sum_{\boldsymbol{e}_{1,i}\in\mathbb{E}_1} P_1(\boldsymbol{e}_{1,i}) \log\left(\frac{P_1(\boldsymbol{e}_{1,i})}{Q_1(\boldsymbol{e}_{1,i})}\right) = \sum_{i=1}^{m} p_i \log\left(\frac{p_i}{q_i}\right) \tag{3.8}$$

and so, the formula (3.7) is true for $n = 1$.

- Let $n \in \mathbb{N}^*$. Suppose Equation (3.7) is true for n and let us show that it is true for $n + 1$. From the classical results of discrete mathematics, we know that the respective cardinals of $\mathbb{E}_n$ and $\mathbb{E}_{n+1}$ are $c_n = C^n_{m+n-1} = \frac{(m+n-1)!}{n!\,(m-1)!}$ and $c_{n+1} = C^{n+1}_{m+n} = \frac{(m+n)!}{(n+1)!\,(m-1)!}$. Note by $e_{n,1}$, ..., e_{n,c_n} and $e_{n+1,1}$, ..., $e_{n+1,c_{n+1}}$ the elements of $\mathbb{E}_n$ and $\mathbb{E}_{n+1}$, respectively. It is known that the distribution $\mathcal{M}(n + 1, \boldsymbol{p})$ is obtained as the probability distribution of the sum of two independent random variables of respective distributions $\mathcal{M}(n, \boldsymbol{p})$ and $\mathcal{M}(1, \boldsymbol{p})$. On the other hand, KL divergence is additive for independent random variables [6]. So, noting $P_{n,1}$ the joint probability function associated with the distributions $\mathcal{M}(n, \boldsymbol{p})$ and $\mathcal{M}(1, \boldsymbol{p})$ and $Q_{n,1}$ the joint probability function associated with the distributions $\mathcal{M}(n, \boldsymbol{q})$ and $\mathcal{M}(1, \boldsymbol{q})$, we have:

$$d_{n+1}(\boldsymbol{p}\|\boldsymbol{q}) = \sum_{i=1}^{c_n}\sum_{j=1}^{m} P_{n,1}(\boldsymbol{e}_{n,i}, \boldsymbol{e}_{1,j}) \log\left(\frac{P_{n,1}(\boldsymbol{e}_{n,i}, \boldsymbol{e}_{1,j})}{Q_{n,1}(\boldsymbol{e}_{n,i}, \boldsymbol{e}_{1,j})}\right)$$

and then, by independence,

$$\begin{aligned} d_{n+1}(\boldsymbol{p}\|\boldsymbol{q}) &= \sum_{i=1}^{c_n}\sum_{j=1}^{m} P_n(\boldsymbol{e}_{n,i})P_1(\boldsymbol{e}_{1,j}) \log\left(\frac{P_n(\boldsymbol{e}_{n,i})P_1(\boldsymbol{e}_{1,j})}{Q_n(\boldsymbol{e}_{n,i})Q_1(\boldsymbol{e}_{1,j})}\right) \\ &= \sum_{i=1}^{c_n}\sum_{j=1}^{m} P_n(\boldsymbol{e}_{n,i})\, p_j \log\left(\frac{P_n(\boldsymbol{e}_{n,i})\, p_j}{Q_n(\boldsymbol{e}_{n,i})\, q_j}\right). \end{aligned}$$

So, we can write

$$\begin{aligned} d_{n+1}(\boldsymbol{p}\|\boldsymbol{q}) &= \sum_{i=1}^{c_n}\sum_{j=1}^{m} P_n(\boldsymbol{e}_{n,i})\, p_j \log\left(\frac{P_n(\boldsymbol{e}_{n,i})}{Q_n(\boldsymbol{e}_{n,i})}\right) + \sum_{i=1}^{c_n}\sum_{j=1}^{m} P_n(\boldsymbol{e}_{n,i})\, p_j \log\left(\frac{p_j}{q_j}\right) \\ &= \sum_{j=1}^{m} p_j \left\{\sum_{i=1}^{c_n} P_n(\boldsymbol{e}_{n,i}) \log\left(\frac{P_n(\boldsymbol{e}_{n,i})}{Q_n(\boldsymbol{e}_{n,i})}\right)\right\} + \sum_{i=1}^{c_n} P_n(\boldsymbol{e}_{n,i}) \left\{\sum_{j=1}^{m} p_j \log\left(\frac{p_j}{q_j}\right)\right\} \\ &= \sum_{j=1}^{m} p_j\, d_n(\boldsymbol{p}\|\boldsymbol{q}) + \sum_{i=1}^{c_n} P_n(\boldsymbol{e}_{n,i})\, d_1(\boldsymbol{p}\|\boldsymbol{q}) \\ &= d_n(\boldsymbol{p}\|\boldsymbol{q}) + d_1(\boldsymbol{p}\|\boldsymbol{q}) \end{aligned}$$

because $\sum_{j=1}^{m} p_j = 1$ and $\sum_{i=1}^{c_n} P_n(\boldsymbol{e}_{n,i}) = 1$ (the total sum of a probability function is equal to 1). Combining Equation (3.8) and the assumption that Equation (3.7) is true for n, we obtain:

$$d_{n+1}(\boldsymbol{p}\|\boldsymbol{q}) = (n + 1) \sum_{i=1}^{m} p_i \log\left(\frac{p_i}{q_i}\right),$$

which completes the proof.

■

We have the following three corollaries.

Corollary 3.3. *Let* $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}_1, \ldots, \hat{\beta}_r)$ *and* $\tilde{\boldsymbol{\theta}} = (\tilde{\alpha}, \tilde{\beta}_1, \ldots, \tilde{\beta}_r)$ *be the MLE of* $\boldsymbol{\theta}$ *in Model 1 and Model 2 respectively. The KL divergences between Model 1 and Model 2 are respectively given by:*

$$D_{\mathrm{KL}}(1\|2) = n \sum_{j=1}^{r} \left\{ \pi_{1j}^{(1)}(\hat{\boldsymbol{\theta}}|\boldsymbol{z}) \log\left(\frac{\pi_{1j}^{(1)}(\hat{\boldsymbol{\theta}}|\boldsymbol{z})}{\pi_{1j}^{(2)}(\tilde{\boldsymbol{\theta}}|\boldsymbol{z})}\right) + \pi_{2j}^{(1)}(\hat{\boldsymbol{\theta}}|\boldsymbol{z}) \log\left(\frac{\pi_{2j}^{(1)}(\hat{\boldsymbol{\theta}}|\boldsymbol{z})}{\pi_{2j}^{(2)}(\tilde{\boldsymbol{\theta}}|\boldsymbol{z})}\right) \right\}$$
$$D_{\mathrm{KL}}(2\|1) = n \sum_{j=1}^{r} \left\{ \pi_{1j}^{(2)}(\tilde{\boldsymbol{\theta}}|\boldsymbol{z}) \log\left(\frac{\pi_{1j}^{(2)}(\tilde{\boldsymbol{\theta}}|\boldsymbol{z})}{\pi_{1j}^{(1)}(\hat{\boldsymbol{\theta}}|\boldsymbol{z})}\right) + \pi_{2j}^{(2)}(\tilde{\boldsymbol{\theta}}|\boldsymbol{z}) \log\left(\frac{\pi_{2j}^{(2)}(\tilde{\boldsymbol{\theta}}|\boldsymbol{z})}{\pi_{2j}^{(1)}(\hat{\boldsymbol{\theta}}|\boldsymbol{z})}\right) \right\}.$$

Proof. This is a direct application of the formula (3.7). For example, to calculate $D_{\mathrm{KL}}(1\|2)$, we apply the formula (3.7) to $\boldsymbol{p} = (\pi_{11}^{(1)}(\hat{\boldsymbol{\theta}}|\boldsymbol{z}), \ldots, \pi_{1r}^{(1)}(\hat{\boldsymbol{\theta}}|\boldsymbol{z}), \pi_{21}^{(1)}(\hat{\boldsymbol{\theta}}|\boldsymbol{z}), \ldots, \pi_{2r}^{(1)}(\hat{\boldsymbol{\theta}}|\boldsymbol{z}))$ and $\boldsymbol{q} = (\pi_{11}^{(2)}(\tilde{\boldsymbol{\theta}}|\boldsymbol{z}), \ldots, \pi_{1r}^{(2)}(\tilde{\boldsymbol{\theta}}|\boldsymbol{z}), \pi_{21}^{(2)}(\tilde{\boldsymbol{\theta}}|\boldsymbol{z}), \ldots, \pi_{2r}^{(2)}(\tilde{\boldsymbol{\theta}}|\boldsymbol{z}))$ where the $\pi_{ij}^{(1)}$'s and $\pi_{ij}^{(2)}$'s, $i = 1, 2$, $j = 1, \ldots, r$, are defined by Equations (2.1) and (2.2). ■

Corollary 3.4. $D_{\mathrm{KL}}(1\|2) \geqslant 0$, $D_{\mathrm{KL}}(2\|1) \geqslant 0$ *with equality if and only if the* z_j*'s are all equal i.e.* $z_1 = \cdots = z_r$.

Proof. The non-negativity of D_{KL} is given by Theorem 3.1 of the second chapter of [6]. According to the same theorem, the KL divergence is zero if and only if the models are the same. In our context, this means that $D_{\mathrm{KL}}(1\|2) = 0$ and $D_{\mathrm{KL}}(2\|1) = 0$ if and only if

$$\pi_{1j}^{(1)}(\hat{\boldsymbol{\theta}}|\boldsymbol{z}) = \pi_{1j}^{(2)}(\tilde{\boldsymbol{\theta}}|\boldsymbol{z}) \quad \text{and} \quad \pi_{2j}^{(1)}(\hat{\boldsymbol{\theta}}|\boldsymbol{z}) = \pi_{2j}^{(2)}(\tilde{\boldsymbol{\theta}}|\boldsymbol{z}), \quad j = 1, \ldots, r.$$

We have:

$$\begin{cases} \dfrac{\hat{\beta}_j}{1 + \hat{\alpha} \sum_{k=1}^{r} z_k \hat{\beta}_k} = \dfrac{\tilde{\beta}_j}{1 + \tilde{\alpha} \sum_{k=1}^{r} z_k \tilde{\beta}_k}, & j = 1, \ldots, r \\[2ex] \dfrac{\hat{\alpha} \hat{\beta}_j z_j}{1 + \hat{\alpha} \sum_{k=1}^{r} z_k \hat{\beta}_k} = \dfrac{\tilde{\alpha} \tilde{\beta}_j \sum_{k=1}^{r} z_k \tilde{\beta}_k}{1 + \tilde{\alpha} \sum_{k=1}^{r} z_k \tilde{\beta}_k}, & j = 1, \ldots, r. \end{cases} \tag{3.9}$$

Dividing the second line by the first, we have:

$$\hat{\alpha} z_j = \tilde{\alpha} \sum_{k=1}^{r} z_k \tilde{\beta}_k, \quad j = 1, \ldots, r.$$

In this last equality, the second member does not depend on the index j so we deduce that $z_1 = \cdots = z_r$. ■

Although interesting, the divergences $D_{\mathrm{KL}}(1\|2)$ and $D_{\mathrm{KL}}(2\|1)$ do not really allow to choose the model that fits better to the observed data. It is known that, for a vector $\boldsymbol{x} = (x_{11}, \ldots, x_{1r}, x_{21}, \ldots, x_{2r})$ such that $\sum_{i=1}^{2} \sum_{j=1}^{r} x_{ij} = n$, the observed distribution is

$$\boldsymbol{\pi}^* = \left(\frac{x_{11}}{n}, \ldots, \frac{x_{1r}}{n}, \frac{x_{21}}{n}, \ldots, \frac{x_{2r}}{n}\right) \tag{3.10}$$

so we found it more interesting to compare the observed distribution and the distributions estimated by the two models using the KL divergence. By applying Equation (3.7) to $\boldsymbol{p} = \boldsymbol{\pi}^*$ and $\boldsymbol{q} = \boldsymbol{\pi}^{(1)}(\hat{\boldsymbol{\theta}}|\boldsymbol{z})$ and $\boldsymbol{q} = \boldsymbol{\pi}^{(2)}(\tilde{\boldsymbol{\theta}}|\boldsymbol{z})$ respectively, we have the following result.

Corollary 3.5. *The KL divergences from the observed distribution to the distributions estimated by the two models are respectively given by:*

$$D_{\mathrm{KL}}(*\|1) = n\sum_{j=1}^{r}\left\{\frac{x_{1j}}{n}\log\left(\frac{x_{1j}}{n\cdot\pi_{1j}^{(1)}(\hat{\boldsymbol{\theta}}|\boldsymbol{z})}\right)+\frac{x_{2j}}{n}\log\left(\frac{x_{2j}}{n\cdot\pi_{2j}^{(1)}(\hat{\boldsymbol{\theta}}|\boldsymbol{z})}\right)\right\} \quad (3.11)$$

$$D_{\mathrm{KL}}(*\|2) = n\sum_{j=1}^{r}\left\{\frac{x_{1j}}{n}\log\left(\frac{x_{1j}}{n\cdot\pi_{1j}^{(2)}(\tilde{\boldsymbol{\theta}}|\boldsymbol{z})}\right)+\frac{x_{2j}}{n}\log\left(\frac{x_{2j}}{n\cdot\pi_{2j}^{(2)}(\tilde{\boldsymbol{\theta}}|\boldsymbol{z})}\right)\right\}. \quad (3.12)$$

The best model will then be the one with a smaller divergence.

4. Empirical comparison on real data

In this section, we compare the fit of the two models on real data. For each dataset, we estimate the parameters of both models using Algorithm (2.5) and Formula (2.6) respectively. Afterwards, we calculate the KL divergences $D_{\mathrm{KL}}(*\|1)$ and $D_{\mathrm{KL}}(*\|2)$ and other indicators among the most used for the comparison of two models which are the Akaike Information Criterion (AIC), the Corrected AIC (AICc) and the Bayesian Information Criterion (BIC) respectively defined by:

$$\mathrm{AIC} = 2k - 2\log L, \quad (4.1)$$

$$\mathrm{AICc} = \mathrm{AIC} + \frac{2k(k+1)}{n-k-1} \quad (4.2)$$

and

$$\mathrm{BIC} = -2\log L + k\log n, \quad (4.3)$$

where k is the number of parameters of the model i.e. $k = r + 1$. The best model is the one with the smallest values for all indicators.

4.1. Evaluation of an unspecified road safety measure

These data (see Table 1) are taken from [1]. Unspecified concrete measures were implemented in 2004 at an experimental site in Accra (Ghana) to improve safety. There are three crash types: Fatal, Hospitalised and Injured.

Table 1: Dataset 1: Before and after crashes data from an experimental site in Ghana

	Before period (3 years)			After period (3 years)		
	Fatal	Hospitalised	Injured	Fatal	Hospitalised	Injured
Experimental site	8	23	23	3	6	16
Control site	33	58	69	27	36	62

Dividing for each type of accident, the number of accidents after by the number of accidents before in the control area, one can obtain the control coefficients in Table 2.

Table 2: Control coefficients related to Dataset 1

z_1 (Fatal)	z_2 (Hospitalised)	z_3 (Injured)
0.8182	0.6207	0.8986

The results of the comparison of the two models are given by Table 3.

According to the results in Table 3, Model 1 is the best because it has smaller values for AIC, AICc, BIC and D_{KL}. According to this model, we have $\hat{\alpha} = 0.5946$ and we can estimate at $(1 - 0.5946) = 0.4054 = 40.54\%$ the reduction in the number of accidents after application of the measure.

Table 3: Models comparison results for Dataset 1 (Standard errors are in parentheses).

	Model 1	Model 2
$\hat{\alpha}$	0.5946 (0.1443)	0.5895 (0.1430)
$\hat{\beta}_1$	0.1370 (0.0386)	0.1392 (0.0390)
$\hat{\beta}_2$	0.3923 (0.0588)	0.3671 (0.0542)
$\hat{\beta}_3$	0.4707 (0.0560)	0.4937 (0.0562)
AIC	261.2306	263.1768
AICc	261.7712	263.7173
BIC	270.7084	272.6546
D_{KL}	$D_{\text{KL}}(*\|1) = 0.7050$	$D_{\text{KL}}(*\|2) = 1.6781$

4.2. Evaluation of the modification of ground markings

The data (see Table 4) come from [13]. A road modification was carried out in 1999 on national road 17 (RN17) in France. It consisted of the modification of ground markings of this three-lane two-way road so that it is impossible to overtake simultaneously in both directions. Accidents are classified into three categories: Fatal, Serious and Minor.

Table 4: Dataset 2

	Before period (4 years)			After period (4 years)		
	Fatal	Serious	Minor	Fatal	Serious	Minor
Treated site	4	4	16	1	1	7
Control site	27	64	182	14	27	102

Table 5: Control coefficients related to Dataset 2.

z_1	z_2	z_3
0.5190	0.4220	0.5600

The results of the comparison of the two models are given by Table 6.

Table 6: Models comparison results for Dataset 2 (Standard errors are in parentheses).

	Model 1	Model 2
$\hat{\alpha}$	0.7054 (0.2760)	0.7037 (0.2753)
$\hat{\beta}_1$	0.1525 (0.0632)	0.1515 (0.0624)
$\hat{\beta}_2$	0.1605 (0.0664)	0.1515 (0.0624)
$\hat{\beta}_3$	0.6870 (0.0854)	0.6970 (0.0800)
AIC	100.8238	101.0209
AICc	102.2524	102.4495
BIC	106.8098	107.0069
D_{KL}	$D_{\text{KL}}(*\|1) = 0.1003$	$D_{\text{KL}}(*\|2) = 0.1988$

According to the results of Table 6, Model 1 is the best because it has smaller values for the AIC, AICc, BIC and D_{KL}. According to this model, we have $\hat{\alpha} = 0.7054$ and we can estimate at $(1-0.7054) = 0.2946 = 29.46\%$ the reduction in the number of accidents after application of the measure.

4.3. Evaluation of the presence of display panels on roadsides

The data (see Table 7) come from [14]. The experimental site is the Turcot Interchange (Canada). The modification consisted of the installation of roadside billboards in July 1995. The control site consists of two interchanges. There are three categories of accidents: Fatal or severe, Minor and Property damages only (PDO).

Table 7: Dataset 3

	Before period (1 year)			Period after (1 year)		
	Fatal or Severe	Minor	PDO	Fatal or Severe	Minor	PDO
Experimental site	4	20	133	3	29	143
Control site	2	26	154	9	37	239

The control coefficients are given by Table 8.

Table 8: Control coefficients related to Dataset 3.

z_1	z_2	z_3
4.5	1.423	1.552

The results of the comparison of the two models are given by Table 9.

Table 9: Models comparison results for Dataset 3 (Standard errors sont entre parenthèses).

	Model 1	Model 2
$\hat{\alpha}$	0.7130 (0.0786)	0.6988 (0.0775)
$\hat{\beta}_1$	0.0106 (0.0040)	0.0211 (0.0079)
$\hat{\beta}_2$	0.1549 (0.0207)	0.1476 (0.0195)
$\hat{\beta}_3$	0.8345 (0.0283)	0.8313 (0.0206)
AIC	814.7443	810.7806
AICc	814.8666	810.9029
BIC	829.9649	826.0011
D_{KL}	$D_{\mathrm{KL}}(*\|1) = 2.5778$	$D_{\mathrm{KL}}(*\|2) = 0.5959$

According to the results of Table 9, Model 2 is the best because it has smaller values for AIC, AICc, BIC and D_{KL}. According to this model, we have $\hat{\alpha} = 0.6988$ and $(1 - 0.6988) = 0.3012 = 30.12\%$ so we can estimate that the measure led to a decrease of 30.12% in the number of accidents.

4.4. Evaluation of the increase in speed limit on Arizona's rural interstate

Data are extracted from [16]. There are three severity levels for accidents: Fatal, Injury and property-damage-only (PDO). The speed limit on Arizona's rural interstate was raised to 65 mph in 1987. The treatment site (rural interstate) represents the portions of the Arizona interstate system that had the speed limit raised to 65 mph and the control site (urban interstate) represents the portions that have the speed limit maintained at 55 mph. In [16], the period before covers about four years and the period after covers one year. In order to respect the basic principle of model construction which requires the periods before and after to have approximately the same duration, we have chosen just one year before the measure and one year after the measure. This gives the following table:

Table 10: Dataset 4 (Arizona)

	Before period (1 year)			After period (1 year)		
	PDO	Injury	Fatal	PDO	Injury	Fatal
Treatment site	1669	1047	97	1969	1322	117
Control site	2105	803	13	2217	737	15

Table 11: Control coefficients related to Dataset 4.

z_1 (PDO)	z_2 (Injury)	z_3 (Fatal)
1.0532	0.9178	1.1538

The control coefficients related to Dataset 4 are given by Table 11.

The results of the comparison of the two models are given by Table 12.

Table 12: Models comparison results for Dataset 4 (Standard errors are in parentheses).

	Model 1	Model 2
$\hat{\alpha}$	1.2087 (0.0308)	1.2054 (0.0307)
$\hat{\beta}_1$	0.5690 (0.0069)	0.5848 (0.0062)
$\hat{\beta}_2$	0.3993 (0.0068)	0.3808 (0.0062)
$\hat{\beta}_3$	0.0318 (0.0021)	0.0344 (0.0023)
AIC	18509.8110	18495.2896
AICc	18509.8174	18495.2960
BIC	18536.7537	18522.2323
D_{KL}	$D_{\mathrm{KL}}(*\|1) = 8.0800$	$D_{\mathrm{KL}}(*\|2) = 0.8193$

According to the results of Table 12, Model 2 is the best because it has smaller values for the AIC, AICc, BIC, and D_{KL}. According to this model, we have $\hat{\alpha} = 1.2054$ and $(1 - 1.2054) = -0.2054 = -20.54\%$ so we can estimate that the measure has led to an increase of 20.54% in the number of accidents.

5. Conclusion

In this paper, we compared two discrete statistical models for the evaluation of a road safety measure applied on an experimental site (treatment site) where the accidents are classified by severity in r categories. In order to take into account the effects external to the measure and which could influence the number of accidents, the treatment site is associated with a control site where the measure was not applied. The two models are multinomial models coming respectively from [10] (Model 1) and [11] (Model 2).

There certainly exist results on the maximum likelihood estimator (MLE) of the parameter vector for each of the models. But for Model 1 (considered in this work as the most complex given the form of the MLE), previous works have used the notion of Schur complement for the exact analytical inversion of a matrix involved in the resolution of likelihood equations. In our work, we have given a much simpler proof of the expression of the MLE for Model 1 without using neither Schur complement nor any other technique of exact analytical inversion of block matrices. We then obtained theoretical results on the measure of divergence between the two models. The results obtained on real data suggest that both models are competitive and that none of them is systematically better than the other.

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Bounds on the covering radius of repetition code in $\mathbb{Z}_2\mathbb{Z}_6$

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Abstract. In this paper, the covering radius of codes over $R = \mathbb{Z}_2\mathbb{Z}_6$ with different weight are discussed. The block repetition codes over R is defined and the covering radius for block repetition codes R are obtained.

AMS Subject Classifications: 16P10,11T71, 94B05, 11H71, 94B65.

Keywords: Finite ring, Additive codes, Covering radius, Different weight.

Contents

1. Introduction and Background

Codes over finite commutative rings have been studied for almost 50 years. The main motivation of studying codes over rings is that they can be associated with codes over finite fields through the Gray map. Recently, coding theory over finite commutative non-chain rings is a hot research topic. Recently, there has been substantial interest in the class of additive codes. In [11, 12], Delsarte contributes to the algebraic theory of association scheme where the main idea is to characterize the subgroups of the underlying abelian group in a given association scheme.

The covering radius is an important geometric parameter of codes. It not only indicates the maximum error correcting capability of codes, but also relates to some practical problems such as the data compression and transmission. Studying of the covering radius of codes has attracted many coding scientists for almost 30 years. The covering radius of linear codes over binary finite fields was studied in [9].

Additive codes over $\mathbb{Z}_2\mathbb{Z}_4$ have been extensively studied in [2, 4–6]. In [7], the authors, in particular, gave lower and upper bounds on the covering radius of codes over the ring $\mathbb{Z}_6$ with respect to different distance. Using above results motivate us to work in this Paper.

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2. Preliminaries

In $\mathbb{Z}_2$ and $\mathbb{Z}_6$ be the rings of integers modulo 2 and 6. Let $\mathbb{Z}_2^n$ and $\mathbb{Z}_6^n$ denote the space of n-tuples over these rings. A ring $R = \mathbb{Z}_2\mathbb{Z}_6 = \{00, 01, 02, 03, 04, 05, 10, 11, 12, 13, 14, 15\}$, with integer modulo is 2 and 6. If C be a non-empty subset of $\mathbb{Z}_2^n$ is called a *code* and if that subcode is a linear space, then C is said to be *linear code*. Similarly, any non-empty subset C of $\mathbb{Z}_6^n$ is called a *linear senary code*.

In this section, some preliminary results are given [4, 6, 15]. A non-empty set C is a R-additive code if it is a subgroup of $\mathbb{Z}_2^\gamma \times \mathbb{Z}_6{}^\delta$. In this case, C is also isomorphic to an abelian structure $\mathbb{Z}_2^\lambda \times \mathbb{Z}_6^\mu$ for some λ and μ and type of C is a $2^\lambda 6^\mu$ as a group. It pursue that it has $\mid C \mid = 2^{\lambda+2\mu}$ codewords and the number of order for two codewords in C is $\mid C \mid = 2^{\lambda+\mu}$.

A linear code C of length n over $\mathbb{Z}_6$ is an additive subgroup of $\mathbb{Z}_6^n$. An element of C is called a *codeword* of C and a generator matrix of C is a matrix whose rows generate C. The Hamming weight $w_H(x)$ of a vector $x \in \mathbb{Z}_6^n$ is the number of non-zero components. The Lee weight $w_L(x)$ of a vector $x = (x_1, x_2, \cdots, x_n)$ is

$$w_L(x_i) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x = 1, 5, \\ 2 & \text{if } x = 2, 4, \\ 3 & \text{if } x = 3. \end{cases}$$

The Euclidean weight $w_E(x)$ of a vector x is

$$w_E(x_i) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x = 1, 5, \\ 4 & \text{if } x = 2, 4, \\ 9 & \text{if } x = 3. \end{cases}$$

The Chinese Euclidean weight $w_E(x)$ of a vector x is

$$w_{CE}(x_i) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x = 1, 5, \\ 3 & \text{if } x = 2, 4, \\ 4 & \text{if } x = 3. \end{cases}$$

The Hamming, Lee, Euclidean, Chinese Euclidean distances $d_H(x,y), d_L(x,y)$, $d_E(x,y)$ and $d_{CE}(x,y)$ between two vectors x and y are $w_H(x-y), w_L(x-y), w_E(x-y)$ and $w_{CE}(x-y)$ respectively. The minimum Hamming, Lee, Euclidean and Chinese Euclidean weights, d_H, d_L, d_E and d_{CE} of C are the smallest Hamming, Lee, Euclidean and Chinese Euclidean weights among all non-zero codewords of C respectively.

The Gray map: $\mu : \mathbb{Z}_6 \to \mathbb{Z}_2\mathbb{Z}_3$ is defined as $\mu(0) = (00), \mu(1) = (11), \mu(2) = (02), \mu(3) = (10), \mu(4) = (01), \mu(5) = (12)$ and the extension of the Gray map
$\rho : \mathbb{Z}_2^\gamma \times \mathbb{Z}_6^\delta \to \mathbb{Z}_2^n\mathbb{Z}_3$, where $n = \gamma + \delta$ is given by

$$\rho(u, w) = (u, \mu(w_1), \cdots, \mu(w_\delta)), \forall u \in \mathbb{Z}_2^\gamma \text{ and } (w_1, \cdots, w_\delta) \in \mathbb{Z}_6^\delta.$$

Then the binary image of a R-additive code under the extended Gray map is called a R-*linear code* of length $n = \gamma + \delta$.

The Hamming weight of u denoted by $w_H(u)$ and $w_L(w), w_E(w), w_{CE}(w)$ be the Lee, Euclidean and Chinese Euclidean weights of w respectively, where $u \in \mathbb{Z}_2^{\gamma}$ and $w \in \mathbb{Z}_6^{\delta}$.

In Lee, Euclidean and Chinese Euclidean weights are x is defined as $w_D(x) = w_H(u) + w_D(w)$, where $D = \{$Lee(L), Euclidean(E), Chinese Euclidean(CE)$\}$, and $x = (u, w) \in \mathbb{Z}_2^{\gamma} \times \mathbb{Z}_6^{\delta}$, and $u = (u_1, \cdots, u_{\gamma}) \in \mathbb{Z}_2^{\gamma}$ and $w = (w_1, \cdots, w_{\delta}) \in \mathbb{Z}_6^{\delta}$. The Gray map defined above is an isometry which transforms the Lee distance defined over $\mathbb{Z}_2^{\gamma} \times \mathbb{Z}_6^{\delta}$ to the Hamming distance defined over $\mathbb{Z}_2^{n}$, with $n = \gamma + \delta$.

3. The covering radius of code and the block repetition codes over R

The covering radius of a code C is the smallest number r such that the spheres of radius r around the codewords cover $\mathbb{Z}_2^{\gamma} \times \mathbb{Z}_6^{\delta} = R$ and thus the covering radius of a code C over R with respect to the different distance, such as(Lee, Euclidean, Chinese Euclidean) is given $r_d(C) = \max_{u \in R}\{\min_{c \in C} d(u, c)\}$.

In $F_q = \{0, 1, \beta_2, \cdots, \beta_{q-1}\}$ is a finite field. Let C be a q-ary *repetition code* C over F_q. That is $C = \{\bar{\beta} = (\beta\beta \cdots \beta) | \beta \in F_q\}$ and the repetition code C is an $[n, 1, n]$ code. Therefore, the covering radius of the code C is $\lceil \frac{n(q-1)}{q} \rceil$ this true for binary repetition code. In [7], the authors studied for different classes of repetition codes over $\mathbb{Z}_6$ and their covering radius has been obtained. Now, generalize those results for codes over R.

Consider the repetition codes over R. For a fixed $1 \leq i \leq 11$. For all $1 \leq j \neq i \leq 11, n_j = 0$, then the code $C^n = C^{n_i}$ is denoted by C_i. Therefore, the eleven basic repetition codes are the following table,

Generator Matrix	Code	Parameters $[n, k, d_*], (n, M, d_*)$
$G_1 = [\overbrace{01(\mathbf{05}) \cdots 01(\mathbf{05})}^{n}] = G_{\mathbf{5}}$	C_1, C_5	$[n, 1, n, n, n]$
$G_2 = [\overbrace{02(\mathbf{04}) \cdots 02(\mathbf{04})}^{n}] = G_{\mathbf{4}}$	C_2, C_4	$(n, 3, 2n, 4n, 3n)$
$G_3 = [\overbrace{03 \cdots 03}^{n}]$	C_3	$(n, 2, 3n, 9n, 4n)$
$G_6 = [\overbrace{10 \cdots 10}^{n}]$	C_6	$(n, 2, n, n, n)$
$G_7 = [\overbrace{11(\mathbf{15}) \cdots 11(\mathbf{15})}^{n}] = G_{\mathbf{11}}$	C_7, C_{11}	$[n, 1, n, n, n]$
$G_8 = [\overbrace{12(\mathbf{14}) \cdots 12(\mathbf{14})}^{n}] = G_{\mathbf{10}}$	C_8, C_{10}	$(n, 6, n, n, n)$
$G_9 = [\overbrace{13 \cdots 13}^{n}]$	C_9	$(n, 4, n, n, n)$

here,$* = L(E)(CE)$
$C_1 = \{c_0, c_1, c_2, c_3, c_4, c_5\} = C_5$,
$C_2 = \{c_0, c_2, c_4\} = C_4$,
$C_3 = \{c_0, c_3\}$,
$C_6 = \{c_0, c_6\}$
$C_7 = \{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}\} = C_{11}$,
$C_8 = \{c_0, c_2, c_4, c_6, c_8\} = C_{10}$,
$C_9 = \{c_0, c_3, c_6, c_9\}$ and $\{c_0 = 00 \cdots 00, c_1 = 01 \cdots 01, c_2 = 02 \cdots 02, c_3 = 03 \cdots 03, c_4 = 04 \cdots 04,$
$c_5 = 05 \cdots 05, c_6 = 10 \cdots 10, c_7 = 11 \cdots 11, c_8 = 12 \cdots 12, c_9 = 13 \cdots 13, c_{10} = 14 \cdots 14, c_{11} = 15 \cdots 15\}$.

Theorem 3.1. *Let $C_{j, 1 \leq j \leq 11,}$ be a code in R. Then,*

1. $\frac{3n}{4} \leq r_L(C_1) = r_L(C_5) \leq \frac{7n}{3}$,

2. $\frac{2n}{3} \leq r_L(C_2) = r_L(C_4) \leq \frac{7n}{3}$,

3. $\frac{3n}{4} \leq r_L(C_3) \leq 2n$,

4. $\frac{n}{4} \le r_L(C_6) \le 3n$,

5. $n \le r_L(C_7) = r_L(C_{11}) \le 2n$,

6. $\frac{11n}{12} \le r_L(C_8) = r_L(C_{10}) \le 2n$ *and*

7. $n \le r_L(C_9) \le 2n$, *where* $r_L(C_j)$ *is a covering radius of* $C_j,\ _{1\le j\le 11}$ *with Lee distance.*

Proof. For $c \in C_{j,1\le j\le 11}$ be a codeword of code C_j in R. Let $t_i(c)_{,0\le i\le 11}$ is the number of occurrences of symbol i in the codeword c.

Let $x \in R^n$ by $(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_8, t_9, t_{10}, t_{11})$, where $\sum_{j=0}^{11} t_j = n$, then

$$\begin{aligned}
d_L(x, \overline{00}) &= n - t_0 + t_2 + 2t_3 + t_4 + t_7 + 2t_8 + 3t_9 + 2t_{10} + t_{11},\\
d_L(x, \overline{01}) &= n - t_1 + t_3 + 2t_4 + t_5 + t_6 + t_8 + 2t_9 + 3t_{10} + 2t_{11},\\
d_L(x, \overline{02}) &= n - t_2 + t_0 + t_4 + 2t_5 + 2t_6 + t_7 + t_9 + 2t_{10} + 3t_{11},\\
d_L(x, \overline{03}) &= n - t_3 + t_1 + t_5 + 3t_6 + 2t_7 + t_8 + t_{10} + 2t_{11} + 2t_0,\\
d_L(x, \overline{04}) &= n - t_4 + t_0 + 2t_1 + t_2 + 2t_6 + 3t_7 + 2t_8 + t_9 + t_{11},\\
d_L(x, \overline{05}) &= n - t_5 + t_1 + 2t_2 + t_3 + t_6 + 2t_7 + 3t_8 + 2t_9 + t_{10},\\
d_L(x, \overline{10}) &= n - t_6 + t_1 + 2t_2 + 3t_3 + 2t_4 + t_5 + t_8 + 2t_9 + t_{10},\\
d_L(x, \overline{11}) &= n - t_7 + t_0 + t_2 + 2t_3 + 3t_4 + 2t_5 + t_9 + 2t_{10} + t_{11},\\
d_L(x, \overline{12}) &= n - t_8 + 2t_0 + t_1 + t_3 + 2t_4 + 3t_5 + t_6 + t_{10} + 2t_{11},\\
d_L(x, \overline{13}) &= n - t_9 + 3t_0 + 2t_1 + t_2 + t_4 + 2t_5 + 2t_6 + t_7 + t_{11},\\
d_L(x, \overline{14}) &= n - t_{10} + 2t_0 + 3t_1 + 2t_2 + t_3 + t_5 + t_6 + 2t_7 + t_8,\\
d_L(x, \overline{15}) &= n - t_{11} + t_0 + 2t_1 + 3t_2 + 2t_3 + t_4 + t_7 + 2t_8 + t_9,
\end{aligned}$$

In code $C_7 = C_{11} \in R$, then $d_L(x, C_7) = d_L(x, C_{11}) = \min\{d_L(x, \overline{00}), d_L(x, \overline{01}), d_L(x, \overline{02}), d_L(x, \overline{03}), d_L(x, \overline{04}), d_L(x, \overline{05}), d_L(x, \overline{10}), d_L(x, \overline{11}) d_L(x, \overline{12}), d_L(x, \overline{13}), d_L(x, \overline{14}), d_L(x, \overline{15})\} \le 2n$ and hence

$$r_L(C_7) = r_L(C_{11}) \le 2n.$$

If $x = (\overbrace{00\cdots00}^{\frac{n}{12}}\overbrace{01\cdots01}^{\frac{n}{12}}\overbrace{02\cdots02}^{\frac{n}{12}}\overbrace{03\cdots03}^{\frac{n}{12}}\overbrace{04\cdots04}^{\frac{n}{12}}\overbrace{05\cdots05}^{\frac{n}{12}}\overbrace{10\cdots10}^{\frac{n}{12}}\overbrace{11\cdots11}^{\frac{n}{12}}\overbrace{12\cdots12}^{\frac{n}{12}}\overbrace{13\cdots13}^{\frac{n}{12}}\overbrace{14\cdots14}^{\frac{n}{12}}\overbrace{15\cdots15}^{\frac{n}{12}}) \in R^n$. Then $d_L(x, \overline{00}) = d_L(x, \overline{01}) = d_L(x, \overline{02}) = d_L(x, \overline{03}) = d_L(x, \overline{04}) = d_L(x, \overline{05}) = d_L(x, \overline{10}) = d_L(x, \overline{11}) = d_L(x, \overline{12}) = d_L(x, \overline{13}) = d_L(x, \overline{14}) = d_L(x, \overline{15}) = \frac{n}{24} + 2(\frac{n}{24}) + 3(\frac{n}{24}) + 2(\frac{n}{24}) + \frac{n}{24} + \frac{n}{24} + 2(\frac{n}{24}) + 3(\frac{n}{24}) + 4(\frac{n}{24}) + 3(\frac{n}{24}) + 2(\frac{n}{24}) = n$. Thus $r_L(C_7) = r_L(C_{11}) \ge n$ and hence, $n \le r_L(C_7) = r_L(C_{11}) \le 2n$.

In Code, $C_3 \in R$, $d_L(x, C_3) = \min\{d_L(x, \overline{00}), d_L(x, \overline{03})\} \le \frac{2n-n+3n}{2} = 2n$. Then $r_L(C_3) \le 2n$.

If $x = (\overbrace{00\cdots00}^{\frac{n}{2}}\overbrace{03\cdots03}^{\frac{n}{2}}) \in R^n$, then $d_L(x, \overline{00}) = d_L(x, \overline{03}) = 3(\frac{n}{4}) = \frac{3n}{4}$. Thus $r_L(C_2) \ge \frac{3n}{4}$ and so $\frac{3n}{4} \le r_L(C_3) \le 2n$.

The remaining part of proof is follows from the above computation with respect to code. ■

Theorem 3.2. *In Euclidean weight for the code* $C_{j,1\le j\le 7}$, *prove the following*

1. $\frac{19n}{12} \le r_E(C_1) = r_{CE}(C_5) \le 4n$,

2. $\frac{4n}{3} \le r_E(C_2) = r_{CE}(C_4) \le \frac{13n}{3}$,

3. $\frac{9n}{4} \le r_E(C_3) \le 5n$,

4. $\frac{n}{4} \leq r_E(C_6) \leq 7n$,
5. $\frac{11n}{6} \leq r_E(C_7) = r_{CE}(C_{11}) \leq \frac{11n}{3}$,
6. $\frac{19n}{12} \leq r_E(C_8) = r_{CE}(C_{10}) \leq 4n$ *and*
7. $\frac{5n}{2} \leq r_E(C_9) \leq 5n$.

Proof. Use to theorem 3.1 and in Code $C_i,\ _{i=1 \text{ to } 11}$ with Euclidean weight. ■

Theorem 3.3. *In Chinese Euclidean weight of code of* $C_{j,1\leq j\leq 11,}$ *to find*

1. $\frac{5n}{6} \leq r_{CE}(C_1) = r_{CE}(C_5) \leq \frac{17n}{6}$,
2. $n \leq r_{CE}(C_2) = r_{CE}(C_4) \leq \frac{8n}{3}$,
3. $n \leq r_{CE}(C_3) \leq \frac{5n}{2}$,
4. $\frac{n}{4} \leq r_{CE}(C_6) \leq 4n$,
5. $r_{CE}(C_7) = r_{CE}(C_{11}) \leq \frac{5n}{2}$,
6. $\frac{5n}{4} \leq r_{CE}(C_8) = r_{CE}(C_{10}) \leq \frac{7n}{3}$ *and*
7. $\frac{5n}{4} \leq r_{CE}(C_9) \leq \frac{5n}{2}$.

Proof. In Code $C_i,\ _{i=1 \text{ to } 11}$ with Chinese Euclidean weight is apply to theorem 3.1. ■

Block repetition code in R

The block repetition code C^n over R is a R-additive code.

Let $G = [\overbrace{0101\cdots01}^{n_1}\overbrace{0202\cdots02}^{n_2}\overbrace{0303\cdots03}^{n_3}\overbrace{0404\cdots04}^{n_4}\overbrace{0505\cdots05}^{n_5}\overbrace{1010\cdots10}^{n_6}$ $\overbrace{1111\cdots11}^{n_7}\overbrace{1212\cdots12}^{n_8}\overbrace{1313\cdots13}^{n_9}\overbrace{1414\cdots14}^{n_{10}}\overbrace{1515\cdots15}^{n_{11}}]$ be a generator matrix with the parameters of C^n : $[n = \sum_{j=1}^{11} n_j, 12, d_L = \min\{\sum_{j=6}^{11} n_j, \sum_{j=1,2,4,5,}^{7,8,10,11} 2n_j\}, d_E = \min\{\sum_{j=6}^{11} n_j, \sum_{j=1,2,4,5,}^{7,8,10,11} 4n_j\}, d_{CE} = \min\{\sum_{j=6}^{11} n_j, \sum_{j=1,2,4,5,}^{7,8,10,11} 3n_j\}]$.

Theorem 3.4. *Let C^n be the block repetition code in R with length is n. Then the covering radius of block repetition code is*

1. $\frac{9(n_1+n_3+n_5)+8(n_2+n_4)+3n_6+12(n_7+n_9+n_{11})+11(n_8+n_{10})}{12} \leq r_L(C^n) \leq \frac{30(n_1+n_3+n_5)+31n_2+24n_4+36n_6+26(n_7+n_8)+24(n_9+n_{11})+25n_{10}}{12}$,
2. $\frac{19(n_1+n_5+n_8+n_{10})+16(n_2+n_4)+27n_3+3n_6+22(n_7+n_{11})+30n_9}{12} \leq r_E(C^n) \leq \frac{52n_1+56(n_2+n_4)+66n_3+50(n_5+n_8+n_{10})+60(n_6+n_9)+45n_7+44n_{11}}{12}$ *and*
3. $\frac{10(n_1+n_5)+12(n_2+n_3)+3n_6+30(n_7+n_{11})+15(n_8+n_9+n_{10})}{12} \leq r_{CE}(C^n) \leq \frac{36(n_1+n_2+n_3+n_4+n_5)+48n_6+32n_7+31(n_8+n_{10})+30(n_9+n_{11})}{12}$.

Proof. Using [9], Theorem 3.1, 3.2 and 3.3, thus

- $\frac{9(n_1+n_3+n_5)+8(n_2+n_4)+3n_6+12(n_7+n_9+n_{11})+11(n_8+n_{10})}{12} \leq r_L(C^n)$,

- $\frac{19(n_1+n_5+n_8+n_{10})+16(n_2+n_4)+27n_3+3n_6+22(n_7+n_{11})+30n_9}{12} \leq r_E(C^n)$ and
- $\frac{10(n_1+n_5)+12(n_2+n_3)+3n_6+30(n_7+n_{11})+15(n_8+n_9+n_{10})}{12} \leq r_{CE}(C^n)$.

Let $x = x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}x_{11} \in R^n$ with $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9,$ x_{10}, x_{11} is $(a_i), (b_i), (c_i), (d_i), (e_i), (f_i), (g_i), (h_i), (k_i), (l_i), (m_i)$, $_{i=0,1,2,3,4,5,6,7,8,9,10,11}$ respectively such that $n_1 = \sum_{j=0}^{11} a_j$, $n_2 = \sum_{j=0}^{11} b_j$, $n_3 = \sum_{j=0}^{11} c_j$, $n_4 = \sum_{j=0}^{11} d_j$, $n_5 = \sum_{j=0}^{11} e_j$, $n_6 = \sum_{j=0}^{11} f_j$, $n_7 = \sum_{j=0}^{11} g_j$, $n_7 = \sum_{j=0}^{11} g_j$, $n_8 = \sum_{j=0}^{11} h_j$, $n_9 = \sum_{j=0}^{11} k_j$, $n_{10} = \sum_{j=0}^{11} l_j$, $n_{11} = \sum_{j=0}^{11} m_j$.

Thus, $r_L(C^n) \leq \frac{30(n_1+n_3+n_5)+31n_2+24n_4+36n_6+26(n_7+n_8)+24(n_9+n_{11})+25n_{10}}{12}$,
$r_E(C^n) \leq \frac{52n_1+56(n_2+n_4)+66n_3+50(n_5+n_8+n_{10})+60(n_6+n_9)+45n_7+44n_{11}}{12}$ and
$r_{CE}(C^n) \leq \frac{36(n_1+n_2+n_3+n_4+n_5)+48n_6+32n_7+31(n_8+n_{10})+30(n_9+n_{11})}{12}$. ■

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Distance strings of the vertices of certain graphs

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Abstract. The notion of the distance string of a vertex $v_i \in V(G)$ which is denoted by, $\tau(v_i)$ is introduced. Distance strings permit a new approach to determining the induced vertex stress, the total induced vertex stress and total vertex stress (sum of vertex stress over all vertices) of a graph. A seemingly under-studied topic i.e. the eccentricity of a vertex of a bipartite Kneser graph $BK(n,k)$, $n \geq 2k+1$ has been furthered. A surprisingly simple result was established, namely for $k \geq 2$, $diam(BK(n,k)) = 5$ if $n = 2k+1$ and $diam(BK(n,k)) = 3$ if $n \neq 2k+1$.

AMS Subject Classifications: 05C12, 05C30, 05C69.

Keywords: Vertex stress, diameter, distance, distance string, induced-stress string.

Contents

1. Introduction

It is assumed that the reader has good knowledge of graph theory. For general notation and concepts in graphs, see [2, 4, 13]. Only finite, undirected and connected simple graphs of order $n \geq 2$ will be considered. For a graph G of order n all vertices will be labeled as v_i, $1 = 1, 2, 3, \dots, n$. Recall that the distance between vertices v_i and v_j is the length of a shortest path between v_i and v_j. The distance is denoted by $d_G(v_i, v_j)$ (or when the context is clear, simply by $d(v_i, v_j)$). A shortest v_iv_j-path is also called a v_iv_j-distance path. Since G is undirected we have that, $d_G(v_i, v_j) = d_G(v_j, v_i)$. However, for purposes of reasoning of proof or motivation of concepts a v_iv_j-distance path and a v_jv_i-distance path will distinguish between the *departure vertex* and the *destination vertex* and possibly, between two distinct shortest paths. This means that a v_iv_j-distance path has the departure vertex v_i and the destination vertex v_j whilst a v_jv_i-distance path has the departure vertex v_j and the destination vertex v_i.

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The vertex stress of vertex $v \in V(G)$ is the number of times v is contained as an internal vertex in all shortest paths between all pairs of distinct vertices in $V(G)\backslash\{v\}$. Formally stated, $\mathcal{S}_G(v) = \sum\limits_{u\neq w\neq v\neq u} \sigma(v)$ with $\sigma(v)$ the number of shortest paths between vertices u, w which contain v as an internal vertex. Such a shortest uw-path is also called a uw-distance path. See [9, 10]. The *total vertex stress* of G is given by $\mathcal{S}(G) = \sum\limits_{v\in V(G)} \mathcal{S}_G(v)$, [5]. From [11] we recall the definition of total induced vertex stress of a vertex v_i denoted by, $\mathfrak{s}_G(v_i)$, $v_i \in V(G)$.

Definition 1.1. *[11] Let $V(G) = \{v_i : 1 \leq i \leq n\}$. For the ordered vertex pair (v_i, v_j) let there be $k_G(i,j)$ distinct shortest paths of length $l_G(i,j)$ from v_i to v_j. Then, $\mathfrak{s}_G(v_i) = \sum\limits_{j=1, j\neq i}^{n} k_G(i,j)(\ell_G(i,j) - 1)$.*

Put differently, imagine a particle ρ moves along all possible shortest v_iv_j-paths, $j = 1, 2, 3, \dots, i-1, i+1, \dots, n$. Definition 1.1 provides the total number of times the particle if departing from vertex v_i will transit through internal vertices.

Let $diam(G) = k \geq 1$. Clearly a vertex v_i has a total number say, $a_{i,1} = deg(v_i)$ of paths of length 1. Similarly the vertex v_i has a total number say, $a_{i,j} \geq 0$ of paths of length $2 \leq j \leq k$. Let the *inductor vector* be $t = (0\ 1\ 2\ \cdots\ k-1)$. Define the $n \times k$ matrix:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,k} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k} \end{pmatrix}.$$

For each vertex v_i there is a corresponding (ordered) row i.e. $\tau(v_i) = (a_{i,1} \triangleright a_{i,s} \triangleright \cdots \triangleright a_{i,k})$. Note that $\triangleright$ serves as a spacer between entries of an ordered row. The ordered row $\tau(v_i)$ is called the *distance string* of v_i. Recall that the transpose t^T is a column vector. It follows from Definition 1.1 that,

$$A \cdot t^T = \begin{pmatrix} \mathfrak{s}_G(v_1) \\ \mathfrak{s}_G(v_2) \\ \vdots \\ \mathfrak{s}_G(v_n) \end{pmatrix}.$$

The *induced-stress string* of graph G is defined by,

$$\tau(G) = (\mathfrak{s}_G(v_1) \triangleright \mathfrak{s}_G(v_2) \triangleright \cdots \triangleright \mathfrak{s}_G(v_n)).$$

A new approach to determine the total induced vertex stress of a graph G denoted and defined by, $\mathfrak{s}(G) = \sum\limits_{i=1}^{n} \mathfrak{s}_G(v_i)$ will be explored. Clearly,

$$\mathfrak{s}(G) = \sum_{\substack{j=1 \\ b_{j,1}\in A\cdot t^T}}^{n} b_{j,1} \text{ and } \mathcal{S}(G) = \tfrac{1}{2}\mathfrak{s}(G).$$

The objective of this paper is limited to determining the distance string of each vertex $v_i \in V(G)$.

2. Distance strings of certain graphs

To ensure clarity of the new concepts we begin with well-known graphs.

Proposition 2.1. *For a path P_n, $n \geq 2$ it follows that:*

$$(i)\ \tau(v_1) = \tau(v_n) = (\underbrace{1 \triangleright 1 \triangleright 1 \triangleright \cdots \triangleright 1}_{(n-1\ entries)}).$$

$$(ii)\ \tau(v_i) = \tau(v_{n-(i-1)}) = (\underbrace{2 \triangleright 2 \triangleright 2 \triangleright \cdots \triangleright 2}_{(i-1\ entries)} \triangleright \underbrace{1 \triangleright 1 \triangleright 1 \triangleright \cdots \triangleright 1}_{(n-(2i-1)\ entries)} \triangleright \underbrace{0 \triangleright 0 \triangleright 0 \triangleright \cdots \triangleright 0}_{(i-1\ entries)})$$
$$\textit{for, } 2 \leq i \leq \lceil \tfrac{n}{2} \rceil.$$

Proof. For convenience of reasoning assume without loss of generality that a path is depicted horizontally with the vertices consecutively labeled from left to right as, v_1, v_2,..., v_n. Note that a result for v_j, $1 \leq j \leq \lceil \frac{n}{2} \rceil$ also yields the corresponding result for $v_{n-(i-1)}$.

(i) The vertex v_1 has a unique $v_1 v_j$-distance path for $2 \leq j \leq n$. Hence, the result as well as for the *mirror image* vertex v_n.

(ii) The upper bound $\lceil \frac{n}{2} \rceil$ with regards to i is required to settle the results for both, n is odd or even. A vertex v_i, $2 \leq i \leq \lceil \frac{n}{2} \rceil$ has a unique $v_1 v_j$-distance path for $2 \leq j \leq i-1$. Similarly, for the mirror image vertices to the right of v_1. The aforesaid observations settle the partial entries $(\underbrace{2 \triangleright 2 \triangleright 2 \triangleright \cdots \triangleright 2}_{(i-1\ entries)} \triangleright \cdots)$. The fact that the vertex v_i has a unique $v_1 v_j$-distance path for $2i \leq j \leq n$ settles the additional partial entries $(\underbrace{2 \triangleright 2 \triangleright 2 \triangleright \cdots \triangleright 2}_{(i-1\ entries)} \triangleright \underbrace{1 \triangleright 1 \triangleright 1 \triangleright \cdots \triangleright 1}_{(n-(2i-1)\ entries)} \triangleright \cdots)$. Since $diam(P_n) = n-1$ and $\tau(v_i)$ is a string with $n-1$ entries, the result is finally obtained. Hence,

$$\tau(v_i) = \tau(v_{n-(i-1)}) = (\underbrace{2 \triangleright 2 \triangleright 2 \triangleright \cdots \triangleright 2}_{(i-1\ entries)} \triangleright \underbrace{1 \triangleright 1 \triangleright 1 \triangleright \cdots \triangleright 1}_{(n-(2i-1)\ entries)} \triangleright \underbrace{0 \triangleright 0 \triangleright 0 \triangleright \cdots \triangleright 0}_{(i-1\ entries)})$$
$$\text{for, } 2 \leq i \leq \lceil \tfrac{n}{2} \rceil.$$

■

Proposition 2.2. *For a cycle C_n, $n \geq 3$ it follows that:*

$$\tau(v_i) = (\underbrace{2 \triangleright 2 \triangleright 2 \triangleright \cdots \triangleright 2}_{(\lfloor \frac{n}{2} \rfloor\ entries)})$$
$$\textit{for, } 1 \leq i \leq n.$$

Proof. For convenience of reasoning assume without loss of generality that a cycle has the vertices labeled clockwise and consecutively as, v_1, v_2,..., v_n. By the symmetric property of a cycle the result for an arbitrary v_i is identical to the result of an arbitrary v_j. Without loss of generality the vertex v_1 is selected to settle the proof. It is known that $diam(C_n) = \lfloor \frac{n}{2} \rfloor$. Hence, a distance string has $\lfloor \frac{n}{2} \rfloor$ entries.

Case 1: Clockwise paths. It is trivial to see that v_1 has a unique $v_1 v_i$-distance path in a clockwise direction to the vertices v_i, $2 \leq i \leq \lceil \frac{n}{2} \rceil$.

Case 2: Anti-clockwise paths. It is trivial to see that v_1 has a unique $v_1 v_j$-distance path in an anti-clockwise direction, to the vertices v_i, $n - \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n$.
Clearly, all distance path from vertex v_1 have been accounted. Therefore,

$$\tau(v_i) = (\underbrace{2 \triangleright 2 \triangleright 2 \triangleright \cdots \triangleright 2}_{(\lfloor \frac{n}{2} \rfloor\ entries)})$$
$$\text{for, } 1 \leq i \leq n.$$

■

Proposition 2.3. *For a complete graph K_n, $n \geq 2$ it follows that:*

$$\tau(v_i) = (n-1)$$
$$for,\ 1 \leq i \leq n.$$

Proof. Since $diam(K_n) = 1$ and $deg(v_i) = n-1$ the result is settled. ∎

Corollary 2.4. *A graph G of order $n \geq 2$ has a singleton distance string if and only if $G \cong K_n$.*

Proof. It is known that a graph G of order $n \geq 2$ has $diam(G) = 1$ if and only if $G \cong K_n$. Hence, the result follows by implication. ∎

The proof of the next proposition is omitted as an exercise for the reader.

Proposition 2.5. *For a complete bipartite graph $K_{n,m}$, $n, m \geq 1$ with partition sets X, $|X| = n$ and Y, $|Y| = m$ it follows that:*

$$(i)\ \tau(v_i) = (m \triangleright m(n-1)),\ v_i \in X.$$
$$(ii)\ \tau(u_i) = (n \triangleright n(m-1)),\ u_i \in Y.$$

Proposition 2.6. *The Petersen graph G has $\tau(v_i) = (3 \triangleright 6)$, $\forall\, v_i \in V(G)$.*

Proof. It is known that for $G \cong$ *Petersen graph*, $diam(G) = 2$, $deg_G(v_i) = 3$, $\forall\, v_i$ and $|V(G)| = 10$. Hence, each vertex v_i has 3 incident edges or 1-distance paths and 6 distance paths of length 2. Therefore, $\tau(v_i) = (3 \triangleright 6)$, $\forall\, v_i \in V(G)$. ∎

3. On bipartite Kneser graphs

It is assumed that the reader has good working knowledge of set theory. For the general notation, notions and important introductory results in set theory, see [3].

Without loss of generality let $n \geq 3$ and let $1 \leq k \leq \lceil \frac{n}{2} \rceil - 1$. Let X_i, $i = 1, 2, 3, \dots, \binom{n}{k}$ be the k-element subsets of the set, $\{1, 2, 3, \dots, n\}$. Let Y_i, $i = 1, 2, 3, \dots, \binom{n}{k}$ be the $(n-k)$-element subsets of the set, $\{1, 2, 3, \dots, n\}$. Let $V_1 = \{v_i : v_i \mapsto X_i\}$ and $V_2 = \{u_i : u_i \mapsto Y_i\}$. A connected bipartite Kneser graph denoted by $BK(n,k)$ is a graph with vertex set,

$$V(BG(n,k)) = V_1 \cup V_2$$

and the edge set,

$$E(BG(n,k)) = \{v_i u_j : X_i \subset Y_j\}.$$

Claim 3.1. *Since vertex adjacency is defined identically for all vertices the, without loss of generality principle (for brevity, the wlg-principle) applies to our method of proof. All results in respect of an arbitrary vertex v_i are (immediately) valid for all $v_j \in V(BG(n,k))$. Such generalization is axiomatically valid and requires no further proof.*

Claim 3.2. *Let G be a graph in a family of graphs of well-defined (or categorized) order $n = f(i)$, $i = 1, 2, 3, \dots$ which has a well-defined adjacency definition (or regime) such that for any arbitrary vertex v_i (selected by the wlg-principle) of G, a graph theoretical parameter (or property) such as valency, eccentricity, centrality and alike is immediately valid for all vertices of G. Then formal mathematical induction on each category of n can be replaced by the principle of immediate induction.*

Theorem 3.3. [6] *A bipartite Kneser graph, $BK(n,1)$, $n \geq 3$ has:*

$$diam(BK(n,1)) = 3.$$

It is known that $BK(n,1)$, $n \geq 3$ is degree regular with $deg(v_i) = \binom{n-1}{1} = n-1$.

Theorem 3.4. *A bipartite Kneser graph, $BK(n,1)$, $n \geq 3$ has:*

$$\tau(v_i) = ((n-1) \triangleright (n-1)(n-2) \triangleright (n-1)(n-2)^2).$$

Proof. Because $diam(BK(n,1)) = 3$ and $BK(n,1)$ is regular it follows immediately the each vertex has exactly $(n-1)$ shortest 1-paths (or edges), exactly $(n-1)(n-2)$ shortest 2-paths and exactly $(n-1)(n-2)^2$ shortest 3-paths. Therefore the result is settled. ∎

Theorem 3.5. *A bipartite Kneser graph $BK(n,2)$, $n \geq 5$ has:*

$$diam(BK(n,2)) = \begin{cases} 5, & \text{if } n = 5; \\ \\ 3, & \text{if } n \geq 6. \end{cases}$$

Proof. The diameter of a connected graph G is equal to the maximum eccentricity $\epsilon(v)$ of some vertex $v \in V(G)$. Since all bipartite Kneser graphs are vertex transitive (see Lemma 3.1 in [7]) and it is degree regular it follows that the diameter is equal to the eccentricity of any vertex. For convenience the eccentricity of $v_1 \in V_1$ will be considered. It is known that for $n > 2k$ (or $n \geq 2k+1$) the bipartite Kneser graph $BK(n,k)$ is connected which implies that a finite diameter exists. Hence, $BK(n,2)$, $n \geq 5$ is connected and has a finite diameter. For $k = 2$ the first bipartite Kneser graph to consider is $BK(5,2)$.

Case 1. Let $n = 2k+1 = 5$. Let V_1 defined as:
$v_1 \mapsto \{1,2\}$, $v_2 \mapsto \{1,3\}$, $v_3 \mapsto \{1,4\}$, $v_4 \mapsto \{1,5\}$, $v_5 \mapsto \{2,3\}$, $v_6 \mapsto \{2,4\}$, $v_7 \mapsto \{2,5\}$, $v_8 \mapsto \{3,4\}$, $v_9 \mapsto \{3,5\}$, $v_{10} \mapsto \{4,5\}$.
Let V_2 be defined as:
$u_1 \mapsto \{1,2,3\}$, $u_2 \mapsto \{1,2,4\}$, $u_3 \mapsto \{1,2,5\}$, $u_4 \mapsto \{1,3,4\}$, $u_5 \mapsto \{1,3,5\}$, $u_6 \mapsto \{1,4,5\}$, $u_7 \mapsto \{2,3,4\}$, $u_8 \mapsto \{2,3,5\}$, $u_9 \mapsto \{2,4,5\}$, $u_{10} \mapsto \{3,4,5\}$.
By utilizing an appropriate shortest path algorithm such as in [8] or [15] or in this case an exhaustive method, it is established that $\epsilon(v_1) = 5$. Without loss of generality one such *diam*-path is $v_1u_3v_7u_9v_{10}u_{10}$. Therefore, $diam(BK(5,2)) = 5$. From the standard heuristic methods used to systematically find the 2-element and the 3-element subsets‡ it follow that the v_1u_{10}-distance path is indeed the eccentricity of v_1.

Case 2. Let $n \geq 2(k+1)$. Construct a bipartite Kneser graph $BK(6,2)$ from the $BK(5,2)$. It requires extension of the set V_1 to obtain say,

$$V_1^\star = V_1 \cup \{\{1,6\},\{2,6\},\{3,6\},\{4,6\},\{5,6\}\}.$$

Let

$$v_{11}^\star \mapsto \{1,6\}, v_{12}^\star \mapsto \{2,6\}, v_{13}^\star \mapsto \{3,6\}, v_{14}^\star \mapsto \{4,6\}, v_{15}^\star \mapsto \{5,6\}.$$

The set V_2 requires extensions as well.

$$V_2^\star = \{u_i^\star \mapsto u_i \cup \{6\} : u_i \in V_2\} \cup$$

$$\{\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\}\}.$$

Let

$$u_{11}^\star \mapsto \{1,2,3,4\}, u_{12}^\star \mapsto \{1,2,3,5\}, u_{13}^\star \mapsto \{1,2,4,5\}, u_{14}^\star \mapsto \{1,3,4,5\}, u_{15}^\star \mapsto \{2,3,4,5\}.$$

‡See https://www.omnicalculator.com,math,subset or [4]

After applying the adjacency regime the first observation is that the $v_1u^\star_{10}$-distance path shortened to 3 i.e. $v_1u^\star_2v^\star_{14}u^\star_{10}$. In $BK(6,2)$ the eccentricity of v_1 is found to be $\epsilon(v_1)=3$. Without loss of generality a *diam*-path which is obtained similar in method to that used for $BK(5,2)$ is given by, $v_1u^\star_3v^\star_{15}u^\star_{10}$. In fact, by first generating the subsets in the conventional algorithmic fashion we have $V_1(BK(6,2))$ defined as:
$v_1\mapsto\{1,2\}$, $v_2\mapsto\{1,3\}$, $v_3\mapsto\{1,4\}$, $v_4\mapsto\{1,5\}$, $v_5\mapsto\{1,6\}$, $v_6\mapsto\{2,3\}$, $v_7\mapsto\{2,4\}$, $v_8\mapsto\{2,5\}$, $v_9\mapsto\{2,6\}$, $v_{10}\mapsto\{3,4\}$, $v_{11}\mapsto\{3,5\}$, $v_{12}\mapsto\{3,6\}$, $v_{13}\mapsto\{4,5\}$, $v_{14}\mapsto\{4,6\}$, $v_{15}\mapsto\{5,6\}$.
Let $V_2(BK(6,2))$ be defined as:
$u_1\mapsto\{1,2,3,4\}$, $u_2\mapsto\{1,2,3,5\}$, $u_3\mapsto\{1,2,3,6\}$, $u_4\mapsto\{1,2,4,5\}$, $u_5\mapsto\{1,2,4,6\}$, $u_6\mapsto\{1,2,5,6\}$, $u_7\mapsto\{1,3,4,5\}$, $u_8\mapsto\{1,3,4,6\}$, $u_9\mapsto\{1,3,5,6\}$, $u_{10}\mapsto\{1,4,5,6\}$, $u_{11}\mapsto\{2,3,4,5\}$, $u_{12}\mapsto\{2,3,4,6\}$, $u_{13}\mapsto\{2,3,5,6\}$, $u_{14}\mapsto\{2,4,5,6\}$, $u_{15}\mapsto\{3,4,5,6\}$.
A *diam*-path which is obtained by similar method to that used for $BK(5,2)$ is given by, $v_1u_6v_{15}u_{15}$.

Assume the result $diam(BK(n,2))=3$ holds for $7\leq n\leq\ell$. Obviously the vertex changes and the addition of exactly 2ℓ new vertices as n progresses consecutively through ℓ to $\ell+1$ (in fact as n progresses consecutively through $\ell,\ell+1,\ell+2\dots$) remain consistent. By similar reasoning to show the result for the progression from $n=5$ to $n=6$, it follows by immediate induction that the results holds for the progression from $n=\ell$ to $n=\ell+1$. Finally by applying the wlg-principle for $n\geq 6$ read with the well-defined 2-element subsets and $(n-2)$-element subsets, the principle of immediately induction is valid. Furthermore, a heuristic method to be used in general for $BK(n,2)$, $n\geq 6$ is:
(a) Select $v_1\mapsto\{1,2\}$ and link to $u_{\binom{n-2}{2}}\mapsto\{1,2,5,6,\dots,n\}$.
(b) From $u_{\binom{n-2}{2}}\mapsto\{1,2,5,6,\dots,n\}$ link to $v_{\binom{n}{2}}\mapsto\{n-1,n\}$ then,
(c) Link $v_{\binom{n}{2}}\mapsto\{n-1,n\}$ to $u_{\binom{n}{2}}\mapsto\{3,4,\dots,n-1,n\}$.
Therefore,

$$diam(BK(n,2))=\begin{cases}5, & \text{if } n=5;\\ \\ 3, & \text{if } n\geq 6.\end{cases}$$

■

Lemma 3.6. *A bipartite Kneser graph $BK(n,k)$, $n=2k+1$, $k\geq 2$ has $diam(BK(2k+1,k))=5$.*

Proof. The result for $BK(5,2)$ follows from Theorem 3.5. For $BK(7,3)$ we utilize

https://www.omnicalculator.com>math>subset

to systematically (conventionally) generate the 3-element and 4-element subsets respectively. Label the respective subsets as generated consecutively as:
$v_1\mapsto\{1,2,3\},\dots,v_{35}\mapsto\{5,6,7\}$ and $u_1\mapsto\{1,2,3,4\},\dots,u_{35}\mapsto\{4,5,6,7\}$.
By utilizing an appropriate shortest path algorithm such as in [8] or [15] or in this case an exhaustive method, it is established that $\epsilon(v_1)=5$. Without loss of generality one such *diam*-path for $BK(7,3)$ is $v_1u_4v_{31}u_{34}v_{35}u_{35}$. Therefore, $diam(BK(7,3))=5$.
By induction reasoning similar to that stated in the proof of Theorem 3.5 and utilizing Claim 3.2 we may utilize immediate induction for the result as stated. Hence, $BK(n,k)$, $n=2k+1$, $k\geq 2$ has:

$$diam(BK(2k+1,k))=5.$$

■

From the proof of Theorem 3.5, Lemma 3.6 read together with Claim 3.3 an immediate generalized result is permitted.

Theorem 3.7. *A bipartite Kneser graph $BK(n,k)$, $n \geq 2k+1$, $k \geq 3$ has:*

$$diam(BK(n,k)) = \begin{cases} 5, & \text{if } n = 2k+1; \\ \\ 3, & \text{if } n \geq 2(k+1) (\text{or } n > 2k+1). \end{cases}$$

The next result is a direct consequence of Theorem 3.7 and the fact that $deg_{BK(n,k)}(v_i) = \binom{n-k}{k}$.

Theorem 3.8. *A bipartite Kneser graph $BK(n,k)$, $n \geq 2k+1$, $k \geq 3$ and $t = \binom{n-k}{k}$ has:*

$$\tau(v_i) = \begin{cases} (t \triangleright t(t-1) \triangleright t(t-1)^2 \triangleright t(t-1)^3 \triangleright t(t-1)^4), & \text{if } n = 2k+1; \\ \\ (t \triangleright t(t-1) \triangleright t(t-1)^2), & \text{if } n \geq 2(k+1). \end{cases}$$

Some authors define the adjacency of bipartite Kneser graphs as:

$$E(BG(n,k)) = \{v_i u_j : X_i \subseteq Y_j\}.$$

It implies that for $n = 2k$ the graph $BK(2k,k)$ is a matching graph. Furthermore, since the empty set is a proper subset of a set, the graph $BK(n,0) \cong K_2$ with $v_1 \mapsto \emptyset$ and $u_1 \mapsto \{1,2,3,\ldots,n\}$. Since $BK(n,k) \cong BK(n,n-k)$, Theorem 3.7 can be stated as:

Theorem 3.9. Alternative *A bipartite Kneser graph $BK(n,k)$, $n \geq 5$, $k \geq 3$ has:*

$$diam(BK(n,k)) = \begin{cases} 5, & \text{if } n = 2k+1; \\ \\ 3, & \text{if } n \neq 2k+1. \end{cases}$$

4. A research avenue

For $n = 0,1,2,3,\ldots$ the integer sequence $A001349$ found in the on-line encyclopedia of integer sequences (OEIS) presents the number of distinct (non-isomorphic) connected simple graphs on n vertices. Let a_n consecutively map onto

$$1, 1, 1, 2, 6, 21, 112, 853, 11117, 261080, 11716571, \ldots$$

This study only considers connected simple graphs of order $n \geq 2$. Consider a graph theoretical property $\sqcup$. Assume that for $n \geq 2$ at least $\ell_n \geq 0$ graphs of order n have property $\sqcup$. Hence, the portion (or ratio) of graphs of order n which have the property is $\frac{\ell_n}{a_n}$. If $\lim\limits_{n\to\infty} \frac{\sum\limits_{m=2}^{n} \ell_m}{\sum\limits_{m=2}^{n} a_m} = 1$, it is said that *almost all* graphs have property $\sqcup$.

Conjecture 4.1. *For almost all graphs G and its distinct vertex pairs $v_i, v_j \in V(G)$ it follow that:*

$$\sum_{a_{i,k} \in \tau(v_i)} a_{i,k} = \sum_{a_{j,k} \in \tau(v_j)} a_{j,k},\ 1 \leq k \leq diam(G).$$

An example to show that Conjecture 4.1 is not valid for all graphs is $K_n - e$, $n \geq 4$. If in K_n, $n \geq 4$ and without loss of generality, the edge $e = v_1 v_2$ is deleted then, $\tau(v_1) = (n-2 \triangleright n-2)$ and $\tau(v_n) = (n-1 \triangleright 0)$. Clearly, $2(n-2) \neq n-1$ for $n \geq 4$.

Conjecture 4.2. *For almost all graphs G it follows that:*

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} \leq \frac{1}{2} \sum_{v_j\in V(G)} deg(v_j) = \varepsilon(G).$$

For certain families of graphs we can show that equality holds in Conjectures 4.1 and 4.2.

Theorem 4.3. *For all paths P_n, $n \geq 2$ and all its distinct vertex pairs $v_i, v_j \in V(P_n)$ it follows that:*
(i)

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \sum_{a_{j,k}\in\tau(v_j)} a_{j,k},\ 1 \leq k \leq diam(P_n) = n-1.$$

(ii)

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \frac{1}{2} \sum_{v_j\in V(G)} deg(v_j).$$

Proof. (i) From Proposition 2.1(i) it follows that:

$$\sum_{a_{1,k}\in\tau(v_1)} a_{1,k} = \sum_{a_{n,k}\in\tau(v_n)} a_{n,k} = n-1,\ 1 \leq k \leq diam(P_n) = n-1.$$

From Proposition 2.1(ii) it follows that:

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = 2(i-1) + \Big[n - (2i-1)\Big] = n-1,\ 1 \leq k \leq diam(P_n) = n-1.$$

The result is settled.
(ii) The sum of vertex degrees in a path is given by, $2 + 2(n-2) = 2n-2 = 2(n-1)$. Therefore,

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \frac{1}{2} \sum_{v_j\in V(G)} deg(v_j).$$

■

The proof of the next result follows similar reasoning to that found in the proof Theorem 4.3. The proof is omitted as an exercise to the reader.

Theorem 4.4. *For all cycles C_n, $n \geq 4$ and all its distinct vertex pairs $v_i, v_j \in V(C_n)$ it follows that:*
(i)

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \sum_{a_{j,k}\in\tau(v_j)} a_{j,k},\ 1 \leq k \leq diam(C_n) = \lfloor \frac{n}{2} \rfloor.$$

(ii)

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \frac{1}{2} \sum_{v_j\in V(G)} deg(v_j)$$

and n is even.

(iii)

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} < \frac{1}{2} \sum_{v_j\in V(G)} deg(v_j)$$

and n is odd.

Theorem 4.5. *For all complete bipartite graphs $K_{n,m}$, $n, m \geq 1$ and all its distinct vertex pairs $v_i, v_j \in V(K_{n,m})$ or $v_i, u_j \in V(K_{n,m})$ or $u_i, u_j \in V(K_{n,m})$ or simply referred to as vertices v_i, v_j, it follows that:*
(i)

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \sum_{a_{j,k}\in\tau(v_j)} a_{j,k},\ 1 \leq k \leq diam(K_{n,m}) = 2.$$

(ii)

$$\sum_{a_{i,k} \in \tau(v_i)} a_{i,k} = \frac{1}{2} \sum_{v_j \in V(G)} deg(v_j).$$

Proof. Both results is a direct result from the fact that, $m + m(n-1) = n + n(m-1) = 2nm$. ■

The notion of *stress regular* graphs was introduced in [10]. A graph G for which $\mathcal{S}_G(v_i) = \mathcal{S}_G(v_j)$ for all distinct pairs $v_i, v_j \in V(G)$ is said to be stress regular.

Theorem 4.6. *For a stress regular graph G it follows that:*

$$\sum_{a_{i,k} \in \tau(v_i)} a_{i,k} = \sum_{a_{j,k} \in \tau(v_j)} a_{j,k},\ 1 \le k \le diam(G).$$

Proof. Since G is stress regular it implies that $\mathcal{S}_G(v_i) = \mathcal{S}_G(v_j) \Leftrightarrow \mathfrak{s}_G(v_i) = \mathfrak{s}_G(v_j), \forall\ v_i, v_j \in V(G)$. The column vector t^T is a "constant" vector hence, $\tau(v_i) = \tau(v_j), \forall\ v_i, v_j \in V(G)$. If the latter is not true then G is not stress regular which is a contradiction. Thus,

$$\sum_{a_{i,k} \in \tau(v_i)} a_{i,k} = \sum_{a_{j,k} \in \tau(v_j)} a_{j,k},\ 1 \le k \le diam(G).$$

■

We recall further results from [10]

Theorem 4.7. [10] *Every distance regular graph is stress regular.*

Corollary 4.8. [10] *Every strongly regular graph is stress regular.*

Corollary 4.9. [10] *Every distance transitive graph is stress regular.*

Let X_i, $i = 1, 2, 3, \ldots, \binom{n}{k}$ be the k-element subsets of the set, $\{1, 2, 3, \ldots, n\}$. A Kneser graph denoted by $KG(n, k)$, $n, k \in \mathbb{N}$ is the graph with vertex set,

$$V(KG(n,k)) = \{v_i : v_i \mapsto X_i\}$$

and the edge set,

$$E(KG(n,k)) = \{v_i v_j : X_i \cap X_j = \emptyset\}.$$

It is known that the family of Kneser graphs $KG(n, 2)$ are distance regular graphs. Therefore, from Theorem 4.7 it follows that the Kneser graphs $KG(n, 2)$ are stress regular. Furthermore, it is known from [1] that every distance regular graph G with $diam(G) = 2$, is strongly regular. We recall some results from [6].

Corollary 4.10. [6] *Kneser graphs $KG(n, k_1)$, $k_1 \in \mathbb{N} \backslash \{1, 2\}$, $n \ge 3k_1 - 1$ are stress regular.*

In fact, a general result (without further proof) is permitted from the knowledge that all Kneser graphs $KG(n, k)$, $n \ge k$ are vertex transitive.

Theorem 4.11. [6] *All Kneser graphs $KG(n, k)$, $n \ge k$ are stress regular.*

The immediate above read together with Theorem 4.6 permits the next corollary without further proof.

Corollary 4.12. *(i) Every distance regular graph has:*

$$\sum_{a_{i,k} \in \tau(v_i)} a_{i,k} = \sum_{a_{j,k} \in \tau(v_j)} a_{j,k},\ 1 \le k \le diam(G).$$

(ii) Every strongly regular graph has:

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \sum_{a_{j,k}\in\tau(v_j)} a_{j,k},\ 1 \leq k \leq diam(G).$$

(iii) Every distance transitive graph has:

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \sum_{a_{j,k}\in\tau(v_j)} a_{j,k},\ 1 \leq k \leq diam(G).$$

5. Conclusion

Sections 1, 2 and 3 laid the foundation for further research related to the notion of distance strings of vertices and the induced-stress string of a graph. A wide scope for further research remains open. Prove or disprove the next conjecture.

Conjecture 5.1. *For the induced-stress string $\tau(G)$ of a graph G it follows that:*

$$\mathfrak{s}_G(v_i) = min\{\mathfrak{s}_G(v_j) : \mathfrak{s}_G(v_j) \in \tau(G)\},\ \textit{is even.}$$

The author holds the view that the eccentricity of a vertex in bipartite Kneser graphs in general did not receive adequate attention in the literature. As stated before, Theorem 21 in [6] established that $diam(BK(n,1)) = 3$, $n \geq 3$. In this paper a result for $diam(BK(n,k))$, $k \geq 2$, $n \geq 2k+1$ was established. With the results of Section 3 it is now possible to determine all the vertex stress related parameters for bipartite Kneser graphs.

Section 4 presents two interesting conjectures. Statements of the form ". . . for almost all graphs (disconnected and connected) . . ." is an interesting field for further research. More so with the vigorous research in experimental mathematics and the introduction of AI-mathematics. For example from [14]:

Theorem 5.2. [14] *Almost all graphs have diameter* 2*.*

Corollary 5.3. [14] *Almost all graphs have every edge in a triangle.*

Corollary 5.4. [14] *Almost all graphs are connected.*

From Corollary 5.1 we state the following.

Corollary 5.5. *Almost all ICT-networks, AI-networks, data structure networks, social media networks and alike will remain incomplete networks.*

From Corollary 5.2 we state the following.

Corollary 5.6. *Almost all graphs G have chromatic number, $\chi(G) \geq 3$.*

From Corollary 5.3 we state that:

Corollary 5.7. *All results which are common for connected graphs are valid for almost all graphs.*

Author is of the view that statements of the form ". . . for almost all graphs (disconnected and connected) . . ." have important implications in the field of experimental mathematics.

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Cone S-metric spaces and some new fixed point results for contractive mapping satisfying ϕ-maps with implicit relation

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Abstract. In this article, we use implicit relations to establish some new fixed point results in the setting of cone S-Metric spaces for phi-map type contractive conditions. An example is provided to support our results. Our results extend, unify, and generalize several results from the current literature. In particular, the results presented in this paper improve and generalize the corresponding results of Sedghi and Dung[8], which used the ideas of Saluja, G. S. [19].

AMS Subject Classifications: 47H10, 54H25.

Keywords: Cone metric spce, Cone S- mtric space, Fixed point, Unique fixed point, Implicit relation.

Contents

1. Introduction and Background

The metric fixed point theory is very important and useful in mathematics. There are a number of generalizations of metric spaces and the Banach contraction principle(1922). From 1922, when Stefen Banach [1] formulated the notion of contraction and proved the famous theorem, scientists around.The world is publishing new results that are connected either to establishing a generalization of metric spaces or to getting an improvement in contractive conditions.

Huang and Zhang [2] recently introduced the concept of cone metric space as a generalization of metric spaces by replacing the set of real numbers with a general Banach space E that is partially ordered with respect to a cone $\mathcal{P} \subset E$ and establishing some fixed point theorems for contractive conditions in normal cone metric spaces. Subsequently, other mathematicians have generalized the results of Huang and Zhang [2].

Very recently, Sedghi, et al. [3] introduced the concept of S-metric space, which is different from other spaces, and proved fixed point theorems in S-metric space. They also give some examples. As a result, numerous authors

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have investigated the fixed point for mapping satisfying contractive conditions in complete S-metric spaces (see, for example, [4,5,6,7,8,9,10]).

Dhamodharan and Krishna Kumar [11] recently introduced the concept of S-metric space and proved some fixed point theorems in the space using various contractive conditions. On the other hand, Berinde and Vetro [12] surveyed an implicit contraction type condition instead of the usual explicit condition.

For this path of research, we prepared a consistent literature on fixed point and common fixed point theorems in various ambient spaces [see, 13,14,15,16,1,18].. Here we prove a paramount result of cone S-metric space and obtain some classical fixed point theorems with corollaries.

In the setting of complete cone S-metric spaces, our results extend and generalize several results from the existing literature, particularly the result of Sedghi Dung [8] and the idea of Saluja, G. S. [19].

2. Basic concept and mathematical preliminaries

Now, we begin with some besic concept, notations, Defifnitions and Lemmas, which needs in the sequel.

Definition 2.1. *Let E be a real Banach space. A sub set $\mathcal{P}$ of E is called a cone whenever the following conditions hold:*

(1). *$\mathcal{P}$ is closed, non empty and $\mathcal{P} \neq \{0\}$;*

(2). *$a, b \in \mathcal{R}$, $a, b \geq 0$ and $x, y \in \mathcal{P} \Rightarrow ax + by \in \mathcal{P}$;*

(3). $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$.

Given a cone $\mathcal{P} \subset E$, we define a partial ordering $\leq$ in E with respect to $\mathcal{P}$ by $x \leq y$ if and only if $y - x \in \mathcal{P}$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \mathcal{P}^0$, where $\mathcal{P}^0$ stands for the interior of $\mathcal{P}$. If $\mathcal{P}^0 \neq \phi$, then $\mathcal{P}$ is called a solid cone (see[20]). There exists two kinds of cones normal (with the normal constant $\mathcal{K}$) and non normal ones([21]).

Let E be a real Banach space,$\mathcal{P} \subset E$ a cone and $\leq$ partial ordering defined by$\mathcal{P}$, then $\mathcal{P}$ is called normal if there is number $\mathcal{K} > 0$ such that for all $x, y \in \mathcal{P}$,

$$0 \leq x \leq y \Rightarrow ||x|| \leq \mathcal{K}||y||$$

(1.1) or equivalently, if for all n, $x_n \leq y_n \leq z_n$ and $\lim\limits_{x \to n} = \lim\limits_{z \to n} x \Rightarrow \lim\limits_{y \to n} = x$.The least positive number $\mathcal{K}$ satisfying (1.1) is called normal constant of $\mathcal{P}$.

The cone P is called regular if every increasing sequence which is bounded from above is convergent, that is , if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq ... \leq x_n \leq ... \leq y_n$ for some $y \in E$, then there is $x \in E$ such that,

$$||x_n - x|| \to 0$$

as $n \to \infty$. Equivalntly, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. it is well known that a regular cone is a normal cone. Suppose E is a Banach space, P is a cone in E with int $P \neq \phi and \leq$ is partial ordering in E with respect to P.

Lemma 1([22]):

Every regular cone is normal.

Definition 2.2 ([2]): Let X be a nonempty set. Let the mapping $d : X \times X \longrightarrow E$ satisfy:

(i) $0 \leq d(x, y)$, for all $x, y \in \mathcal{X}$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2 ([2]): Let $E = R^2$, $P = \{(x,y) \in R^2 : x \geq 0, y \geq 0\}$, $X = R$ and $d : X \times X \to E$ defined by $d(x,y) = (|x-y), \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space with normal cone P, where $K = 1$.

Definition 2.3 ([3,6]): Let X be a non empty set and $S : X^3 \to [0,\infty)$ be a function satisfying the following conditions for all $x, y, z, t \in X$;

1. $S(x,y,z) \geq 0$;
2. $S(x,y,z) = 0$if and only if $x = y = z$;
3. $S(x,y,z) \leq S(x,x,t) + S(y,y,t) + S(z,z,t)$.

Then the function S is called S-metric on X and the pair (X,S) is called an S-metric space or simplySMS.
Example 3([23]): Let X be a nonempty set and d be the ordering metric on X. Then $S(x,y,z) = d(x,z) + d(y,z)$ is an S-metric on X.

Definition 2.4 ([11]): Suppose that, E is a real Banach space. P is a cone E with int $P \neq \phi$ and $\leq$ is partial ordering with respect to P. Let X be a nonempty set and let the function $S : X^3 \to E$ satisfy the following condtions:

1. $S(x,y,z) \geq 0$;
2. $S(x,y,z) = 0 if and only if x = y = z$;
3. $S(x,y,z) \leq S(x,xa) + S(y,y,a) + (z,z,a)$, for all $x, y, z, a \in X$.

Thenthefunction S is called a cone S-metric on X and the pair (X,S) is called a cone S-metric space or simply$CSMS$.

Example 4 ([11]): Let $E = R^2$, $P = \{(x,y) \in R^2 : x \geq 0, y \geq 0\}$, $X = R$ and d be the ordering metric on X. Then the dunction $S : X^3 \to E$ defined by $S(x,y,z) = (d(x,y) + d(y,z)), \alpha(d(x,z) + d(y,z))$, where $\alpha > 0$ is a cone S-metric space on X.
Lemma 2([11]), Let $S(X,S)$ be a cone S-metric spaces. Then we have $S(x,x,y) = S(y,y,x)$.
Definition 2.5([11]), Let (X,S) be a cone S-metric space Then a

1. sequence $\{x_n\}$ in X converges to $x \in X$if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$, that is, there exists $n_0 \in N$ such that for all $n \geq n_0, S(x_n,x_n,x) \ll c$ for each $c \in E, 0 \ll c$. We denote this
$$\lim_{n\to\infty} x_n = x, or \lim_{n\to\infty} S(x_n,x_n,x) = 0$$
.
2. A sequence $\{x_n\}$ in X is called a Cauchy sequence if $(x_n,x_n,x_m) \to 0$ as $n,m \to \infty$.That is, there exists $n_0 \in N$ such that for all $n,m \geq n_0, S(x_n,x_n,x_m) \ll$, for each $c \in E, 0 \ll c$.
3. The cone S-metric space (X,S) is called complete if every Cauchy sequence is convergent.

Lemma 3 ([8]): Let $T : X \times X \to Y$ be a map from an S-metric space X to an S-metric space Y. Then T is continuous at $x \in X$ iff $T(x_n) \to T(x)$, whenever $x_n \to x$.
Definition 2.6 ([24]), ϕ Maps
: In 1977, Matkowski[24] introduced the ϕ maps as the following:
Let Φ be the set of all functions ϕ such that $\phi : [0,\infty) \to [0,\infty)$ is a nondecreasing function function satisfying
$$\lim_{n\to\infty} \phi^n(t) = 0, forall, t \in (0,\infty)$$
. if $\phi \in \Phi$, then ϕ is called Φ-maps. Furthermore, if ϕ is a Φ map, then

1. $\phi(t) < t$ for all $t \in (0, \infty)$;
2. $\phi(0) = 0$.
   From now on, unless otherwise stated ϕ is meant the Φ-map.

Now, we'll look at some fixed point theorems on cone S-metric spaces using an implicit relation. **Definition 2.6**: Let Ψ be the family of all continuous functions of the five variables $\phi : R_+^5 \to R_+$, for some $K \in [0, 1)$, we consider the following conditions:

1. For all $x, y, z \in R_+$, if $Y \leq \phi(x, x, 0, z, y)$ with $z \leq 2x + y$, then $Y \in kx$.
2. For all $y \in R_+$, if $y \leq \phi(y, 0, y, y, 0)$. Then $y = 0$.
3. If $x_i \leq y_i + z_i$, for all $x_i, y_i, z_i \in R_+$, $i \leq 5$, then
   $\phi(x_1, ...x_5) \leq \phi(y_1....y_5) + \phi(z_1...z_5)$. Moreover, for all $y \in X, \phi(0, 0, 0, y, 2y) \leq 1 < y$.

Remark 1: Note the coefficient $\mathcal{K}$ in conditions (1) and (3) may be different for k_1 and k_3 respectively, But we may assume that they are equal by taking $k = max\{k_1, k_3\}$.

3. Main Results

In this section, we will prove some new unique fixed point theorems in the framework of cone S-metric space for the Phi type of contractive conditions.

Theorem 3.1. *Let (X, S) be a complete cone S-metric space and P be a normal cone with normal constant K. Suppose the mapping $f : X \times X \to X$ satifies the following conditions*

$$S(fx, fx, fy) \leq \phi(S(x, x, y), S(x, fx, fx), S(y, fx, fx), S(x, fy, fy), S(y, fy, fy)), \tag{3.1}$$

for all $x, y \in X$ and some $\phi \in \psi$.
Then we have

1. *If ϕ satisfies the condition (1). Then f has a fixed point. Moreover, for any $w_0 \in X$ and the fixed point w, we have*
$$S(fw_n, fw_n, w) \leq \frac{2(\lambda)^n}{1-\lambda}$$
2. *If ϕ satisfies the condition (2) and f has a fixed point, then the fixed point is unique.*
3. *If ϕ satisfies the condition (3) and f has a fixed point w, f is continuous at w.*

Proof. (i) For each $w_0 \in X$ and $n \in N$, put $w_{n+1} = fw_n$. It follows from (2) and Lemma 2, that

$$\begin{aligned}
S(w_{n+1}, w_{n+1}, w_{n+2} &= S(fw_n, fw_n, fw_{n+1}) \\
&\leq \phi(S(w_n, w_n, w_{n+1}), S(w_n, fw_n, fw_n), S(w_{n+1}, fw_n, fw_n), \\
&\quad S(w_n, fw_{n+1}, fw_{n+1}), S(w_{n+1}, fw_{n+1}, fw_{n+1}) \\
&= \phi(S(w_n, w_n, w_{n+1}), S(w_n, w_{n+1}, w_{n+1}), S(w_{n+1}, w_{n+1}, w_{n+1}), \\
&\quad S(w_{n+1}, w_{n+2}, w_{n+2}), S(w_{n+1}, w_{n+2}, w_{n+2}) \\
&= \phi(S(w_n, w_n, w_{n+1}), S(w_n, w_n, w_{n+1}), 0, S(w_n, w_n, w_{n+2}), \\
&\quad S(w_{n+1}, w_{n+1}, w_{n+2}).
\end{aligned}$$

By the definition 2.4(3) and Lemma 2, we have

$S(w_n, w_n, w_{n+2}) \leq 2S(w_n, w_n, w_{n+1}) + S(w_{n+2}, w_{n+2}, w_{n+1}). \quad = 2S(w_n, w_n, w_{n+1}) + S(w_{n+1}, w_{n+1}, w_{n+2})$

Since ϕ satisfies the condition (1), there exists $\lambda \in [0, 1)$ such that

$$\begin{aligned} S(w_{n+1}, w_{n+1}, w_{n+2}) &\leq S(w_n, w_n, w_{n+1}) \\ &\leq \lambda^{n+1} S(w_0, w_0, w_1). \end{aligned} \tag{3.2}$$

Thus for all $n < m$, by using $Definition 2.4(3)$, Lemma 2 and equation (5), we have

$$\begin{aligned} S(w_n, w_n, w_m) &\leq 2S(w_n, w_n, w_{n+1}) + S(w_m, w_m, w_{n+1}) \\ &= 2S(w_n, w_n, w_{n+1}) + S(w_{n+1}, w_{n+1}, w_m) \\ &\leq ... \leq \\ &\leq 2[\lambda^n + ... + \lambda^{m-1}] S(w_0, w_0, w_1) \\ &\leq \frac{2(\lambda)^n}{1-\lambda} \end{aligned} \tag{3.3}$$

This implies that $||S(w_n, w_n, w_m)|| \leq \frac{2(\lambda)^n}{1-\lambda}.K||S(w_0, w_0, w_1)||$.
Taking the limit as $n, m \to \infty$. we get

$$||(w_n, w_n, w_m)|| \to 0$$

. Since $0 < \lambda < 1$. Thus, we have $S(w_n, w_n, w_m) \to 0$ as $n, m \to \infty$.
This proves that w_n is a Cauchy sequence in the complete S-metric space (X, S). By the completeness of the space, we have

$$\lim_{n\to\infty} w_n = w \in X$$

. Moreover, taking limit as $n \to \infty$, we get

$$S(w_n, w_n, w) \leq \frac{2\lambda^{n+1}}{1-\lambda} S(w_0, w_0, w_1)$$

. It implies that

$$S(fw_n, fw_n, w) \leq \frac{2\lambda^n}{1-\lambda} S(w_0, w_0, fw_0)$$

. Now we prove that w is a fixed point of f. By using inequality (2) again we obtain

$$\begin{aligned} S(w_{n+1}, w_{n+1}, fw &= S(fw_n, fw_n, fw) \\ &\leq \phi(S(w_n, w_n, w), S(w, fw_n, fw_n), S(w_n, fw_n, fw_n), \\ &\quad S(w_n, fw, fw), S(w, fw, fw)) \\ &= \phi(S(w_n, w_n, w), S(w, w_{n+1}, w_{n+1}), \\ &\quad S(w_n, w_{n+1}, w_{n+1}), \end{aligned}$$

Note that $\phi \in \psi$, then using Lemma (3) and taking the limit as $n \to \infty$, we get

$$S(w, w, fw) \leq \phi(0, 0, 0, S(w, fw, fw), S(w, fw, fw))$$

Since ϕ satisfies the definition (3.6) of condition (1), then $S(w, w, fw) \leq \lambda 0 = 0$. This proves that $fw = w$. Thus w is a fixed point of f.

(ii) Let w_1, w_2 be fixed points of f. We shall prove that $w_1 = w_2$. It follows from (2) and Lemma 2 that

$$\begin{aligned} S(w_1,w_1,w_2) &= S(fw_1,fw_1,fw_2) \\ &\leq \phi(S(w_1,w_1,w_2),S(w_1,fw_1,fw_1),S(w_2,fw_1,fw_1), \\ &\quad S(w_1,fw_2,fw_2),S(w_2,fw_2,fw_2)) \\ &= (S(w_1,w_1,w_2),S(w_1,w_1,w_1),S(w_2,w_1,w_1),S(w_1,w_2,w_2),S(w_2,w_2,w_2)) \\ &= (S(w_1,w_1,w_2),0,S(w_2,w_1,w_1),S(w_1,w_2,w_2),0) \\ &= (S(w_1,w_1,w_2),0,S(w_1,w_1,w_2),S(w_1,w_1,w_2),0) \end{aligned}$$

Sinc ϕ satisfies the condition (2), then $S(w_1, w_1, w_2) = 0$. This prove that $w_1 = w_2$.
Thus the fixed point of f is unique.
(iii) Let w be the fixed point of f and $x_n \to w \in X$. By Lemma 3, we need to prove that $fw_n \to fw$. It follows from (2) and Lemma 2 that

$$\begin{aligned} S(w,w,fx_n) &= S(fw,fw,fx_n) \\ &\leq \phi(S(w,w,x_n),S(w,fw,fw),S(x_n,fw,fw),S(w,fx_n,fx_n), \\ &\quad S(x_n,fx_n,fx_n)) \\ &\leq \phi(S(w,w,x_n),S(w,w,w),S(x_n,w,w),S(w,fx_n,fx_n),S(x_n,fx_n,fx_n) \\ &\qquad \text{(by Lemma 2)} \end{aligned}$$

Since ϕ satisfies the condition (3), by Lemma 2 and (CSM_3), we have
$S(fx_n, fx_n, x_n) \leq 2S(w, fx_n, fx_n) + S(w, x_n, x_n)$, then we have

$$\begin{aligned} S(w,w,fx_n) &\leq \phi(S(w,w,x_n),0,S(x_n,w,w),0S(x_n,w,w) \\ &\quad + \phi(0,0,0,S(w,fx_n,fx_n),2S(w,fx_n,fx_n) \\ &\leq \phi(S(w,w,x_n),0,S(x_n,w,w),0,S(x_n,w,w)+\lambda S(w,fx_n,fx_n) \end{aligned}$$

Therefore,
$S(w, w, fx_n) \leq \frac{1}{1-\lambda}.\phi(S(x_n, w, w), 0, S(x_n, w, w,), 0, S(x_n, w, w))$.
Note that $\phi \in \psi$, hence taking limit as $n \to \infty$, we get

$$(fx_n, w, w) \to \infty.$$

This proves that $fx_n \to w = fw$. This completes the proof.

■

Next, we give some anaogues of fixed point theorems in metric spaces for cone S-metric spaces by combining Theorem 1 with $\phi \in \psi$ and ϕ satisfies the definition (2.6) of conditions (1), (2), and (3).
The following Corollary is analogues of Hardy and Rogers result in [25].
Corollary3.2: Let (X, S) be a complete cone S-metric space and P be a normal cone with normal constant K. Suppose the mapping $f : X \times X \to X$ satisfies the followingcondition:
$S(fx, fx, fy) \leq a_1S(x, x, y) + a_2S(x, fx, fx) + a_3S(y, fx, fx) + a_4S(x, fy, fy) + a_5S(y, fy, fy)$ for some $a_1...a_5 \geq 0$ Such that

$$max\{a_1 + a_2 + 3a_4 + a_5, a_1 + a_3 + a_4, a_4 + 2a_5\} < 1$$

and for all $x, y \in X$. Then f has a unique fixed point in X. Moreover, f is continuous at the fixed point.

Proof. The assertion follows using Theorem 1 with $\phi(x,y,z,t) = a_1x + a_2y + a_3z + a_4s + a_5t \geq 0$ for some $a_1...a_5 \geq 0$ such that

$$max\{a_1 + a_2 + 3a_4 + a_5, a_1 + a_3 + a_4 + 2a_5\} < 1$$

and all $x, y, z, t \in R_+$. Indeed, ϕ is continuous.
First we have

$$\phi(x,x,0,z,) = a_1x + a_2x + a_4z + a_5y.$$

So, if $y \leq \phi(x,x,0,z,y)$ with $z \leq 2x + y$, then

$$\begin{aligned} y &\leq a_1x + a_2x + a_4z + a_5y \\ &\leq a_1x + a_2x + a_4(2x+y) + a_5y.Then \\ y &\leq \frac{a_1 + a_2 + 2a_4}{(1 - a_4 - a_5)x} with \frac{a_1 + a_2 + 2a_4}{1 - a_4 - a_5} < 1. \end{aligned} \tag{3.4}$$

Therefore, f satisfies by definition $(3,6)$ of the condition (1). Next

$$\begin{aligned} y &\leq \phi(y,0,y,y,0) \\ &= a_1y + a_3y + a_4y \\ &= (a_1 + a_3 + a_4)y, \end{aligned} \tag{3.5}$$

then $y = 0$. Since $a_1 + a_3 + a_4 < 1$. Therefore, f is satisfies by Definition (3.6) of the condition (2).
Finally, if $x_i \leq y \leq +z_i$, for $i \leq 5$, then

$$\begin{aligned} \phi(x_1,...x_5) &= a_1x_1 + ... + a_5x_5 \\ &\leq a_1(y_1 + z_1) + ... + a_5(y_5 + z_5) \\ &= (a_1y_1 + a_1z_1 + ... + a_5y_5) \\ &= (a_1y_1 + ... + a_5y_5) + (a_1z_1 + ... + a_5z_5) \\ &= \phi(y_1,...,y_5) + \phi(z_1,...,z_5) \end{aligned} \tag{3.6}$$

Moreover,

$$\begin{aligned} \phi(0,0,0,y,2y) &= a_4y + 2a_5y \\ &= (a_4 + 2a_5)y, \text{ where } a_4 + 2a_5 < 1. \end{aligned}$$

Therefore, f is satisfies by definition(3.6) of the condition(3).

■

The following Corollary is an analogue of Reich, S. result in [26].

Corollary 3.3: Let (X,S) be a complete cone S-metric space and$P$ be a normal cone with normal constant K. Suppose the mapping $f : X \times X \to X$ satisfies the followingcondition:
$S(fx,fx,fy) \leq aS(x,x,y) + bS(x,fx,fx) + cS(y,fy,fy)$ for all $x,y \in X$ where $a,b,c,\geq 0$ are constants with $a + b + c < 1$. Then f has a unique fixed point in X. Moreover, if $c < \frac{1}{2}$, then f is continuous at the fixed point.
The following Corollary is an analogur of Kannans resultin [27]:

Corollary 3.4: Let (X,S) be a complete cone S-metric space and P be a normal cone with normal constant K. Suppose the mapping $f : X\times X \to X$ satisfies the followingcondition:
$S(fx,fx,fy) \leq \alpha[S(x,fx,fx)+S(y,fy,fy)]$ for some $\alpha \in [0,\frac{1}{2}$ for all $x,y \in X$. Then f has a unique fixed point in X. Moreover, f is continuous at the fixed point.

Next, the following Corollary is an analogue of Banach contraction principle:

Corollary 3.5: Let (X,S) be a complete cone S-metric space and P be a normal cone with normal constant K. Suppose the mapping $f : X\times X \to X$ satisfies the following condition:

$$S(fx,fx,fy) \leq \lambda S(x,x,y)$$

for some $\lambda \in [0,1)$ for all $x,y \in X$. Then f has a unique fixed point in X. Moreover, f is continuous at the fixed point.

Example 5 Let $E = R^2$, the ecludian plane, $P = (x,y) \in R^2 : x \geq 0, y \geq 0$ a normal cone in E and $X = R$. Then the function $S : X^3 \to E$ defined by $(x,y,z) = |x-z|+|y-z|$ for all $x,y,z \in X$. Then (X,S) is a cone S-metric space. Now we consider the mapping $f : X\times X \to X$ by $fx = \frac{x}{5}$. Then

$$\begin{aligned}
S(fx,fx,fy) &= |fx-fy|+|fx-fy| \\
&= 2|fx-fy| \\
&= 2\left|\left(\frac{x}{5}\right)-\left(\frac{x}{5}\right)\right| \\
&= \frac{2}{5}|x-y| \\
&\leq \frac{1}{4}\left(2|x-y|\right) \\
&= hS(x,x,y),\ \text{where}\ \frac{1}{4} < 1.
\end{aligned}$$

Thus f satisfies all the conditions of the theore 3.1 Corollary with 3.2 and clearly $0 \in X$ is the unique fixed point of f.

Example 3.6 Let $E = R^2$, the ecludian plane, $P = (x,y) \in R^2 : x \geq 0, y \geq 0$ a normal cone in E and $X = R$. Then the function $S : X^3 \to E$ defined by $(x,y,z) = |x-2|+|y-2|$ for all $x,y,z \in X$. Then (X,S) is a cone S-metric space. Now we consider the mapping $f : X\times X \to X$ by $fx = \frac{x}{2}$ and $\{x_n\} = \{\frac{1}{2^n}\}$ for all $n \in N$ is a sequence converging to zero.

4. Conclusion

In the present work, we obtained some unique fixed pont results by using the ϕ- maps type contractive condition in cone S -metric space with an implicit function.Our theorem 1 with corollaries 1,2,3,and 4 of extend and improve some recent results of Sedghi and Dung [8] by idea of Saluja,G.S.[19]. Also gave the example to support our result. Further scope for higher teaching and higher learning mathematics, in particular nonlinear space transformation.

5. Acknowledgement

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Simpson type Katugampola fractional integral inequalities via Harmonic convex functions

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Abstract. In this paper, we attain the new lemma of Simpson type Katugampola fractional integral equality for harmonically convex functions. With the help of this equality, we obtain some new results related to Simpson-like type Katugampola fractional integral inequalities using some inequalities for example power mean inequality and Hölder inequality. Then, we give some conclusions for some special cases of Katugampola fractional integrals when $\rho \to 1$.

AMS Subject Classifications: 26D15, 26D10, 34A08.

Keywords: Simpson inequality, Katugampola fractional integral, harmonic convex.

Contents

1. Introduction

In mathematics, an inequality is a relationship that makes an unequal comparison between two numbers or other mathematical expressions. Inequalities are used in many different areas in real life to facilitate the complexity. For example, businesses use inequalities to control inventory, plan production lines, create pricing models, and move store goods and materials. On the other hand, inequalities are used in engineering and production quality assurance. Therefore, almost all higher mathematical science makes extensive use of inequalities. In the literature, there are some inequalities such as Hermite-Hadamard type inequality, Simpson's type inequality. Simpson's inequality are significantly studied by many mathematicians. It is adapted some kinds of functions for example, convex functions, s-convex functions, harmonic convex functions, readers can see in [1, 2, 7, 9, 10, 12–15, 18, 21].

Now, let we give the following Simpson's inequality we inspire.

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Theorem 1.1. *Let $\Psi : [\varepsilon, \eta] \to \mathbb{R}$ be a four times continuously differentiable mapping on (ε, η) and $\left\|\Psi^{(4)}\right\|_{\infty} = \sup\left|\Psi^{(4)}(x)\right| < \infty$. Then, the following inequality holds:*

$$\left|\int_{\varepsilon}^{\eta} \Psi(w)\, dw - \frac{\eta-\varepsilon}{3}\left[\frac{\Psi(\varepsilon)+\Psi(\eta)}{2} + 2\Psi\left(\frac{\varepsilon+\eta}{2}\right)\right]\right| \leq \frac{1}{2880}\left\|\Psi^{(4)}\right\|_{\infty} \cdot (\eta-\varepsilon)^4. \tag{1.1}$$

In the following definition readers can find the definition of harmonically convex functions.

Definition 1.2. *[3].Let $A \subset \mathbb{R}\setminus\{0\}$ and $\Psi : A \to \mathbb{R}$ be a function. Ψ is said to be harmonically convex, if*

$$\Psi\left(\frac{uv}{tu+(1-t)v}\right) \leq t\Psi(v) + (1-t)\Psi(u) \tag{1.2}$$

for all $u, v \in A$ and $t \in [0,1]$. Otherwise, Ψ is said to be harmonically concave.

Using the above definition, many authors obtained several inequalities for harmonic convex functions [3, 8, 16]. In the literature, one of the most studied inequalities for harmonic convex functions is Hermite-Hadamard, which is stated as follows:

Theorem 1.3. *[3]Let $\Psi : A \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonically convex function and $u, v \in A$ with $u < v$. If $\Psi \in L[u,v]$ then the following inequalities hold:*

$$\Psi\left(\frac{2uv}{u+v}\right) \leq \frac{uv}{v-u}\int_{u}^{v} \frac{\Psi(\eta)}{\eta^2} d\eta \leq \frac{\Psi(u)+\Psi(v)}{2}. \tag{1.3}$$

The main aim of this paper is to establish Simpson type Katugampola fractional integral inequalities for harmonic convex functions.

2. Preliminaries

In this section, we give some definitions and fundamental results we use in our results.

Definition 2.1. *Let $u, v \in \mathbb{R}$ with $u < v$ and $\Psi \in L[u,v]$. The left and right Riemann- Liouville fractional integrals $J^{\alpha}_{u+}\Psi$ and $J^{\alpha}_{v-}\Psi$ of order $\alpha > 0$ are defined by*

$$J^{\alpha}_{u+}\Psi(\varepsilon) = \frac{1}{\Gamma(\alpha)}\int_{u}^{\varepsilon} (\varepsilon-\eta)^{\alpha-1}\Psi(\eta)\, d\eta, \ \ \varepsilon > u$$

and

$$J^{\alpha}_{v-}\Psi(\varepsilon) = \frac{1}{\Gamma(\alpha)}\int_{\varepsilon}^{v} (t-\varepsilon)^{\alpha-1}\Psi(\eta)\, d\eta, \ \ \varepsilon < v$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_{0}^{\infty} e^{-t}t^{\alpha-1}dt$ (see [6], p. 69).

In 2011, Katugampola [4] introduced a new fractional integral operator which generalizes the Riemann-Liouville and Hadamard fractional integrals as follows.

Definition 2.2. *Let $[u, v] \subset \mathbb{R}$ be a finite interval. Then the left and right-side Katugampola fractional integrals of order $\alpha > 0$ of $\Psi \in X_c^p(u, v)$ are defined by*

$$ {}^{\rho}I_{a+}^{\alpha}\Psi(\varepsilon) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{\varepsilon} \frac{\eta^{\rho-1}}{(\varepsilon^{\rho}-\eta^{\rho})^{1-\alpha}}\Psi(\eta)\,d\eta, $$

and

$$ {}^{\rho}I_{b-}^{\alpha}\Psi(\varepsilon) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\varepsilon}^{v} \frac{\eta^{\rho-1}}{(\eta^{\rho}-\varepsilon^{\rho})^{1-\alpha}}\Psi(\eta)\,d\eta, $$

with $u < \varepsilon < v$ and $\rho > 0$, respectively.

If we take $\rho \to 1$ in the Definition 2.2, we obtain the Definition 2.1. For more information about the Katugampola fractional integrals, readers can see the papers [5, 11, 16, 19].

3. Main Results

Along this study, we will use the following notations to make the article easier to read and to avoid the complexity of the calculations.

$$ u_1(t) = \frac{2a^{\rho}b^{\rho}}{(1-t^{\rho})a^{\rho} + (1+t^{\rho})b^{\rho}}, $$

$$ u_2(t) = \frac{2a^{\rho}b^{\rho}}{(1+t^{\rho})a^{\rho} + (1-t^{\rho})b^{\rho}}, $$

$$ H = \frac{2a^{\rho}b^{\rho}}{a^{\rho}+b^{\rho}}. $$

Let's start the following Lemma which helps us to obtain the main results:

Lemma 3.1. *Let $\varphi : I \subset (0,\infty) \to \mathbb{R}$, be a differentiable function on $I^{\circ}, a^{\rho}, b^{\rho} \in I^{\circ}$ and $a < b$. If φ' $\in L[a^{\rho}, b^{\rho}]$, then the following equality holds:*

$$ \begin{aligned} &\frac{1}{6}\left[\varphi(a^{\rho}) + 4\varphi(H) + \varphi(b^{\rho})\right] - 2^{\alpha-1}\rho^{\alpha}\left(\frac{a^{\rho}b^{\rho}}{b^{\rho}-a^{\rho}}\right)^{\alpha}\Gamma(\alpha+1)\left[\begin{array}{c} {}^{\rho}I_{\frac{1}{b}+}^{\alpha}(\varphi\circ\phi)\left(\frac{1}{H}\right) \\ + {}^{\rho}I_{\frac{1}{a}-}^{\alpha}(\varphi\circ\phi)\left(\frac{1}{H}\right) \end{array}\right] \\ &= \rho\frac{b^{\rho}-a^{\rho}}{2a^{\rho}b^{\rho}}\int_0^1 \left(\tfrac{1}{3} - \tfrac{t^{\alpha\rho}}{2}\right)t^{\rho-1}\left[(u_1(t))^2\,\varphi'(u_1(t)) - (u_2(t))^2\,\varphi'(u_2(t))\right]dt \end{aligned} \tag{3.1} $$

and $\phi(x) = \frac{1}{x}, \alpha > 0$.

Proof. We start by considering the following computations which follows from change of variables and using the definition of the Katugampola fractional integrals.

$$
\begin{aligned}
I_1 &= \int_0^1 \left(\frac{1}{3} - \frac{t^{\alpha\rho}}{2}\right) t^{\rho-1} \left(u_1(t)\right)^2 \varphi'\left(u_1(t)\right) dt \\
&= \frac{1}{3\rho} \frac{2a^\rho b^\rho}{a^\rho - b^\rho} \varphi\left(u_1(t)\right)|_0^1 \\
&- \frac{1}{2\rho} \frac{2a^\rho b^\rho}{a^\rho - b^\rho} \left(t^{\alpha\rho} \varphi\left(u_1(t)\right)|_0^1 - \alpha\rho \int_0^1 t^{\alpha\rho-1} \varphi\left(u_1(t)\right) dt \right) \\
&= \frac{1}{3\rho} \frac{2a^\rho b^\rho}{a^\rho - b^\rho} \left(\varphi(a^\rho) - \varphi(H)\right) \\
&- \frac{1}{2\rho} \frac{2a^\rho b^\rho}{a^\rho - b^\rho} \varphi(a^\rho) + \frac{\rho^{\alpha-1}}{2} \left(\frac{2a^\rho b^\rho}{a^\rho - b^\rho}\right)^{\alpha+1} \Gamma(\alpha+1)\ {}^\rho I^\alpha_{\frac{1}{a}-} (\varphi \circ \phi)\left(\tfrac{1}{H}\right) \\
&= \frac{1}{\rho} \frac{2a^\rho b^\rho}{a^\rho - b^\rho} \left(\frac{1}{6}\varphi(a^\rho) + \frac{1}{3}\varphi(H)\right) - \frac{\rho^{\alpha-1}}{2} \left(\frac{2a^\rho b^\rho}{a^\rho - b^\rho}\right)^{\alpha+1} \Gamma(\alpha+1)\ {}^\rho I^\alpha_{\frac{1}{a}-} (\varphi \circ \phi)\left(\tfrac{1}{H}\right)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
I_2 &= \int_0^1 \left(\frac{1}{3} - \frac{t^{\alpha\rho}}{2}\right) t^{\rho-1} \left(u_2(t)\right)^2 \varphi'\left(u_2(t)\right) dt \\
&= \frac{1}{\rho} \frac{2a^\rho b^\rho}{b^\rho - a^\rho} \left(-\frac{1}{6}\varphi(b^\rho) - \frac{1}{3}\varphi(H)\right) + \frac{\rho^{\alpha-1}}{2} \left(\frac{2a^\rho b^\rho}{b^\rho - a^\rho}\right)^{\alpha+1} \Gamma(\alpha+1)\ {}^\rho I^\alpha_{\frac{1}{b}+} (\varphi \circ \phi)\left(\tfrac{1}{H}\right).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\rho \frac{b^\rho - a^\rho}{2a^\rho b^\rho} (I_1 - I_2) \\
&= \frac{1}{6}\left[\varphi(a^\rho) + 4\varphi(H) + \varphi(b^\rho)\right] - 2^{\alpha-1}\rho^\alpha \left(\frac{a^\rho b^\rho}{b^\rho - a^\rho}\right)^\alpha \Gamma(\alpha+1) \begin{bmatrix} {}^\rho I^\alpha_{\frac{1}{b}+} (\varphi \circ \phi)\left(\frac{1}{H}\right) \\ + {}^\rho I^\alpha_{\frac{1}{a}-} (\varphi \circ \phi)\left(\frac{1}{H}\right) \end{bmatrix}.
\end{aligned}
$$

■

Remark 3.2. *If we take $\rho \to 1$ in Lemma 3.1, we have the following equality*

$$
\begin{aligned}
&\frac{1}{6}\left[\varphi(a) + 4\varphi\left(\frac{2ab}{a+b}\right) + \varphi(b)\right] - 2^{\alpha-1} \left(\frac{ab}{b-a}\right)^\alpha \Gamma(\alpha+1) \begin{bmatrix} J^\alpha_{1/b^+} (\varphi \circ \phi)\left(\frac{a+b}{2ab}\right) \\ + I^\alpha_{1/a^-} (\varphi \circ \phi)\left(\frac{a+b}{2ab}\right) \end{bmatrix} \\
&= \frac{b-a}{2ab} \int_0^1 \left(\frac{1}{3} - \frac{t^\alpha}{2}\right) \begin{bmatrix} \left(\frac{2ab}{(1-t)a+(1+t)b}\right)^2 \varphi'\left(\frac{2ab}{(1-t)a+(1+t)b}\right) \\ - \left(\frac{2ab}{(1+t)a+(1-t)b}\right)^2 \varphi'\left(\frac{2ab}{(1+t)a+(1-t)b}\right) \end{bmatrix} dt.
\end{aligned}
\tag{3.2}
$$

Remark 3.3. *If we take $\alpha = 1$ in Remark 3.2, we have the equality [[17], Remark 1].*

Theorem 3.4. *Let $\varphi : I \subset (0, \infty) \to \mathbb{R}$, be a differentiable function on $I^\circ, a^\rho, b^\rho \in I^\circ$ and $a < b$. If $\varphi' \in L[a^\rho, b^\rho]$ and $|\varphi'|$ is a harmonic convex function on $[a^\rho, b^\rho]$, then the following inequality holds:*

$$
\begin{aligned}
&\left| \frac{1}{6}\left[\varphi(a^\rho) + 4\varphi(H) + \varphi(b^\rho)\right] - 2^{\alpha-1}\rho^\alpha \left(\frac{a^\rho b^\rho}{b^\rho - a^\rho}\right)^\alpha \Gamma(\alpha+1) \begin{bmatrix} {}^\rho I^\alpha_{\frac{1}{b}+} (\varphi \circ \phi)\left(\frac{1}{H}\right) \\ + {}^\rho I^\alpha_{\frac{1}{a}-} (\varphi \circ \phi)\left(\frac{1}{H}\right) \end{bmatrix} \right| \\
&\leq \frac{b^\rho - a^\rho}{6a^\rho b^\rho} \left(\left|\varphi'(a^\rho)\right| \left(K_1(w;\alpha) + K_2(w;\alpha)\right) + \left|\varphi'(b^\rho)\right| \left(K_3(w;\alpha) + K_4(w;\alpha)\right) \right)
\end{aligned}
\tag{3.3}
$$

where $\alpha > 0$ and $K_1(w;\alpha), K_2(w;\alpha), K_3(w;\alpha), K_4(w;\alpha)$ are the same as in [[17], Theorem 3.]

Proof. Using Lemma 3.1 and harmonic convexity of $|\varphi'|$, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\varphi(a^\rho)+4\varphi(H)+\varphi(b^\rho)\right]-2^{\alpha-1}\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma(\alpha+1)\left[\begin{array}{c}{}^\rho I^\alpha_{\frac{1}{b}+}(\varphi\circ\phi)\left(\frac{1}{H}\right)\\+{}^\rho I^\alpha_{\frac{1}{a}-}(\varphi\circ\phi)\left(\frac{1}{H}\right)\end{array}\right]\right|\\
&\le \rho\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left((u_1(t))^2\left|\varphi'(u_1(t))\right|+(u_2(t))^2\left|\varphi'(u_2(t))\right|\right)dt\\
&\le \rho\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(\begin{array}{c}(u_1(t))^2\left(\frac{1+t^\rho}{2}\left|\varphi'(a^\rho)\right|+\frac{1-t^\rho}{2}\left|\varphi'(b^\rho)\right|\right)\\+(u_2(t))^2\left(\frac{1-t^\rho}{2}\left|\varphi'(a^\rho)\right|+\frac{1+t^\rho}{2}\left|\varphi'(b^\rho)\right|\right)\end{array}\right)dt\\
&= \frac{b^\rho-a^\rho}{2a^\rho b^\rho}\int_0^1\left|\frac{1}{3}-\frac{w^\alpha}{2}\right|\left(\begin{array}{c}(u_1(w))^2\left(\frac{1+w}{2}\left|\varphi'(a^\rho)\right|+\frac{1-w}{2}\left|\varphi'(b^\rho)\right|\right)\\+(u_2(w))^2\left(\frac{1-w}{2}\left|\varphi'(a^\rho)\right|+\frac{1+w}{2}\left|\varphi'(b^\rho)\right|\right)\end{array}\right)dw\\
&\le \frac{b^\rho-a^\rho}{6a^\rho b^\rho}\left(\left|\varphi'(a^\rho)\right|(K_1(w;\alpha)+K_2(w;\alpha))+\left|\varphi'(b^\rho)\right|(K_3(w;\alpha)+K_4(w;\alpha))\right).
\end{aligned}$$

The last inequality is obtained using where $\left|\frac{1}{3}-\frac{w^\alpha}{2}\right|\le\frac{1}{3}$ for all $u\in[0,1]$. This completes the proof. ■

Remark 3.5. *If we take $\rho\to 1$, we have the inequality [[17], Theorem 3].*

Theorem 3.6. *Let $\varphi : I\subset(0,\infty)\to\mathbb{R}$, be a differentiable function on $I^\circ, a^\rho, b^\rho\in I^\circ$ and $a<b$. If $\varphi'\in L[a^\rho,b^\rho]$ and $|\varphi'|^q$ is a harmonic convex function on $[a^\rho,b^\rho]$ for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\varphi(a^\rho)+4\varphi(H)+\varphi(b^\rho)\right]-2^{\alpha-1}\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma(\alpha+1)\left[\begin{array}{c}{}^\rho I^\alpha_{\frac{1}{b}+}(\varphi\circ\phi)\left(\frac{1}{H}\right)\\+{}^\rho I^\alpha_{\frac{1}{a}-}(\varphi\circ\phi)\left(\frac{1}{H}\right)\end{array}\right]\right| \qquad (3.4)\\
&\le \frac{b^\rho-a^\rho}{6a^\rho b^\rho}\left[\begin{array}{c}\left(X_1(q;a,b)\left|\varphi'(a)\right|^q+X_2(q;a,b)\left|\varphi'(b)\right|^q\right)^{\frac{1}{q}}\\+\left(X_3(q;a,b)\left|\varphi'(a)\right|^q+X_4(q;a,b)\left|\phi'(b)\right|^q\right)^{\frac{1}{q}}\end{array}\right]
\end{aligned}$$

where $\alpha>1$ and $X_1(q;a,b), X_2(q;a,b), X_3(q;a,b), X_4(q;a,b)$ are the same as in [[17], Theorem 4].

Proof. From Lemma 3.1 and using the Hölder's integral inequality and the harmonic convexity of $|\varphi'|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\varphi(a^\rho)+4\varphi(H)+\varphi(b^\rho)\right]-2^{\alpha-1}\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma(\alpha+1)\left[\begin{array}{c}{}^\rho I^\alpha_{\frac{1}{b}+}(\varphi\circ\phi)\left(\frac{1}{H}\right)\\+{}^\rho I^\alpha_{\frac{1}{a}-}(\varphi\circ\phi)\left(\frac{1}{H}\right)\end{array}\right]\right|\\
&\le\rho\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left((u_1(t))^2|\varphi'(u_1(t))|+(u_2(t))^2|\varphi'(u_2(t))|\right)dt\\
&=\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\int_0^1\left|\frac{1}{3}-\frac{w^\rho}{2}\right|\left((u_1(w))^2|\varphi'(u_1(w))|+(u_2(w))^2|\varphi'(u_2(w))|\right)\\
&\le\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\left(\int_0^1\left|\frac{1}{3}-\frac{w^\rho}{2}\right|^p dw\right)^{\frac{1}{p}}\left\{\begin{array}{c}\left(\int_0^1(u_1(w))^{2q}\,|\varphi'(u_1(w))|^q\,dt\right)^{\frac{1}{q}}\\+\left(\int_0^1(u_2(w))^{2q}\,|\varphi'(u_2(w))|^q\,dt\right)^{\frac{1}{q}}\end{array}\right\}dw\\
&\le\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\left\{\begin{array}{c}\left(\int_0^1\left|\frac{1}{3}-\frac{w^\rho}{2}\right|^p dw\right)^{\frac{1}{p}}\\\times\left[\begin{array}{c}\left(\int_0^1(u_1(w))^{2q}\left[|\varphi'(a^\rho)|^q\left(\frac{1+w}{2}\right)+|\varphi'(b^\rho)|^q\left(\frac{1-w}{2}\right)\right]dt\right)^{\frac{1}{q}}\\+\left(\int_0^1(u_2(t))^{2q}\left[|\varphi'(a^\rho)|^q\left(\frac{1-w}{2}\right)+|\varphi'(b^\rho)|^q\left(\frac{1+w}{2}\right)\right]dt\right)^{\frac{1}{q}}\end{array}\right]\end{array}\right\}\\
&\le\frac{b^\rho-a^\rho}{6a^\rho b^\rho}\left[\begin{array}{c}\left(X_1(q;a,b)\,|\varphi'(a^\rho)|^q+X_2(q;a,b)\,|\varphi'(b^\rho)|^q\right)^{\frac{1}{q}}\\+\left(X_3(q;a,b)\,|\varphi'(a^\rho)|^q+X_4(q;a,b)\,|\phi'(b^\rho)|^q\right)^{\frac{1}{q}}\end{array}\right].
\end{aligned}$$

The last inequality is obtained using where $\left|\frac{1}{3}-\frac{w^\alpha}{2}\right|\le\frac{1}{3}$ for all $u\in[0,1]$. This completes the proof. ■

Remark 3.7. *If wetake $\rho\to 1$, we have the inequality [[17], Theorem 4].*

Theorem 3.8. *Let $\varphi: I\subset(0,\infty)\to\mathbb{R}$, be a differentiable function on I°, $a^\rho, b^\rho\in I^\circ$ and $a<b$. If $\varphi'\in$ If φ' $\in L[a^\rho,b^\rho]$ and $|\varphi'|^q$ is a harmonic convex function on $[a^\rho,b^\rho]$ for $q\ge 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\varphi(a^\rho)+4\varphi(H)+\varphi(b^\rho)\right]-2^{\alpha-1}\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma(\alpha+1)\left[\begin{array}{c}{}^\rho I^\alpha_{\frac{1}{b}+}(\varphi\circ\phi)\left(\frac{1}{H}\right)\\+{}^\rho I^\alpha_{\frac{1}{a}-}(\varphi\circ\phi)\left(\frac{1}{H}\right)\end{array}\right]\right| \qquad (3.5)\\
&\le\rho\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\left(\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}dt\right)^{1-\frac{1}{q}}\left[\begin{array}{c}\left(\zeta_1(q,t;\alpha,n)\,|\varphi'(a)|^q+\zeta_2(q,t;\alpha,n)\,|\varphi'(b)|^q\right)^{\frac{1}{q}}\\+\left(\zeta_3(q,t;\alpha,n)\,|\varphi'(a)|^q+\zeta_4(q,t;\alpha,n)\,|\varphi'(b)|^q\right)^{\frac{1}{q}}\end{array}\right]
\end{aligned}$$

where

$$\zeta_1(q,t;\alpha,\rho)=\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}(u_1(t))^{2q}\,\frac{1+t}{2}dt,$$

$$\zeta_2(q,t;\alpha,\rho)=\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(u_1(t)\right)^{2q}\frac{1-t}{2}dt,$$

$$\zeta_3(q,t;\alpha,\rho)=\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(u_2(t)\right)^{2q}\frac{1-t}{2}dt,$$

$$\zeta_4(q,t;\alpha,\rho)=\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(u_2(t)\right)^{2q}\frac{1-t}{2}dt,$$

$\alpha>1$ and $\zeta_1(q,t;\alpha,\rho),\zeta_2(q,t;\alpha,\rho),\zeta_3(q,t;\alpha,\rho),\zeta_4(q,t;\alpha,\rho)$ are the same as in [[17], Theorem 5].

Proof. From Lemma 3.1 and using the power mean inequality, we have that the following inequality holds:

$$\begin{aligned}
&\left|\frac{1}{6}\left[\varphi(a^\rho)+4\varphi(H)+\varphi(b^\rho)\right]-2^{\alpha-1}\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma(\alpha+1)\left[\begin{array}{c}{}^\rho I^\alpha_{\frac{1}{b}+}(\varphi\circ\phi)\left(\frac{1}{H}\right)\\+{}^\rho I^\alpha_{\frac{1}{a}-}(\varphi\circ\phi)\left(\frac{1}{H}\right)\end{array}\right]\right|\\
&\leq\rho\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\int_0^1\left|\left(\tfrac{1}{3}-\tfrac{t^{\alpha\rho}}{2}\right)\right|t^{\rho-1}\left[(u_1(t))^2\left|\varphi'(u_1(t))\right|+(u_2(t))^2\left|\varphi'(u_2(t))\right|\right]dt\\
&\leq\rho\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\left\{\begin{array}{c}\left(\displaystyle\int_0^1\left|\tfrac{1}{3}-\tfrac{t^{\alpha\rho}}{2}\right|t^{\rho-1}dt\right)^{1-\frac{1}{q}}\\ \times\left[\begin{array}{c}\left(\displaystyle\int_0^1\left|\tfrac{1}{3}-\tfrac{t^{\alpha\rho}}{2}\right|t^{\rho-1}(u_1(t))^{2q}\left|\varphi'(u_1(t))\right|^q dt\right)^{\frac{1}{q}}\\+\left(\displaystyle\int_0^1\left|\tfrac{1}{3}-\tfrac{t^{\alpha\rho}}{2}\right|t^{\rho-1}(u_2(t))^{2q}\left|\varphi'(u_1(t))\right|^q dt\right)^{\frac{1}{q}}\end{array}\right]\end{array}\right\}.
\end{aligned}$$

By the harmonic convexity of $|\varphi'|^q$

$$\begin{aligned}
&\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}(u_1(t))^{2q}\left|\varphi'(u_1(t))\right|^q dt\\
&\leq\left|\varphi'(a)\right|^q\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}(u_1(t))^{2q}\frac{1+t}{2}dt\\
&+\left|\varphi'(b)\right|^q\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}(u_1(t))^{2q}\frac{1-t}{2}dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right| t^{\rho-1}\left(u_2(t)\right)^{2q}\left|\varphi'\left(u_2(t)\right)\right|^q dt \\
&\leq \left|\varphi'(a)\right|^q \int_0^1 \left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right| t^{\rho-1}\left(u_2(t)\right)^{2q}\frac{1-t}{2}dt \\
&+\left|\varphi'(b)\right|^q \int_0^1 \left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right| t^{\rho-1}\left(u_2(t)\right)^{2q}\frac{1+t}{2}dt.
\end{aligned}$$

Using the last two inequalities we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[\varphi(a^\rho)+4\varphi(H)+\varphi(b^\rho)\right]-2^{\alpha-1}\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma(\alpha+1)\left[\begin{array}{c} {}^\rho I^\alpha_{\frac{1}{b}+}(\varphi\circ\phi)\left(\frac{1}{H}\right) \\ +{}^\rho I^\alpha_{\frac{1}{a}-}(\varphi\circ\phi)\left(\frac{1}{H}\right)\end{array}\right]\right| \\
&\leq \rho\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\left(\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right| t^{\rho-1}dt\right)^{1-\frac{1}{q}} \\
&\times\left[\begin{array}{c}\left(\begin{array}{c}\left|\varphi'(a)\right|^q\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(u_1(t)\right)^{2q}\frac{1+t}{2}dt \\ +\left|\varphi'(b)\right|^q\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(u_1(t)\right)^{2q}\frac{1-t}{2}dt\end{array}\right)^{\frac{1}{q}} \\ +\left(\begin{array}{c}\left|\varphi'(a)\right|^q\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(u_2(t)\right)^{2q}\frac{1-t}{2}dt \\ +\left|\varphi'(b)\right|^q\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(u_2(t)\right)^{2q}\frac{1+t}{2}dt\end{array}\right)^{\frac{1}{q}}\end{array}\right].
\end{aligned}$$

■

Remark 3.9. *If wetake $\rho\to 1$, we have the inequality [[17], Theorem 5].*

4. Conclusion

In this paper, using a new identity of Simpson-like type for Katugampola fractional integral for harmonic convex functions, we obtained some new integral inequalities related to Simpson inequalities. Furthermore, some interesting conclusions were obtained for some special values of ρ. This study generalizes the paper [17].

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Some properties of b-linear functional in linear n-normed space

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Abstract. Some results in linear n-normed space have been discussed. Several nice properties of bounded b-linear functional in linear n-normed space are presented. We see that the collection of all bounded b-linear functionals after introducing suitable operations, is a normed space.

AMS Subject Classifications: 46A22, 46A32, 46B07, 46B25.

Keywords: n-normed space, n-Banach space, b-linear functional, hyperplane, convex set, sequentially continuous.

Contents

1. Introduction and Background

If one is interested in the study of a certain class of mathematical objects, it is natural and fruitful to investigate the set of operators from one such class to another which preserve some or all of the structures defined on the objects. The linear operators from one normed space to another normed space over the same scalar field preserve the algebraic structure. The fact that a normed space gives rise to a metric, induced by the norm, provides naturally to the extremely important classification of linear operators into continuous and discontinuous ones. In normes spaces, this distinction is facilitated by a very simple criterion for continuity; namely, any linear operator between normed space is continuous if and only if it is bounded. The collection of all these bounded linear operators is a normed space.

The idea of linear 2-normed space was first introduced by S. Gahler [5] and thereafter the geometric structure of linear 2-normed spaces was developed by A. White, Y. J. Cho, R. W. Freese [1, 9]. The concept of 2-Banach space is briefly discussed in [9]. In recent times, some important results in classical normed spaces have been proved into 2-norm setting by many researchers. P. Ghosh et al. [4] studied some fundamental theorem in classical normed space into 2-normed space. H. Gunawan and Mashadi [6] developed the generalization of a linear 2-normed space for $n \geq 2$. Some results of classical normed space with respect to b-linear functional in linear n-normed space were established by P. Ghosh and T. K. Samanta [2]. They also studied the reflexivity of linear n-normed space with respect to b-linear functional [3].

In this paper, some results in linear n-normed space are being described. We shall verify that the collection of all bounded b-linear functionals after introducing suitable operations, is a normed space. Some properties of bounded b-linear functional are going to be established.

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Definition 1.1. *[7] Let X be a linear space and M be subspace of X. Then M is said to be a convex set if $x, y \in M$, $tx + (1-t)y \in M$ for $0 \le t \le 1$.*

Definition 1.2. *[7] A set M in a linear space X is called a hyperplane if X can be expressed as a direct sum of M and one-dimensional subspace of X, i. e., $X = M + \langle x \rangle$, for some $x \in X$.*

Definition 1.3. *[6] Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the real or complex numbers field with dim $X \ge n$, where n is a positive integer. A real valued function $\| \cdot, \cdots, \cdot \| : X^n \to \mathbb{R}$ is called an n-norm on X if*

(i) $\| x_1, x_2, \cdots, x_n \| = 0$ *if and only if $x_1, \cdots, x_n$ are linearly dependent,*

(ii) $\| x_1, x_2, \cdots, x_n \|$ *is invariant under permutations of $x_1, x_2, \cdots, x_n$,*

(ii) $\| \alpha x_1, x_2, \cdots, x_n \| = |\alpha| \, \| x_1, x_2, \cdots, x_n \| \quad \forall \; \alpha \in \mathbb{K}$,

(iv) $\| x + y, x_2, \cdots, x_n \| \le \| x, x_2, \cdots, x_n \| + \| y, x_2, \cdots, x_n \|$

hold for all $x, y, x_1, x_2, \cdots, x_n \in X$. The pair $(X, \| \cdot, \cdots, \cdot \|)$ is then called a linear n-normed space.

Throughout this paper, X will denote linear n-normed space over the field $\mathbb{K}$ of complex or real numbers, associated with the n-norm $\| \cdot, \cdots, \cdot \|$.

Definition 1.4. *[6] A sequence $\{x_k\} \subseteq X$ is said to converge to $x \in X$ if*

$$\lim_{k \to \infty} \| x_k - x, e_2, \cdots, e_n \| = 0$$

for every $e_2, \cdots, e_n \in X$ and it is called a Cauchy sequence if

$$\lim_{l, k \to \infty} \| x_l - x_k, e_2, \cdots, e_n \| = 0$$

for every $e_2, \cdots, e_n \in X$. The space X is said to be complete or n-Banach space if every Cauchy sequence in this space is convergent in X.

Definition 1.5. *[8] For $a \in X$, define the following open and closed ball in X:*

$$B_{\{e_2, \cdots, e_n\}}(a, \delta) = \{ x \in X : \| x - a, e_2, \cdots, e_n \| < \delta \} \text{ and}$$

$$B_{\{e_2, \cdots, e_n\}}[a, \delta] = \{ x \in X : \| x - a, e_2, \cdots, e_n \| \le \delta \},$$

for every $e_2, \cdots, e_n \in X$ and δ be a positive number.

Definition 1.6. *[8] A subset G of X is said to be open in X if for all $a \in G$, there exist $e_2, \cdots, e_n \in X$ and $\delta > 0$ such that $B_{\{e_2, \cdots, e_n\}}(a, \delta) \subseteq G$.*

Definition 1.7. *[8] Let $A \subseteq X$. Then the closure of A is defined as*

$$\overline{A} = \left\{ x \in X \mid \exists \, \{x_k\} \in A \text{ with } \lim_{k \to \infty} x_k = x \right\}.$$

The set A is said to be closed if $A = \overline{A}$.

Definition 1.8. *[2] Let W be a subspace of X and $b_2, b_3, \cdots, b_n$ be fixed elements in X and $\langle b_i \rangle$ denote the subspaces of X generated by b_i, for $i = 2, 3, \cdots, n$. Then a map $T : W \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle \to \mathbb{K}$ is called a b-linear functional on $W \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$, if for every $x, y \in W$ and $k \in \mathbb{K}$, the following conditions hold:*

$(i)\quad T(x+y, b_2, \cdots, b_n) = T(x, b_2, \cdots, b_n) + T(y, b_2, \cdots, b_n)$

$(ii)\quad T(kx, b_2, \cdots, b_n) = k\,T(x, b_2, \cdots, b_n)$.

A b-linear functional is said to be bounded if $\exists$ a real number $M > 0$ such that

$$|T(x, b_2, \cdots, b_n)| \leq M\,\|x, b_2, \cdots, b_n\| \;\; \forall\, x \in W.$$

The norm of the bounded b-linear functional T is defined by

$$\|T\| = \inf\{M > 0 : |T(x, b_2, \cdots, b_n)| \leq M\,\|x, b_2, \cdots, b_n\| \;\; \forall\, x \in W\}.$$

2. Main Results

Theorem 2.1. *Let X be a linear n-normed space. Then*

$$|\,\|x, e_2, \cdots, e_n\| - \|y, e_2, \cdots, e_n\|\,| \leq \|x - y, e_2, \cdots, e_n\|$$

hold for all $x, y, e_2, e_3, \cdots, e_n \in X$.

Proof. Take $x, y, e_2, \cdots, e_n \in X$. Then

$$\begin{aligned}
&\|x, e_2, \cdots, e_n\| = \|x - y + y, e_2, \cdots, e_n\| \\
&\leq \|x - y, e_2, \cdots, e_n\| + \|y, e_2, \cdots, e_n\| \\
&\Rightarrow \|x, e_2, \cdots, e_n\| - \|y, e_2, \cdots, e_n\| \leq \|x - y, e_2, \cdots, e_n\|.
\end{aligned}$$

Also interchanging x and y we get

$$-(\|x, e_2, \cdots, e_n\| - \|y, e_2, \cdots, e_n\|) \leq \|x - y, e_2, \cdots, e_n\|.$$

Combining the above two inequality the result follows. ■

Theorem 2.2. *Let M be a subspace of a linear n-normed space X. Then the closure $\overline{M}$ of M is also subspace.*

Proof. Let $x, y \in \overline{M}$. Then corresponding to $\epsilon > 0$, $\exists\, u, v \in M$ such that

$$\|x - u, e_2, \cdots, e_n\| < \epsilon \text{ and } \|y - v, e_2, \cdots, e_n\| < \epsilon$$

for every $e_2, e_3, \cdots, e_n \in X$. Now, for non zero scalars α, β,

$$\begin{aligned}
&\|(\alpha x + \beta y) - (\alpha u + \beta v), e_2, \cdots, e_n\| \\
&= \|\alpha(x - u) + \beta(y - v), e_2, \cdots, e_n\| \\
&\leq |\alpha|\,\|x - u, e_2, \cdots, e_n\| + |\beta|\,\|y - v, e_2, \cdots, e_n\| \\
&< \epsilon(|\alpha| + |\beta|) = \epsilon', \text{ say}
\end{aligned}$$

This shows that $\alpha u + \beta v \in B_{\{e_2, \cdots, e_n\}}(\alpha x + \beta y, \epsilon')$. As $\alpha u + \beta v \in M$ and $\epsilon' > 0$ is arbitrary, it follows that $\alpha x + \beta y \in \overline{M}$. Hence the proof. ■

Theorem 2.3. *The sets $B_{\{e_2, \cdots, e_n\}}[a, \delta]$ and $B_{\{e_2, \cdots, e_n\}}(a, \delta)$ in a linear n-normed space X are convex.*

Proof. Let $x, y \in B_{\{e_2,\cdots,e_n\}}[a, \delta]$. Then

$$\|x - a, e_2, \cdots, e_n\| \leq \delta \text{ and } \|y - a, e_2, \cdots, e_n\| \leq \delta.$$

Take $z = tx + (1-t)y \in M$ for $0 \leq t \leq 1$. Then we have

$$\begin{aligned}
\|z - a, e_2, \cdots, e_n\| &= \|tx + (1-t)y - a, e_2, \cdots, e_n\| \\
&= \|tx + (1-t)y - ta - (1-t)a, e_2, \cdots, e_n\| \\
&\leq \|tx - ta, e_2, \cdots, e_n\| + \|(1-t)y - (1-t)a, e_2, \cdots, e_n\| \\
&= t\|x - a, e_2, \cdots, e_n\| + (1-t)\|y - a, e_2, \cdots, e_n\| \leq t\delta + (1-t)\delta = \delta.
\end{aligned}$$

So, $z \in B_{\{e_2,\cdots,e_n\}}[a, \delta]$. This shows that $B_{\{e_2,\cdots,e_n\}}[a, \delta]$ is a convex set. Similarly, it can be shown that $B_{\{e_2,\cdots,e_n\}}(a, \delta)$ is also a convex set. This completes the proof. ■

Theorem 2.4. *Let X be a linear n-normed space and M be a convex subset of X. Then the closure of M, $\overline{M}$ is convex.*

Proof. Let $x, y \in \overline{M}$. Then corresponding to $\epsilon > 0$, $\exists\, u, v \in M$ such that

$$\|x - u, e_2, \cdots, e_n\| < \epsilon \text{ and } \|y - v, e_2, \cdots, e_n\| < \epsilon$$

for every $e_2, e_3, \cdots, e_n \in X$. Let $0 \leq t \leq 1$, then

$$\|\{tx + (1-t)y\} - \{tu + (1-t)v\}, e_2, \cdots, e_n\|$$

$$\leq t\|x - u, e_2, \cdots, e_n\| + (1-t)\|y - v, e_2, \cdots, e_n\| < t\epsilon + (1-t)\epsilon = \epsilon.$$

Since M is convex, $tu + (1-t)v \in M$ and because $\epsilon > 0$ is arbitrary, $tx + (1-t)y \in \overline{M}$. Hence, $\overline{M}$ is convex. ■

Theorem 2.5. *Let X be a linear n-normed space and x_0 be a fixed element in X and $\alpha \neq 0$ be a fixed scalar. Then the mappings $x \to x_0 + x$ and $x \to \alpha x$ are sequentially continuous.*

Proof. Let $\{x_k\}$ be a sequence in X such that $x_k \to x$ as $k \to \infty$. Then

$$\lim_{k\to\infty} \|x_k - x, e_2, \cdots, e_n\| = 0 \;\forall\; e_2, \cdots, e_n \in X. \tag{2.1}$$

Firstly, we consider the mapping, $f(x) = x_0 + x$. Then,

$$\begin{aligned}
\|f(x_k) - f(x), e_2, \cdots, e_n\| &= \|(x_0 + x_k) - (x_0 + x), e_2, \cdots, e_n\| \\
&= \|x_k - x, e_2, \cdots, e_n\|
\end{aligned}$$

Taking limit both sides as $k \to \infty$, we get

$$\begin{aligned}
\lim_{k\to\infty} \|f(x_k) - f(x), e_2, \cdots, e_n\| &= \lim_{k\to\infty} \|x_k - x, e_2, \cdots, e_n\| \\
&= 0 \;[\text{using } (2.1)].
\end{aligned}$$

This shows that $f(x_k) \to f(x)$ as $k \to \infty$.

Now, take $g(x) = \alpha x$. Then, for each $e_2, \cdots, e_n \in X$, we have

$$\begin{aligned}
\|g(x_k) - g(x), e_2, \cdots, e_n\| &= \|\alpha x_k - \alpha x, e_2, \cdots, e_n\| \\
&= \|\alpha(x_k - x), e_2, \cdots, e_n\| \\
&= |\alpha|\,\|x_k - x, e_2, \cdots, e_n\|
\end{aligned}$$

So by (2.1), $g(x_k) \to g(x)$ as $k \to \infty$. Therefore, the mappings $x \to x_0 + x$ and $x \to \alpha x$ are sequentially continuous. ■

3. Main results

In this section, some properties of bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ are discussed.

Theorem 3.1. *Let T be a bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. Then*

(i) $|T(x, b_2, \cdots, b_n)| \leq \|T\| \|x, b_2, \cdots, b_n\| \;\; \forall x \in X$.

(ii) $\|T\| = \sup\{|T(x, b_2, \cdots, b_n)| : \|x, b_2, \cdots, b_n\| \leq 1\}$.

(iii) $\|T\| = \sup\{|T(x, b_2, \cdots, b_n)| : \|x, b_2, \cdots, b_n\| = 1\}$.

(iv) $\|T\| = \sup\left\{\dfrac{|T(x, b_2, \cdots, b_n)|}{\|x, b_2, \cdots, b_n\|} : \|x, b_2, \cdots, b_n\| \neq 0\right\}$.

Proof. (i) For arbitrary $\epsilon > 0$, it follows by the definition of norm of T that

$$|T(x, b_2, \cdots, b_n)| \leq (\|T\| + \epsilon)\|x, b_2, \cdots, b_n\| \;\; \forall x \in X. \tag{3.1}$$

If possible, suppose that there exists $x_1 \in X$ such that

$$|T(x_1, b_2, \cdots, b_n)| > \|T\| \, \|x_1, b_2, \cdots, b_n\|.$$

Then for some $\epsilon > 0$,

$$\begin{aligned} |T(x_1, b_2, \cdots, b_n)| &> \|T\| \, \|x_1, b_2, \cdots, b_n\| + \epsilon \, \|x_1, b_2, \cdots, b_n\| \\ &= (\|T\| + \epsilon)\|x_1, b_2, \cdots, b_n\| \end{aligned}$$

which contradicts (3.1). Hence

$$|T(x, b_2, \cdots, b_n)| \leq \|T\| \|x, b_2, \cdots, b_n\| \;\; \forall x \in X.$$

(ii) If $\|x, b_2, \cdots, b_n\| \leq 1$, then

$$\begin{aligned} &|T(x, b_2, \cdots, b_n)| \leq \|T\| \|x, b_2, \cdots, b_n\| \leq \|T\| \\ \Rightarrow\; &\sup\{|T(x, b_2, \cdots, b_n)| : \|x, b_2, \cdots, b_n\| \leq 1\} \leq \|T\|. \end{aligned} \tag{3.2}$$

On the other hand, by definition, for every $\epsilon > 0$ there exists $x' \neq \theta$ such that

$$|T(x', b_2, \cdots, b_n)| > (\|T\| - \epsilon) \, \|x', b_2, \cdots, b_n\|.$$

Take $x_1 = \dfrac{x'}{\|x', b_2, \cdots, b_n\|}$, then we get

$$\begin{aligned} |T(x_1, b_2, \cdots, b_n)| &= \frac{1}{\|x', b_2, \cdots, b_n\|} |T(x', b_2, \cdots, b_n)| \\ &> \frac{1}{\|x', b_2, \cdots, b_n\|} (\|T\| - \epsilon) \, \|x', b_2, \cdots, b_n\| = \|T\| - \epsilon. \end{aligned}$$

Since $\|x_1, b_2, \cdots, b_n\| = 1$, we get

$$\sup\{|T(x, b_2, \cdots, b_n)| : \|x, b_2, \cdots, b_n\| \leq 1\} \geq |T(x_1, b_2, \cdots, b_n)|$$

$$> \| T \| - \epsilon \geq \| T \| \ [\text{ since } \epsilon > 0 \text{ is arbitrary }].$$

Combining this with (3.2), the proof of (II) is complete.

(iii) The proof is follows from (II), replacing $\| x, b_2, \cdots, b_n \| \leq 1$ by $\| x, b_2, \cdots, b_n \| = 1$.

(iv) Let $\alpha = \sup \left\{ \dfrac{| T(x, b_2, \cdots, b_n) |}{\| x, b_2, \cdots, b_n \|} : \| x, b_2, \cdots, b_n \| \neq 0 \right\}$. Now, for arbitrary $\epsilon > 0$, there exists an elements $x_1 \neq \theta$ with $x_1, b_2, \cdots, b_n$ are linearly independent such that

$$| T(x_1, b_2, \cdots, b_n) | > (\alpha - \epsilon) \| x_1, b_2, \cdots, b_n \|.$$

It follows from the definition of norm that $\| T \| > \alpha - \epsilon$ and since $\epsilon > 0$ is arbitrary, we obtain $\| T \| \geq \alpha$. If possible, suppose that $\| T \| > \alpha$.
Let $\epsilon = | T(x_1, b_2, \cdots, b_n) | - \alpha$, then $\alpha < \| T \| - \dfrac{\epsilon}{2}$. So, for arbitrary x,

$$\frac{| T(x, b_2, \cdots, b_n) |}{\| x, b_2, \cdots, b_n \|} \leq \alpha < \| T \| - \frac{\epsilon}{2}$$

$$\Rightarrow | T(x, b_2, \cdots, b_n) | < \left(\| T \| - \frac{\epsilon}{2} \right) \| x, b_2, \cdots, b_n \| \ \ \forall x \in X.$$

This contradicts the fact that $\| T \|$ is the lower bound of all those M for which

$$| T(x, b_2, \cdots, b_n) | \leq M \| x, b_2, \cdots, b_n \| \ \ \forall x \in X.$$

So $\| T \|$ cannot be greater that α, i. e., $\| T \| = \alpha$. This proves the theorem. ■

Theorem 3.2. *The set X_F^* of all bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ is a linear space.*

Proof. Let $S, T \in X_F^*$. Then there exists $L, M > 0$ such that

$$| S(x, b_2, \cdots, b_n) | \leq L \| x, b_2, \cdots, b_n \|, \text{ and}$$

$$| T(x, b_2, \cdots, b_n) | \leq M \| x, b_2, \cdots, b_n \| \ \ \forall x \in X.$$

$$\Rightarrow | (S + T)(x, b_2, \cdots, b_n) | \leq | S(x, b_2, \cdots, b_n) | + | T(x, b_2, \cdots, b_n) |$$
$$\leq (L + M) \| x, b_2, \cdots, b_n \| \ \ \forall x \in X, \text{ and}$$
$$\Rightarrow | (\lambda T)(x, b_2, \cdots, b_n) | \leq | \lambda | M \| x, b_2, \cdots, b_n \| \ \ \forall x \in X \text{ and } \lambda \in \mathbb{K}.$$

This shows that $S + T \in X_F^*$ and $\lambda T \in X_F^*$. Hence, X_F^* is a linear space. ■

Theorem 3.3. *Let X_F^* be the linear space of all bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. Define $\| \cdot \| : X_F^* \to \mathbb{R}$ by*

$$\| T \| = \sup \{ | T(x, b_2, \cdots, b_n) | : x \in X, \| x, b_2, \cdots, b_n \| \leq 1 \}.$$

Then $(X_F^*, \| \cdot \|)$ is a Banach space.

Proof. Since every $T \in X_F^*$ is bounded b-linear functional, the norm $\| \cdot \|$ on X_F^* is well defined.

(i) $\| T \| \geq 0 \ \forall T \in X_F^*$.

(ii) $\| T \| = 0 \Rightarrow | T(x, b_2, \cdots, b_n) | = 0, \forall x \in X, \| x, b_2, \cdots, b_n \| \leq 1$

$$\Rightarrow T(x, b_2, \cdots, b_n) = 0, \forall x \in X, \| x, b_2, \cdots, b_n \| \leq 1.$$

Let $x \in X$ with $\| x, b_2, \cdots, b_n \| = \alpha > 1$ and suppose $y = \dfrac{x}{\beta}$, where $\beta > \alpha$. Then

$$\| y, b_2, \cdots, b_n \| = \frac{\| x, b_2, \cdots, b_n \|}{\beta} = \frac{\alpha}{\beta} < 1$$

and so $T(y, b_2, \cdots, b_n) = 0$. But

$$0 = T(y, b_2, \cdots, b_n) = T\left(\frac{x}{\beta}, b_2, \cdots, b_n\right) = \frac{1}{\beta} T(x, b_2, \cdots, b_n).$$

Therefore $T(x, b_2, \cdots, b_n) = 0$. So, $T(x, b_2, \cdots, b_n) = 0, \forall x \in X$ and therefore $T = 0$. Conversely, if $T = 0$, then clearly $\| T \| = \| 0 \| = 0$.

(iii) For any $\lambda \in \mathbb{K}$, we have

$$\| \lambda T \| = \sup \{ | (\lambda T)(x, b_2, \cdots, b_n) | : x \in X, \| x, b_2, \cdots, b_n \| \leq 1 \}$$
$$= \sup \{ |\lambda| \, | T(x, b_2, \cdots, b_n) | : x \in X, \| x, b_2, \cdots, b_n \| \leq 1 \} = |\lambda| \, \| T \|.$$

(iv) For $T, S \in X_F^*$, we have

$$\| T + S \| = \sup \{ | (T + S)(x, b_2, \cdots, b_n) | : x \in X, \| x, b_2, \cdots, b_n \| \leq 1 \}$$
$$\leq \sup \{ | T(x, b_2, \cdots, b_n) | : x \in X, \| x, b_2, \cdots, b_n \| \leq 1 \} +$$
$$\sup \{ | S(x, b_2, \cdots, b_n) | : x \in X, \| x, b_2, \cdots, b_n \| \leq 1 \}$$
$$= \| T \| + \| S \|.$$

Therefore $\| \cdot \|$ defines a norm on X_F^*.

To prove the completeness of X_F^* under the norm $\| \cdot \|$, let $\{ T_k \}$ be a Cauchy sequence in X_F^*. Now, for every $x \in X$, we have

$$| T_l(x, b_2, \cdots, b_n) - T_k(x, b_2, \cdots, b_n) | = | (T_l - T_k)(x, b_2, \cdots, b_n) |$$
$$\leq \| T_l - T_k \| \, \| x, b_2, \cdots, b_n \|.$$

This calculation shows that $\{ T_k(x, b_2, \cdots, b_n) \}$ is a Cauchy sequence in $\mathbb{K}$ for each $x \in X$. Since $\mathbb{K}$ is complete, $\{ T_k(x, b_2, \cdots, b_n) \}$ converges in $\mathbb{K}$. Let $T_k(x, b_2, \cdots, b_n) \to T(x, b_2, \cdots, b_n)$. We shall now show that $T \in X_F^*$.

(i) T is b-linear: For $x, y \in X$ and $\lambda \in \mathbb{K}$, we have

$$T(x + y, b_2, \cdots, b_n) = \lim_{k \to \infty} T_k(x + y, b_2, \cdots, b_n)$$
$$= \lim_{k \to \infty} T_k(x, b_2, \cdots, b_n) + \lim_{k \to \infty} T_k(y, b_2, \cdots, b_n) \; [\text{ since } T_k \text{ is } b\text{-linear }]$$
$$= T(x, b_2, \cdots, b_n) + T(y, b_2, \cdots, b_n),$$

and $T(\lambda x, b_2, \cdots, b_n) = \lim_{k \to \infty} T_k(\lambda x, b_2, \cdots, b_n)$

$$= \lambda \lim_{k \to \infty} T_k(x, b_2, \cdots, b_n) = \lambda T(x, b_2, \cdots, b_n).$$

This verifies that T is a b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$.

(ii) Since for each k, T_k is bounded b-linear functional, it follows that

$$|T_k(x, b_2, \cdots, b_n)| \leq \|T_k\| \; \|x, b_2, \cdots, b_n\| \;\; \forall\, x \in X.$$

Also, the sequence $\{T_k\}$ being Cauchy sequence in X_F^*, it is bounded and hence there is a constant $K > 0$ such that $\|T_k\| \leq K \;\forall\, k \in \mathbb{N}$. Consequently,

$$|T_k(x, b_2, \cdots, b_n)| \leq K \,\|x, b_2, \cdots, b_n\| \;\; \forall\, x \in X \text{ and } k \in \mathbb{N}.$$

Thus, for each $x \in X$ and $k \in \mathbb{N}$, we have $|T\,(x, b_2, \cdots, b_n)|$

$$\leq |T\,(x, b_2, \cdots, b_n) - T_k(x, b_2, \cdots, b_n)| + |T_k(x, b_2, \cdots, b_n)|$$

$$\leq |T\,(x, b_2, \cdots, b_n) - T_k(x, b_2, \cdots, b_n)| + K\,\|x, b_2, \cdots, b_n\|.$$

Since k is arbitrary, letting $k \to \infty$ and using $T_k(x, b_2, \cdots, b_n) \to T(x, b_2, \cdots, b_n)$, we obtain

$$|T(x, b_2, \cdots, b_n)| \leq K\,\|x, b_2, \cdots, b_n\| \;\; \forall\, x \in X.$$

Hence, T is bounded. Therefore (i) and (ii) verify that $T \in X_F^*$.

Finally, we show that $T_k \to T$ in $(X_F^*, \|\cdot\|)$. Since $\{T_k\}$ is a Cauchy sequence in X_F^*, for each $\epsilon > 0$, there exists an integer $N > 0$ such that

$$\|T_l - T_k\| < \epsilon \;\forall\; k, l \leq N. \text{ Therefore}$$

$$|T_l\,(x, b_2, \cdots, b_n) - T_k(x, b_2, \cdots, b_n)| < \epsilon\,\|x, b_2, \cdots, b_n\| \;\; \forall\, x \in X.$$

Taking $l \to \infty$, for all $k \geq N$, we get

$$|T\,(x, b_2, \cdots, b_n) - T_k(x, b_2, \cdots, b_n)| < \epsilon\,\|x, b_2, \cdots, b_n\| \;\; \forall\, x \in X.$$

This gives, $\|T_k - T\|$

$$= \sup\{|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| : x \in X, \|x, b_2, \cdots, b_n\| \leq 1\}$$

$$\leq \epsilon \;\forall\, k \geq N.$$

Hence, $T_k \to T$, as $k \to \infty$, in $(X_F^*, \|\cdot\|)$. This completes the proof. ■

Theorem 3.4. *Let X be a linear n-normed space and T be a b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. Then T is bounded if and only if T maps bounded sets in X into bounded sets in $\mathbb{K}$.*

Proof. Suppose T is bounded and S is any bounded subset of X. Then, there exists $M_1 > 0$ such that

$$|T(x, b_2, \cdots, b_n)| \leq M_1\,\|x, b_2, \cdots, b_n\| \;\; \forall\, x \in X$$

and in particular, $\forall\, x \in S$. The set S being bounded, for some real number $M > 0$, we have

$$|T(x, b_2, \cdots, b_n)| \leq M \;\; \forall\, x \in S \Rightarrow \text{ the set } \{T(x, b_2, \cdots, b_n) : x \in S\}$$

is bounded in $\mathbb{K}$ and hence T maps bounded sets in X into bounded sets in $\mathbb{K}$.

Conversely, for the closed unit ball

$$B_{\{e_2, \cdots, e_n\}}[0, 1] = \{x \in X : \|x, e_2, \cdots, e_n\| \leq 1\},$$

the set $\{T(x, b_2, \cdots, b_n) : x \in B_{\{e_2, \cdots, e_n\}}[0, 1]\}$ is bounded set in $\mathbb{K}$. Therefore, there exists $K > 0$ such that

$$|T(x, b_2, \cdots, b_n)| \leq K \quad \forall x \in B_{\{e_2, \cdots, e_n\}}[0, 1].$$

If $x = 0$, then $T(x, b_2, \cdots, b_n) = 0$ and the assertion

$$|T(x, b_2, \cdots, b_n)| \leq K \|x, b_2, \cdots, b_n\|$$ is obviouly true. If $x \neq 0$, then

$\dfrac{x}{\|x, e_2, \cdots, e_n\|} \in B_{\{e_2, \cdots, e_n\}}[0, 1]$, and for particular $e_2 = b_2, \cdots, e_n = b_n$

$$\left| T\left(\frac{x}{\|x, b_2, \cdots, b_n\|}, b_2, \cdots, b_n \right) \right| \leq K$$

$$\Rightarrow |T(x, b_2, \cdots, b_n)| \leq K \|x, b_2, \cdots, b_n\| \quad \forall x \in X.$$

Hence, T is a bounded b-linear functional. ∎

Theorem 3.5. *Let X be a linear n-normed space and T be a b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. Then T is bounded if the set Ker$(T) = \{x \in X : T(x, b_2, \cdots, b_n) = 0\}$ is closed.*

Proof. Let $T : X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle \to \mathbb{K}$ be a b-linear functional. Suppose T is not the zero b-linear functional. Clearly, $X - \text{Ker}(T) \neq \phi$. Then, there exists $y \in X - \text{Ker}(T)$ such that $T(y, b_2, \cdots, b_n) \neq 0$. Letting $z = \dfrac{y}{T(y, b_2, \cdots, b_n)}$, note that $T(z, b_2, \cdots, b_n) = 1$ and $z \in X - \text{Ker}(T)$. Since $X - \text{Ker}(T)$ is open, there exist $e_2, \cdots, e_n \in X$ and $r > 0$ such that

$$B_{\{e_2, \cdots, e_n\}}(z, r) \subset X - \text{Ker}(T).$$

Now, we shall first prove that

$$|T(x, b_2, \cdots, b_n)| < 1 \quad \forall x \in B_{\{e_2, \cdots, e_n\}}(0, r).$$

If possible suppose there exists some $x_1 \in B_{\{e_2, \cdots, e_n\}}(0, r)$ such that

$$|T(x_1, b_2, \cdots, b_n)| \geq 1.$$

Then, for $t = \dfrac{-x_1}{T(x_1, b_2, \cdots, b_n)}$, we have $T(t, b_2, \cdots, b_n) = -1$ and

$$\|t, e_2, \cdots, e_n\| = \left\| \frac{-x_1}{T(x_1, b_2, \cdots, b_n)}, e_2, \cdots, e_n \right\|$$

$$= \frac{1}{|T(x_1, b_2, \cdots, b_n)|} \|x_1, e_2, \cdots, e_n\| \leq \|x_1, e_2, \cdots, e_n\|.$$

Thus, $t \in B_{\{e_2, \cdots, e_n\}}(0, r)$ and this implies $z + t \in B_{\{e_2, \cdots, e_n\}}(z, r)$. Also,

$$T(z + t, b_2, \cdots, b_n) = T(z, b_2, \cdots, b_n) + T(t, b_2, \cdots, b_n) = 0$$

$$\Rightarrow z + t \in \text{Ker}(T)$$

Therefore,

$$\text{Ker}(T) \cap B_{\{e_2, \cdots, e_n\}}(z, r) \neq \phi$$

This contradicts the fact that $B_{\{e_2, \cdots, e_n\}}(z, r) \subset X - \text{Ker}(T)$. Thus

$$|T(x, b_2, \cdots, b_n)| < 1 \ \forall x \in B_{\{e_2, \cdots, e_n\}}(0, r). \tag{3.3}$$

Now, take any $x \neq 0$. Then

$$\left\| \frac{r\,x}{2\,\|\,x,\,e_2,\,\cdots,\,e_n\,\|},\,e_2,\,\cdots,\,e_n \right\| = \frac{r}{2\,\|\,x,\,e_2,\,\cdots,\,e_n\,\|}\,\|\,x,\,e_2,\,\cdots,\,e_n\,\| = \frac{r}{2} < r$$

$$\Rightarrow \frac{r\,x}{2\,\|\,x,\,e_2,\,\cdots,\,e_n\,\|} \in B_{\{e_2,\cdots,e_n\}}(0,\,r).$$

Then by (3.3), for particular $e_2 = b_2,\,\cdots,\,e_n = b_n$

$$\left| T\left(\frac{r\,x}{2\,\|\,x,\,b_2,\,\cdots,\,b_n\,\|},\,b_2,\,\cdots,\,b_n \right) \right| < 1$$

$$\Rightarrow \frac{r}{2\,\|\,x,\,b_2,\,\cdots,\,b_n\,\|}\,|\,T(x,\,b_2,\,\cdots,\,b_n)\,| < 1$$

$$\Rightarrow |\,T(x,\,b_2,\,\cdots,\,b_n)\,| < \frac{2}{r}\,\|\,x,\,b_2,\,\cdots,\,b_n\,\|.$$

Also, if $x = 0$, we have

$$|\,T(x,\,b_2,\,\cdots,\,b_n)\,| = 0 = \|\,x,\,b_2,\,\cdots,\,b_n\,\|$$

$$\Rightarrow |\,T(x,\,b_2,\,\cdots,\,b_n)\,| \leq \frac{2}{r}\,\|\,x,\,b_2,\,\cdots,\,b_n\,\|.$$

Thus, we have shown that

$$|\,T(x,\,b_2,\,\cdots,\,b_n)\,| \leq \frac{2}{r}\,\|\,x,\,b_2,\,\cdots,\,b_n\,\| \quad \forall\, x \in X.$$

Hence, T is a bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. ■

Theorem 3.6. *Let T be a non-zero b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$, where X be a linear n-normed space and let $x_0 \in X - Ker(T)$. Then any $x \in X$ can be expressed uniquely in the form $x = y + \alpha\,x_0$, where $y \in Ker(T)$ and α is some scalar.*

Proof. Since $x_0 \in X - \text{Ker}(T)$, so $T(x_0,\,b_2,\,\cdots,\,b_n) \neq 0$. Take

$$\alpha = \frac{T(x,\,b_2,\,\cdots,\,b_n)}{T(x_0,\,b_2,\,\cdots,\,b_n)} \quad \text{and define} \quad y = x - \frac{T(x,\,b_2,\,\cdots,\,b_n)}{T(x_0,\,b_2,\,\cdots,\,b_n)}\,x_0.$$

Then $x = y + \alpha\,x_0$ and $T(y,\,b_2,\,\cdots,\,b_n)$

$$= T(x,\,b_2,\,\cdots,\,b_n) - \frac{T(x,\,b_2,\,\cdots,\,b_n)}{T(x_0,\,b_2,\,\cdots,\,b_n)}\,T(x_0,\,b_2,\,\cdots,\,b_n) = 0.$$

Thus $y \in \text{Ker}(T)$. For the uniqueness, we assume that $x = y + \alpha\,x_0$ and $x = y_1 + \alpha_1\,x_0$. If $\alpha = \alpha_1$, then $y = y_1$. If $\alpha \neq \alpha_1$, then $x_0 = \dfrac{y - y_1}{\alpha - \alpha_1}$ and

$$T(x_0,\,b_2,\,\cdots,\,b_n) = \frac{1}{\alpha - \alpha_1}\,\{T(y,\,b_2,\,\cdots,\,b_n) - T(y_1,\,b_2,\,\cdots,\,b_n)\} = 0.$$

Therefore $x_0 \in \text{Ker}(T)$, which contradicts the assumption that $x_0 \in X - \text{Ker}(T)$. This completes the proof. ■

Theorem 3.7. *Let X be a linear n-normed space and $0 \neq T \in X_F^*$ and*

$$M_T = \{\,x \in X : T(x,\,b_2,\,\cdots,\,b_n) = 1\,\}.$$

Then M_T is a hyperplane and $\displaystyle\inf_{x \in M_T} \|\,x,\,b_2,\,\cdots,\,b_n\,\| = \frac{1}{\|\,T\,\|}$.

Proof. Since T be a non-zero b-linear functional, there exists $x_1 \in X - \text{Ker}(T)$ such that $T(x_1, b_2, \cdots, b_n) \neq 0$. Take $x_0 = \dfrac{x_1}{T(x_1, b_2, \cdots, b_n)}$, then $T(x_0, b_2, \cdots, b_n) = 1$. Now,

$$M_T = \{x \in X : T(x, b_2, \cdots, b_n) = 1 = T(x_0, b_2, \cdots, b_n)\}$$
$$= \{x \in X : T(x - x_0, b_2, \cdots, b_n) = 0\} = x_0 + \text{Ker}(T)$$

and therefore M_T is a hyperplane. Since T is a bounded b-linear functional,

$$|T(x, b_2, \cdots, b_n)| \leq \|T\| \, \|x, b_2, \cdots, b_n\| \quad \forall\, x \in X.$$

In particular, for all $x \in M_T$,

$$\|T\| \, \|x, b_2, \cdots, b_n\| \geq 1 \Rightarrow \|x, b_2, \cdots, b_n\| \geq \frac{1}{\|T\|}$$
$$\Rightarrow \inf_{x \in M_T} \|x, b_2, \cdots, b_n\| \geq \frac{1}{\|T\|}.$$

Further,

$$\|T\| = \sup\left\{\frac{|T(x, b_2, \cdots, b_n)|}{\|x, b_2, \cdots, b_n\|} : x \in X, \|x, b_2, \cdots, b_n\| \neq 0\right\}$$

it follows that, there exists $0 \neq y \in X$ such that

$$\frac{1}{\|T\|} > \left\| \frac{y}{|T(y, b_2, \cdots, b_n)|}, b_2, \cdots, b_n \right\|$$
$$\geq \inf_{x \in M_T} \|x, b_2, \cdots, b_n\| \left[\text{since } \frac{y}{|T(y, b_2, \cdots, b_n)|} \in M_T\right]$$

and hence the result follows. ■

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