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Existence of mild solutions of second-order impulsive differential equations in Banach spaces

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Abstract. We discuss the existence of solutions for second-order impulsive differential equation with nonlocal conditions in Banach spaces. Our approach is based on the generalization of Schauder fixed point principle that is Darbo fixed point theorem. An example is also presented for illustration.

AMS Subject Classifications: 34G20, 35R10.

Keywords: Second order differential equations, mild solution, impulse, nonlocal condition, Kuratowski measures of noncompactness.

Contents

1. Introduction

In the present paper we consider the abstract second-order nonlinear impulsive differential equation with non local condition

$$\begin{cases} x''(t) = Ax(t) + f(t, x(t), x'(t)),\ t \in J = [0,T],\ t \neq t_i, i = 0, ..., p \\ x(0) = x_0 + g(x),\ x'(0) = x_1 \\ \triangle(x(t_i)) = I_i(x(t_i)),\ i = 0, ..., p \\ \triangle(x'(t_i)) = D_i(x(t_i), x'(t_i)),\ i = 0, ..., p. \end{cases} \tag{1.1}$$

Where A is a linear operator from a Banach space E into itself, $\triangle x(t_i) = x(t_i^+) - x(t_i^-)$, $\triangle x'(t_i) = x'(t_i^+) - x'(t_i^-)$, $0 < t_1 < t_2 < ... < t_p < T$ are the instants of impulse effect, $f : [0,T] \times E \times E \to E$, $I_i : E \to E$, $D_i : E \times E \to E$, $x_0, x_1 \in E$ and $g(x)$ is a function with values in E to be specified later.
For the basic theory on impulsive differential equations in infinite dimensional spaces, the reader is referred to the literature [2, 3]. The impulsive differential equations has become an important area of investigation by many authors because of their applications. For more details, we refer the reader to [3, 11, 15]. In [4], Peng and

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Xiang discuss the existence of optimal controls for a Lagrange problem of systems governed by the second-order nonlinear impulsive differential equations in infinite dimensional spaces:

$$\begin{cases} x''(t) = Ax(t) + f(t, x(t), x'(t)) + B(t),\ t \in J = [0, T],\ t \neq t_i, i = 0, ..., p \\ x(0) = x_0,\ x'(0) = x_1 \\ \triangle(x(t_i)) = I_i(x(t_i), x'(t_i)),\ i = 0, ..., p \\ \triangle(x'(t_i)) = D_i(x(t_i), x'(t_i)),\ i = 0, ..., p. \end{cases}$$

They apply a direct approach to derive the maximum principle for the problem at hand. The authors [6] considered the existence of mild solutions for a class of abstract impulsive second-order neutral functional differential equations. In [10], the authors studied the abstract second-order nonlinear impulsive differential equation with nonlocal condition

$$\begin{cases} x''(t) = Ax(t) + f(t, x(t), x'(t)), x(b_1(t)), x(b_2(t)), ..., \\ x(b_m(t)), x'(b_1(t)), ..., x'(b_m(t)))\ t \in J = [0, T], \\ x(0) = x_0,\ x'(0) + g(x) = x_1 \\ \triangle(x(t_i)) = I_i(x(t_i),),\ i = 0, ..., m \\ \triangle(x'(t_i)) = D_i(x(t_i), x'(t_i)),\ i = 0, ..., m. \end{cases}$$

In the present work, the existence of a mild solution for problem (1.1) is obtained by the cosine family theory, measure of non-compactness and the the well known Schauder fixed point principle. Its generalization, called the Darbo fixed point theorem. It should be pointed out that the restrictive condition on the impulsive term is removed. The work is organized as follows: In Section two, we recall some definitions and facts about the cosine family and facts concerning the Kuratowski measures of noncompactness in the Banach space $PC([0, T], E)$. In Section three, we give the existence of mild solutions to the problem (1.1). In Section four we present an example to illustrate our main result.

2. Preliminaries

We begin by giving some notation. Let E be a Banach space with the norm $\|\cdot\|$. We use θ to present the zero element in E. For any constant $T > 0$, denote $J = [0, T]$. Let $C(J, E)$ and be the Banach space of all continuous functions from J into E endowed with the supremum-norm $\|x\|_C = \sup_{t \in J} \|x(t)\|$ for every $x \in C(J, E)$. From the associate literature, we consider the following space of piecewise continuous functions,

$$PC(J, E) = \big\{u : J \to E : x \text{ is continuous for } t \neq t_k, \\ \text{left continuous at } t = t_k \text{ and } x(t_k^+) \text{ exists for } k = 1, 2, \ldots, m\big\}.$$

It easy to see that $PC(J, E)$ is a Banach space endowed with the PC-norm

$$\|x\|_{PC} = \max\Big\{\sup_{t \in J} \|x(t^+)\|,\ \sup_{t \in J} \|x(t^-)\|\Big\}, \quad x \in PC(J, E),$$

where $x(t^+)$ and $x(t^-)$ represent respectively the right and left limits of $x(t)$ at $t \in J$. Similarly, PC^1 will be the space of the functions $x(\cdot) \in PC$ such that $x(\dot)$ is continuously differentiable on J, t_i, $i = 1, 2, \ldots, n$ and the derivatives

$$x'_r(t) = \lim_{s \to 0} \frac{x(t + s) - x(t^+)}{s}, \quad x'_l(t) = \lim_{s \to 0} \frac{x(t + s) - x(t^-)}{s}$$

are continuous on $[0, T[$ and $]0, T]$, respectively. Next, for $x \in PC^1$, we represent, by $x'(t)$, the left derivative at $t \in]0, T]$ and, by $x'(0)$, the right derivative at zero. It is easy to see that PC^1, provided with the norm

$$\|x\|_{PC^1} := \max\{\|x\|_{PC}, \|x'\|_{PC}\}$$

is a Banach space. For each finite constant $r > 0$, let

$$\Omega_r = \{u \in PC(J, E) : \|u(t)\| \leq r,\ t \in J\},$$

then Ω_r is a bounded closed and convex set in $PC(J, E)$.

Let $\mathcal{L}(E)$ be the Banach space of all linear and bounded operators on E. Since the semigroup $T(t)(t \geq 0)$ generated by A is a C_0-semigroup in E, denoting

$$M := \sup_{t \in J} \|T(t)\|_{\mathcal{L}(E)}, \tag{2.1}$$

then $M \geq 1$ is a finite number.

Definition 2.1. A C_0-semigroup $T(t)(t \geq 0)$ in E is said to be equicontinuous if $T(t)$ is continuous by operator norm for every $t > 0$.

Now we introduce some basic definitions and properties about Kuratowski measure of noncompactness that will be used in the proof of our main results.

Definition 2.2. *[1, 8] The Kuratowski measure of noncompactness $\alpha(\cdot)$ defined on a bounded set S of Banach space E is*

$$\alpha(S) := \inf\{\delta > 0 : S = \cup_{i=1}^{m} S_i \text{ with } \operatorname{diam}(S_i) \leq \delta \text{ for } i = 1, 2, \ldots, m\}.$$

The following properties about the Kuratowski measure of noncompactness are well known.

Lemma 2.3. *[1, 8] Let E be a Banach space and $S, U \subset E$ be bounded. The following properties are satisfied:*

(i) $\alpha(S) = 0$ *if and only if* $\overline{S}$ *is compact, where* $\overline{S}$ *means the closure hull of* S*;*

(ii) $\alpha(S) = \alpha(\overline{S}) = \alpha(\operatorname{conv} S)$*, where* $\operatorname{conv} S$ *means the convex hull of* S*;*

(iii) $\alpha(\lambda S) = |\lambda| \alpha(S)$ *for any* $\lambda \in \mathbb{R}$*;*

(iv) $S \subset U$ *implies* $\alpha(S) \leq \alpha(U)$*;*

(v) $\alpha(S \cup U) = \max\{\alpha(S), \alpha(U)\}$*;*

(vi) $\alpha(S + U) \leq \alpha(S) + \alpha(U)$*, where* $S + U = \{x \mid x = y + z, y \in S, z \in U\}$*;*

(vii) If the map $Q : \mathcal{D}(Q) \subset E \to X$ *is Lipschitz continuous with constant* k*, then* $\alpha(Q(V)) \leq k\alpha(V)$ *for any bounded subset* $V \subset \mathcal{D}(Q)$*, where* X *is another Banach space.*

In this work, we denote by $\alpha(\cdot)$, $\alpha_c(\cdot)$, $\alpha_{pc}(\cdot)$ and $\alpha_{pc^1}(\cdot)$ the Kuratowski measure of noncompactness on the bounded set of E, $C(J, E)$, $PC(J, E)$ and $PC^1(J, E)$, respectively.
In the following, let $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], ..., J_{p-1} = (t_{p-1}, t_p]$ and $J_p = (t_p, 1]$, $t_{p+1} = 1$. For any $X \subset PC(J, E)$, we denote by $X' = \{x' \ : \ x \in X\} \subset PC(J, E)$ and by $X(t) = \{x(t) : \ x \in X\} \subset E$ and by $X'(t) = \{x(t) : \ x \in X\} \subset E$ for $t \in J$. To discuss the problem (1.1), we also need the following lemma [12].

Lemma 2.4. *[12] If $X \subset PC^1(J, E)$ is bounded and the elements of X' are equicontinuous on each $J_k, k = 0, 1, ..., p$ then*

$$\alpha_{pc^1}(X) = \max\{\sup_{t \in J} \alpha(X(t)), \sup_{t \in J} \alpha(X'(t))\} \tag{2.2}$$

Obviously the following formulated theorem constitutes the well known Schauder fixed point principle. Its generalization, called the Darbo fixed point theorem, is formulated below.

Lemma 2.5. *[8] Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that $\mu(T(X)) = k\mu(X)$ for any nonempty subset X of Ω, where μ is a measure of noncompactness defined in E. Then T has a fixed point in the set Ω.*

Lemma 2.6. *[5, 16] Let E be a Banach space, and let $X \subset E$ be bounded. Then there exists a countable set $X_0 \subset X$, such that $\alpha(X) \leq 2\alpha(X_0)$.*

Lemma 2.7. *[13] Let E be a Banach space, and let $X = \{u_n : n = 0, 1, ...\} \subset PC([0, T], E)$ be a bounded and countable set for constants $-\infty < 0 < T < +\infty$. Then $\alpha(X(t))$ is Lebesgue integral on $[0, T]$, and*

$$\alpha\Big(\Big\{\int_0^T u_n(t)dt : n \in \mathbb{N}\Big\}\Big) \leq 2\{\int_0^T \alpha(u_n(t))dt : n = 0, 1, ...\}.$$

Lemma 2.8. *[1] Let E be a Banach space, and let $X \subset C([0, T], E)$ be bounded and equicontinuous. Then $\alpha(X(t))$ is continuous on $[0, T]$, and*

$$\alpha_c(X) = \max_{t\in[0,T]} \alpha(X(t)).$$

Next, we shall need the following definitions [25].

Definition 2.9. A one parameter family $\{C(t), t \in J\}$ of bounded linear operators in the Banach space X is called a strongly continuous cosine family if

(i) $C(s + t) + C(s - t) = 2C(s)C(t)$, for all $s, t \in J$;

(ii) $C(0) = I$;

(iii) $C(t)x$ is continuous in t on J, for each $x \in X$.

Define the associated sine family $S(t), t \in J$ by

$$S(t)x := \int_0^t C(s)xds, \quad x \in X,\ t \in J$$

The infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in J\}$ is the operator $A : X \to X$, defined by

$$Ax = \lim_{t\to 0} \frac{d^2}{dt^2} C(t)x, \quad x \in D(A),$$

where $D(A) := \{x \in X : C(t)x$ is twice continuously differentiable in $t\}$.

Define $E := \{x \in X : C(t)x$ is twice continuously differentiable in $t\}$. We assume

(H_A) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in J\}$ of bounded linear operators in the Banach space X.

To establish our main theorem, we need the following lemmas.

Lemma 2.10. *Let (H_A) hold. Then*

(i) there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|C(t)\| \leq Me^{\omega|t|}$ and

$$\|S(b) - S(a)\| \leq M\Big|\int_b^a e^{\omega|s|}ds\Big|, \quad \textit{for } a, b \in J;$$

(ii) $S(t)X \subset E$ and $S(t)E \subset D(A)$, for $t \in J$;

(iii) $\frac{d}{dt}C(t)x = AS(t)x$, *for* $x \in E$ *and* $t \in J$*;*

(iv) $\frac{d^2}{dt^2}C(t)x = AC(t)x$, *for* $x \in D(A)$ *and* $t \in J$.

Further we denote by $\|C(t)\|$ and $\|S(t)\|$ the operators norm of $C(t), S(t)$ for $t \in [0,T]$ in the Banach space E, respectively. From assumption (H_A) it follows that there is a constant $M \geq 1$ such that

$$\|C(t)\| \leq M \text{ and } \|S(t)\| \leq M \text{ for } t \in [0,T].$$

Lemma 2.11. *[25] Let* (H_A) *hold and* $v : \mathcal{R} \to X$ *be such that* v *is continuous and let* $q(t) = \int_0^t S(t-s)v(s)ds$*. Then* q *is twice continuously differentiable and, for* $t \in \mathcal{I}$*:* $q(t) \in D(A)$*,* $q'(t) = \int_0^t C(t-s)v(s)ds$ *and*

$$q''(t) = \int_0^t C(t-s)v'(s)ds + C(t)v(0) = Aq(t) + v(t).$$

3. Main results

We first give the following hypotheses:

(H_A) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in \mathcal{I}\}$ of bounded linear operators in the Banach space X.

(H_f) (i) $(t,x,y) \mapsto f(t,x,y)$ satisfies the Carathéodory conditions, i.e. $f(.,x,y)$ is measurable for $x, y \in E$ and $f(t,.,.)$ is continuous for a.e. $t \in [0,T]$

(ii) There exist $m \in L^1([0,T], \mathbb{R}_+)$ such that $\|f(t,x,y)\| \leq m(t)(\|x\| + \|y\|)$ for a.e. $t \in [0,T]$ and all $x \in E$.

(iii) There exists a function $l \in L^1([0,T], \mathbb{R}_+)$ such that for any bounded subset $B, D \subset E, \alpha(f(t,B,D)) = l(t)\max\{\alpha(B), \alpha(D)\}$ for a.e. $t \in [0,T]$.

(H_g) (i) g is continuous.

(ii) There is nonnegative constant q such that $\alpha(g(D)) \leq q\alpha_{pc^1}(D)$ for any bounded set $D \subset PC^1([0,T], E)$.

(H) (i) I_i and D_i are continuous.

(ii) There exist nonnegative constants k_i^1 and k_i^2 such that $\alpha(I_i(B)) \leq k_i^1\alpha(B)$ and $\alpha(D_i(B,D)) \leq k_i^2 \max(\alpha(B), \alpha(D))$ for any nonempty and bounded subset $B, D \subset E$ and $i = 1, ..., p$.

(H_R) There exists a number $R > 0$ such that

$$\max(\eta_1(R), \eta_1(R)) \leq R,$$

where,

$$\eta_1(R) = M\big[\|x_0\| + \|x_1\| + C_1\big] + 2MR \sup_{t\in[0,T]} \big(\int_0^t m(s)ds\big) + Mp(C_2 + C_3)$$

and

$$\eta_2(R) = M\big[\|A\|(\|x_0\| + C_1) + \|x_1\|\big] + 2MR \sup_{t\in[0,T]} \big(\int_0^t m(s)ds\big) + Mp(\|A\|C_2 + C_3),$$

where

$$C_1 = \sup_{x\in B_{pc^1}(R)} g(\|x\|),$$

$$C_2 = \sup_{x\in B_{pc^1}(R)} \|I_i(x(t_i))\|$$

and

$$C_3 = \sup_{x\in B_{pc^1}(R)} \|D_i(x(t_i), x'(t_i))\|.$$

Next, let us start by defining what we mean by a solution of the problem (1.1)(see [6]).

Definition 3.1. *A function $x \in PC^1([0,T],E)$ is said to be a mild solution of the problem (1.1) if x satisfies the equation*

$$x(t) = C(t)[x_0 + g(x)] + S(t)x_1 + \int_0^t S(t-s)f(s,x(s),x'(s))ds$$
$$+ \sum_{0<t_i<t} C(t-t_i)I_i(x(t_i)) + \sum_{0<t_i<t} S(t-t_i)D_i(x(t_i),x'(t_i)),\ t\in[0,T]. \tag{3.1}$$

Remark 3.2. *Assumptions $(H_f)(i)$, $(H_g)(ii)$ and $(H)(ii)$ imply that mappings f, g, I_i and I_i are bounded on bounded subsets of $PC^1([0,T],E)$ and E, respectively.*

To simplify the writing and the calculation one poses

$$\widetilde{L} = \int_0^T l(s)ds,\ S_1 = \sum_{0<t_i<t} k_i^1 \text{ and } S_2 = \sum_{0<t_i<t} k_i^2$$

.

Theorem 3.3. *Let E be a separable Banach space. Assume that the assumptions (H_A), (H_f), (H_g), (H) and (H_R) are satisfied. If*

$$\max\{q + \widetilde{L} + S_1 + S_2; \|A\|q + \widetilde{L} + \|A\|S_1 + S_2\} < \frac{1}{M},$$

then for each $x_0 \in E$, the equation (1.1) has at least one mild solution x in $PC^1(J,E)$.

Proof. Consider the operator $Fx : PC^1([0,T],E) \to PC^1([0,T],E)$ define by

$$(Fx)(t) = C(t)[x_0 + g(x)] + S(t)x_1 + \int_0^t S(t-s)f(s,x(s),x'(s))ds$$
$$+ \sum_{0<t_i<t} C(t-t_i)I_i(x(t_i)) + \sum_{0<t_i<t} S(t-t_i)D_i(x(t_i),x'(t_i)),\ t\in[0,T]. \tag{3.2}$$

It easy to see that $(Fx) \in PC([0,T],E)$ for $x \in PC^1([0,T],E)$. Moreover,

$$(Fx)'(t) = \frac{\partial(Fx)}{\partial t}(t) = AS(t)[x_0 + g(x)] + C(t)x_1 + \int_0^t C(t-s)f(s,x(s),x'(s))ds$$
$$+ \sum_{0<t_i<t} AS(t-t_i)I_i(x(t_i)) + \sum_{0<t_i<t} C(t-t_i)D_i(x(t_i),x'(t_i)),\ t\in[0,T]. \tag{3.3}$$

Then, we get that $(Fx)' \in PC([0,T],E)$ and therefore, $Fx \in PC^1([0,T],E)$. So, F maps the Banach space $PC^1([0,T],E)$ into itself. Next, Let R be a positive number satisfying the inequality from assumption (H_R). Taking an arbitrary function $x \in B_{pc^1}(R)$, we get

$$\begin{aligned}
\|Fx(t)\|_{pc} &\le M\big[\|x_0\| + g(\|x\|)\big] + M\|x_1\| + M\int_0^t m(s)(\|x(s)\| + \|x'(s)\|)ds \\
&+ M\sum_{0<t_i<t}\|I_i(x(t_i))\| + M\sum_{0<t_i<t}\|D_i(x(t_i), x'(t_i))\| \\
&\le M\big[\|x_0\| + \sup_{x\in B_{pc^1}(R)} g(\|x\|)\big] + M\|x_1\| \\
&+ 2M\sup_{t\in[0,T]}\big(\int_0^t m(s)\max\{\|x(s)\|, \|x'(s)\|\}ds\big) \\
&+ M\sum_{0<t_i<t}\sup_{x\in B_{pc^1}(R)}\|I_i(x(t_i))\| + M\sum_{0<t_i<t}\sup_{x\in B_{pc^1}(R)}\|D_i(x(t_i), x'(t_i))\| \\
&\le \eta_1(R). \qquad (3.4)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|(Fx)'(t)\|_{pc} &\le M\|A\|\big[\|x_0\| + g(\|x\|)\big] + M\|x_1\| + M\int_0^t m(s)(\|x(s)\| + \|x'(s)\|)ds \\
&+ M\|A\|\sum_{0<t_i<t}\|I_i(x(t_i))\| + M\sum_{0<t_i<t}\|D_i(x(t_i), x'(t_i))\| \\
&\le M\big[\|x_0\| + \sup_{x\in B_{pc^1}(R)} g(\|x\|)\big] + M\|x_1\| \\
&+ 2M\sup_{t\in[0,T]}\big(\int_0^t m(s)\max\{\|x(s)\|, \|x'(s)\|\}ds\big) \\
&+ M\|A\|\sum_{0<t_i<t}\sup_{x\in B_{pc^1}(R)}\|I_i(x(t_i))\| + M\|\sum_{0<t_i<t}\sup_{x\in B_{pc^1}(R)}\|D_i(x(t_i), x'(t_i))\| \\
&\le \eta_2(R), \qquad (3.5)
\end{aligned}$$

and thus,

$$\begin{aligned}
\|(Fx)(t)\|_{pc^1} &= \max\Big\{\|(Fx)(t)\|_{pc}, \|(Fx)'(t)\|_{pc}\Big\} \\
&\le \max\Big\{\eta_2(R), \eta_2(R)\Big\} = \eta(R) \le R. \qquad (3.6)
\end{aligned}$$

The last inequality shows that $(Fx) \in B_{pc^1}(R)$ for $x \in B_{pc^1}(R)$, that is $F(B_{pc^1}(R)) \subset B_{pc^1}(R)$. Now, we prove that operator F is continuous in $B_{pc^1}(R)$. To do this, let us fix $x \in B_{pc^1}(R)$ and take an arbitrary sequence $(x_n) \in B_{pc^1}(R)$ such that $x_n \to x$ in $B_{pc^1}(R)$. It also implies that the family $\{Fx \;\; x \in B_{pc^1}(R)\}$ is equibounded. Next, we shall show that the family $\{Fx \;\; x \in B_{pc^1}(R)\}$ is equicontinuous on each interval of continuity J_k, $k = 0, 1, ..., p$. For this, let $x \in B_{pc^1}(R)$ and $0 \le t_1 < t_2 \le T$. Then we have

$$\begin{aligned}
&(Fx)'(t_2) - (Fx)'(t_1) = A[S(t_2) - S(t_1)][x_0 + g(x)] + [C(t_2) - C(t_2)]x_1 \\
&+ \int_0^{t_1}[C(t_2 - s) - C(t_2 - s)]f(s, x(s), x'(s))ds + \int_{t_1}^{t_2} C(t_2 - s)f(s, x(s), x'(s))ds \\
&+ \sum_{0<t_i<t_1} A[S(t_2 - t_i) - S(t_1 - t_i)]I_i(x(t_i)) + \sum_{t_1<t_i<t_2} A[S(t_2 - t_i)]I_i(x(t_i)) \\
&+ \sum_{0<t_i<t_1}[C(t_2 - t_i) - C(t_1 - t_i)]D_i(x(t_i), x'(t_i)) + \sum_{t_1<t_i<t_2}[C(t_2 - t_i)]D_i(x(t_i), x'(t_i).
\end{aligned}$$

So,

$$\|(Fx)'(t_2)-(Fx)'(t_1)\| \leq \|A\|\|S(t_2)-S(t_1)\|[\|x_0\|+\|g(x)\|]+[\|C(t_2)-C(t_2)\|]\|x_1\|$$
$$+\int_0^{t_1}[\|C(t_2-s)-C(t_2-s)\|]\|f(s,x(s),x'(s))\|ds+\int_{t_1}^{t_2}\|C(t_2-s)\|\|f(s,x(s),x'(s))\|ds$$
$$+\sum_{0<t_i<t_1}\|A\|[\|S(t_2-t_i)-S(t_1-t_i)\|]\|I_i(x(t_i))\|+\sum_{t_1<t_i<t_2}\|A\|\|S(t_2-t_i)\|\|I_i(x(t_i))\|$$
$$+\sum_{0<t_i<t_1}\|C(t_2-t_i)-C(t_1-t_i)\|\|D_i(x(t_i),x'(t_i))\|+\sum_{t_1<t_i<t_2}\|C(t_2-t_i)\|\|D_i(x(t_i),x'(t_i))\|.$$

Then,

$$\|(Fx)'(t_2)-(Fx)'(t_1)\| \leq \|A\|\|S(t_2)-S(t_1)\|[\|x_0\|+C_1]+\|C(t_2)-C(t_2)\|\|x_1\|$$
$$+R\int_0^{t_1}[\|C(t_2-s)-C(t_2-s)\|]m(s)ds+MR\int_{t_1}^{t_2}m(s)ds$$
$$+\|A\|C_2\sum_{0<t_i<t_1}\|S(t_2-t_i)-S(t_1-t_i)\|+\|A\|MC_2 i(t_1,t_2)$$
$$+C_3\sum_{0<t_i<t_1}\|C(t_2-t_i)-C(t_1-t_i)\|+Mi(t_1,t_2). \tag{3.7}$$

where, $i(t_1,t_2)$ is the number of instants of impulse effect in the interval $[t_1,t_2)$. First, notice that the right-hand side of inequality is independant of the choose of $x \in B_{pc^1}(R)$. Further, from the uniform continuity of $C(t)$ and $S(t)$ on J in the operator norm, all norm in the right-hand side converge to 0 as $t_1 \to t_2$. Finally $i(t_1,t_2)$ is zero for t_1, t_2 both in one of the intervals of continuity $J_k, k=0,1,...,p$. This, prove that the family of functions $\{(Fx)' : x \in B_{pc^1}(R)\}$ is equicontinuous on each interval J_k, $k=0,1,...,p$. In what follows, we will show that F is a strict set contraction from $PC^1(J,E)$ into itself. Let Q be a bounded set of $PC^1(J,E)$. Then $F(Q) \subset PC^1(J,E)$ is bounded and by (3.7) the elements of $(F(Q))'$ are equicontinuous on each interval J_k, $k=0,1,...,p$. Hence by lemma 2.4, we get

$$\alpha_{pc^1}(FQ)=\max\{\sup_{t\in J}\alpha((FQ)(t)),\sup_{t\in J}\alpha((FQ)'(t))\}. \tag{3.8}$$

Firstly,

$$\alpha((FQ)(t)) \leq M\alpha(g(Q))+M\int_0^t\alpha(f(s,Q(s),Q'(s)))ds$$
$$+M\sum_{0<t_i<t}\alpha(I_i(Q(t_i)))+M\sum_{0<t_i<t}\alpha(D_i(Q(t_i),Q'(t_i))$$
$$\leq Mq\alpha_{pc^1}(Q)+M\alpha_{pc^1}(Q)\int_0^t l(s)ds$$
$$+M\sum_{0<t_i<t}k_i^1\alpha(Q(t_i)))+M\alpha_{pc^1}(Q)\sum_{0<t_i<t}k_i^2$$
$$\leq M(q+\widetilde{L}+S_1+S_2)\alpha_{pc^1}(Q). \tag{3.9}$$

Similarly,

$$\alpha((FQ)(t)) \leq M(\|A\|q+\widetilde{L}+\|A\|S_1+S_2)\alpha_{pc^1}(Q). \tag{3.10}$$

Finally, inequalities (3.8), (3.9) and (3.10) imply that

$$\alpha_{pc^1}((FQ)) \leq MK\alpha_{pc^1}(Q),$$

where $K=\max(\|A\|q+\widetilde{L}+\|A\|S_1+S_2, q+\widetilde{L}+S_1+S_2)$

By lemma 2.5 the theorem (3.3) is proved.

4. Application

Consider the following impulse scalar second order differential equation with nonlocal conditions

$$\begin{cases} x''(t) = x(t) + \frac{\arctan(t)}{18+t^2}\left[x(t) + x'(t)\right],\ t \in J = (0,1]\backslash\{\beta_1, \beta_2, ..., \beta_5\} \\ x(0) = x_0 + \frac{1}{9}\sum_{j=1}^{3} 2^{-j}x(t_j),\ x'(0) = x_1 \\ \triangle(x(\frac{1}{4})) = \frac{1}{30}x(\frac{1}{4}),\ i = 0, ..., 5 \\ \triangle(x'(\frac{1}{4})) = \frac{1}{100}(x(\frac{1}{4}) + x'(\frac{1}{4})), \end{cases} \tag{4.1}$$

where $0 < \beta_1 < \beta_2 < ... < \beta_5 < 1$ and $t_j \in (0,1], j = 1, 2, ..., p$. Here $E = \mathbb{R}$, $C(t) = \cosh(t)$, $S(t) = \sinh(t)$, $\max\limits_{t\in[0,1]} \cosh(t) = \cosh(1) < 3$. Since arcosh(3) = 1,7627, $\max\limits_{t\in[0,1]} \sinh(t) = \sinh(1) < 3$, thus we can choose $M = 3$. It is easy to see that $f(t, x, y) = \frac{1}{1+t^2}\sqrt{x^2 + y^2}$ satisfies to the inequality $|f(t, x, y)| \leq \frac{1}{18+t^2}(|x| + |y|)$ for all $t \in [0, 1]$ and $x, y \in \mathbb{R}$. Similarly, it is not difficult to show that
$q = \dfrac{1}{9}\sum\limits_{j=1}^{3}(\dfrac{1}{2})^j$, $k_i^1 = \frac{1}{9}$, $k_i^2 = \frac{1}{9}$ and $l(s) = \frac{\pi}{2(18+s^2)}$. If we take $R = 3$ it is easy to see that when $\|x_0\| + \|x_1\| < \frac{13}{30}$ and $q + L + S_1 + S_2 < \frac{1}{3}$. Then all conditions of theorem (3.3) are satisfied. Thus, our conclusion follows from the main theorem.

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On Berezin radius inequalities via Cauchy-Schwarz type inequalities

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Abstract. A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Theta \subseteq \mathbb{C}$ that the evaluation functionals are continuous for each $\tau \in \Theta$ on $\mathcal{H}$. The Berezin transform $\widetilde{S}$ and the Berezin radius of an operator S on the functional Hilbert space (or reproducing kernel Hilbert space) over some set Θ with the reproducing kernel k_τ are defined, respectively, by

$$\widetilde{S}(\tau) = \left\langle S\widehat{k}_\tau, \widehat{k}_\tau \right\rangle, \ \tau \in \Theta \text{ and } \operatorname{ber}(S) := \sup_{\tau \in \Theta} \left| \widetilde{S}(\tau) \right|.$$

Using this limited function $\widetilde{S}$, we investigate several novel inequalities that include improvements to some Berezin radius inequalities for operators working on the functional Hilbert space.

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Keywords: Berezin symbol, Berezin radius, Cauchy-Schwarz inequality, triangle inequality, reproducing kernel.

Contents

1. Introduction

Let $\mathbb{L}(\mathcal{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathcal{H}, \langle .,. \rangle)$. Throughout the paper, we work on functional Hilbert space (FHS), which are complete inner product spaces made up of complex-valued functions defined on a non-empty set Θ with bounded point evaluation. Recall that a functional Hilbert space $\mathcal{H} = \mathcal{H}(\Theta)$ is a complex Hilbert space on a (nonempty) Θ, which has the property that point evaluatians are continuons for each $\tau \in \Theta$ there is an unique element $k_\tau \in \mathcal{H}$ such that $f(\tau) = \langle f, k_\tau \rangle$, for all $f \in \mathcal{H}$. The family $\{k_\tau : \tau \in \Theta\}$ is called the reproducing kernel $\mathcal{H}$. If $\{e_n\}_{n \geq 0}$ is an orthonormal basis for FHS, the reproducing kernel is showed by $k_\tau = \sum_{n=0}^{\infty} \overline{e_n(\tau)} e_n(z)$. For $\tau \in \Theta$, $\widehat{k}_\tau = \frac{k_\tau}{\|k_\tau\|_{\mathcal{H}}}$ is called the normalized reproducing kernel.

Definition 1.1. *(i) For $S \in \mathbb{L}(\mathcal{H})$, the function $\widetilde{S}$ defined on Θ by*

$$\widetilde{S}(\tau) = \left\langle S\widehat{k}_\tau, \widehat{k}_\tau \right\rangle_{\mathcal{H}}$$

is the Berezin symbol (or Berezin transform) of S.

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(ii) The Berezin range of S (or Berezin set of S) is

$$\operatorname{Ber}(S) := \operatorname{Range}(\widetilde{S}) = \left\{ \widetilde{S}(\tau) : \tau \in \Theta \right\}.$$

(iii) The Berezin radius of S (or Berezin number of S) is

$$\operatorname{ber}(S) := \sup \left\{ \left| \widetilde{S}(\tau) \right| : \tau \in \Theta \right\}.$$

The Berezin transform $\widetilde{S}$ is a bounded real-analytic function on for each bounded operator S on $\mathcal{H}$. The Berezin transform $\widetilde{S}$ frequently reflects the characteristics of the operator S. A key tool in operator theory is the Berezin transform, which Berezin first described in [10]. This is because the Berezin transforms of many significant operators include information on their fundamental characteristics. The Berezin range and Berezin radius of the operator were defined by Karaev in [25].

Recall that the numerical range and numerical radius number of $S \in \mathbb{L}(\mathcal{H})$ are denoted respectively, by

$$W(S) = \{\langle Su, u\rangle : u \in \mathcal{H} \text{ and } \|u\| = 1\} \text{ and,}$$

$$w(S) = \sup\{|\langle Su, u\rangle| : u \in \mathcal{H} \text{ and } \|u\| = 1\}.$$

The absolute value of positive operator is denoted by $|S| = (S^*S)^{\frac{1}{2}}$. The numerical range has several intriguing features. For example, it is usually assumed that an operator's spectrum is confined in the closure of its numerical range. For an illustration of how this and other numerical radius inequalities were addressed in those sources, we urge the reader read [1, 14, 28, 29]. For $S, T \in \mathbb{L}(\mathcal{H})$ it is clear from the definition of the Berezin number and the Berezin norm that the following properties hold:

(B1) $\operatorname{ber}(zS) = |z| \operatorname{ber}(S)$ for all $z \in \mathbb{C}$,
(B2) $\operatorname{ber}(S + T) \leq \operatorname{ber}(S) + \operatorname{ber}(T)$,
(B3) $\operatorname{ber}(S) \leq \|S\|_{\text{ber}}$,
(B4) $\|zS\|_{\text{ber}} = |z| \|S\|_{\text{ber}}$ for all $z \in \mathbb{C}$,
(B5) $\|S + T\|_{\text{ber}} \leq \|S\|_{\text{ber}} + \|T\|_{\text{ber}}$.

It is clear from the definition that $\operatorname{Ber}(S) \subseteq W(S)$ and so

$$\operatorname{ber}(S) \leq w(S) \leq \|S\| \tag{1.1}$$

for any $S \in \mathbb{L}(\mathcal{H}(\Theta))$.

In [24], Huban et al. obtained the following result:

$$\operatorname{ber}(S) \leq \frac{1}{2}\left(\|S\|_{\text{ber}} + \left\|S^2\right\|_{\text{ber}}^{1/2}\right). \tag{1.2}$$

After that, in [22], and [9], respectively, the same authors proved for $S \in \mathbb{L}(\mathcal{H}(\Theta))$

$$\frac{1}{4}\left\||S|^2 + |S^*|^2\right\|_{\text{ber}} \leq \operatorname{ber}^2(S) \leq \frac{1}{2}\left\||S|^2 + |S^*|^2\right\|_{\text{ber}} \tag{1.3}$$

where $|S| = (S^*S)^{1/2}$ is the acsolute value of S, and

$$\operatorname{ber}^{2\alpha}(S) \leq \frac{1}{2}\left\||S|^{2\alpha} + |S^*|^{2\alpha}\right\|_{\text{ber}} \tag{1.4}$$

where $\alpha \geq 1$.

Huban et al. demonstrated the following Berezin radius estimate for the product of two functional Hilbert space operators

$$\operatorname{ber}^{\alpha}(T^*S) \leq \frac{1}{2}\left\||S|^{2\alpha} + |T|^{2\alpha}\right\|,\ \alpha \geq 1, \tag{1.5}$$

in [22, Theorem 3.11].

On Bergman and Hardy spaces, the Berezin symbol (or transform) has been thoroughly investigated for Hankel and Toeplitz operators. Several mathematical works have examined the Berezin symbol and Berezin radius throughout the years; a few of them are [6, 7, 12, 19, 20, 25, 26, 32]. In order to functional Hilbert space (reproducing kernel Hilbert space) operators, this study establishes numerous improvements of the aforementioned Berezin radius inequalities. In specifically, it is demonstrated that

$$\operatorname{ber}^2(S) \leq \frac{1}{6}\left\| |S|^2 + |S^*|^2 \right\|_{\operatorname{ber}} + \frac{1}{3}\operatorname{ber}(S) \left\| |S| + |S^*| \right\|_{\operatorname{ber}} \tag{1.6}$$

for the arbitrary bounded linear operator $S \in \mathbb{L}(\mathcal{H}(\Theta))$. Furthermore covered are a few additional connected issues. The related results are obtained in [4].

2. Known Lemmas

The following series of corollaries are necessary for us to succeed in our mission.

According to the Cauchy-Schwarz inequality,

$$|\langle u, v\rangle| \leq \|u\| \|v\| \tag{2.1}$$

holds true for every vectors u and v in an inner product space.

Contrarily, the traditional Schwarz inequality for positive operators states that for any $u, v \in \mathcal{H}$, if $S \in \mathbb{L}(\mathcal{H})$ is a positive operators, then

$$|\langle Su, v\rangle|^2 \leq \langle Su, u\rangle \langle Sv, v\rangle . \tag{2.2}$$

A companion of Schwarz inequality (2.2) known as the Kato's inequality or the so called mixed Cauchy Schwarz inequality was first proposed by Kato [27] in 1952. It states:

$$|\langle Su, v\rangle|^2 \leq \left\langle |S|^{2r} u, u \right\rangle \left\langle |S^*|^{2(1-r)} v, v \right\rangle, 0 \leq r \leq 1 \tag{2.3}$$

for any operators $S \in \mathcal{B}(\mathcal{H})$ and any vectors $u, v \in \mathcal{H}$.

$$|\langle Su, u\rangle| \leq \sqrt{\langle |S| u, u\rangle \langle |S^*| u, u\rangle}. \tag{2.4}$$

in particular is present.

$$\begin{aligned} |\langle Su, u\rangle|^2 &\leq \frac{1}{3}\langle |S| u, u\rangle \langle |S^*| u, u\rangle + \frac{2}{3}|\langle Su, u\rangle| \sqrt{\langle |S| u, u\rangle \langle |S^*| u, u\rangle} \\ &\leq \langle |S| u, u\rangle \langle |S^*| u, u\rangle \end{aligned} \tag{2.5}$$

was proven to be the refinement of (2.4) in [30].

The following well-known lemmas will make it necessary to demonstrate our findings. The Power-Mean (PM) inequality comes first.

Lemma 2.1. *([31]) According to the PM inequality,*

$$x^r y^{1-r} \leq rx + (1-r) y \leq (rx^\alpha + (1-r) y^\alpha)^{\frac{1}{\alpha}} \tag{2.6}$$

holds for every $0 \leq r \leq 1$, $x, y \geq 0$ *and* $\alpha \geq 1$.

The McCarty inequality for positive operators is the following lemma.

Lemma 2.2. *([15]) If* $S \in \mathbb{L}(\mathcal{H})$ *is a positive operator and* $u \in \mathcal{H}$ *is an unit vector, then we have*

$$\langle Su, u\rangle^\alpha \leq (\geq) \langle S^\alpha u, u\rangle, \alpha \geq 1 \; (0 \leq \alpha \leq 1). \tag{2.7}$$

Lemma 2.3. *([5]) If $S, T \in \mathbb{L}(\mathcal{H})$ and f is a non-negative convex function on $[0, \infty)$, then we have*

$$\left\| f\left(\frac{S+T}{2}\right) \right\| \leq \left\| \frac{f(S)+f(T)}{2} \right\|. \tag{2.8}$$

Lemma 2.4. *If $u, v \in \mathcal{H}$ and $0 \leq \xi \leq 1$, then we have*

$$|\langle u, v\rangle|^2 \leq (1-\xi)\, |\langle u, v\rangle|\, \|u\|\, \|v\| + \xi\, \|u\|^2\, \|v\|^2 \leq \|u\|^2\, \|v\|^2. \tag{2.9}$$

Lemma 2.5. *Let $u, v \in \mathcal{H}$. Then*

$$|\langle u, v\rangle| \leq (1-\xi)\sqrt{|\langle u, v\rangle|\, \|u\|\, \|v\|} + \xi\, \|u\|\, \|v\| \leq \|u\|\, \|v\|. \tag{2.10}$$

The next finding expands and clarifies Kato's inequality (2.3), which in turn expands and clarifies (2.5).

Lemma 2.6. *([4]) If $S \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \leq \xi, r \leq 1$ and $\alpha \geq 1$, then we have*

$$\begin{aligned} \left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon \right\rangle\right|^{2\alpha} &\leq \xi \left\langle |S|^{2\alpha r}\, \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle |S^*|^{2\alpha(1-r)}\, \widehat{k}_\upsilon, \widehat{k}_\upsilon \right\rangle \\ &\quad + (1-\xi) \left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon \right\rangle\right|^{\alpha} \sqrt{\left\langle |S|^{2\alpha r}\, \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle |S^*|^{2\alpha(1-r)}\, \widehat{k}_\upsilon, \widehat{k}_\upsilon \right\rangle} \\ &\leq \left\langle |S|^{2\alpha r}\, \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle |S^*|^{2\alpha(1-r)}\, \widehat{k}_\upsilon, \widehat{k}_\upsilon \right\rangle. \end{aligned} \tag{2.11}$$

Proof. Let $\tau, \upsilon \in \Theta$ be an arbitrary. By using (2.7), we get

$$\begin{aligned} &\xi \left\langle |S|^{2\alpha r}\, \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle |S^*|^{2\alpha(1-r)}\, \widehat{k}_\upsilon, \widehat{k}_\upsilon \right\rangle \\ &+ (1-\xi) \left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon \right\rangle\right|^{\alpha} \sqrt{\left\langle |S|^{2\alpha r}\, \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle |S^*|^{2\alpha(1-r)}\, \widehat{k}_\upsilon, \widehat{k}_\upsilon \right\rangle} \\ &\geq \xi \left\langle |S|^{2r}\, \widehat{k}_\tau, \widehat{k}_\tau \right\rangle^{\alpha} \left\langle |S^*|^{2(1-r)}\, \widehat{k}_\upsilon, \widehat{k}_\upsilon \right\rangle^{\alpha} \\ &+ (1-\xi) \left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon \right\rangle\right|^{\alpha} \sqrt{\left\langle |S|^{2r}\, \widehat{k}_\tau, \widehat{k}_\tau \right\rangle^{\alpha} \left\langle |S^*|^{2(1-r)}\, \widehat{k}_\upsilon, \widehat{k}_\upsilon \right\rangle^{\alpha}} \\ &= \xi \left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon \right\rangle\right|^{2\alpha} + (1-\xi) \left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon \right\rangle\right|^{\alpha} \left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon \right\rangle\right|^{\alpha} \\ &= \left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon \right\rangle\right|^{2\alpha} \end{aligned} \tag{2.12}$$

for every $0 \leq \xi \leq 1$ and $\alpha \geq 1$. As opposed to that, we get

$$\begin{aligned} &\xi \left\langle |S|^{2\alpha r}\, \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle |S^*|^{2\alpha(1-r)}\, \widehat{k}_\upsilon, \widehat{k}_\upsilon \right\rangle \\ &+ (1-\xi) \left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon \right\rangle\right|^{\alpha} \sqrt{\left\langle |S|^{2\alpha r}\, \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle |S^*|^{2\alpha(1-r)}\, \widehat{k}_\upsilon, \widehat{k}_\upsilon \right\rangle} \\ &\leq \xi \left\langle |S|^{2\alpha r}\, \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle |S^*|^{2\alpha(1-r)}\, \widehat{k}_\upsilon, \widehat{k}_\upsilon \right\rangle \\ &+ (1-\xi) \sqrt{\left\langle |S|^{2\alpha r}\, \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle |S^*|^{2\alpha(1-r)}\, \widehat{k}_\upsilon, \widehat{k}_\upsilon \right\rangle} \sqrt{\left\langle |S|^{2\alpha r}\, \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle |S^*|^{2\alpha(1-r)}\, \widehat{k}_\upsilon, \widehat{k}_\upsilon \right\rangle} \\ &= \xi \left\langle |S|^{2\alpha r}\, \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle |S^*|^{2\alpha(1-r)}\, \widehat{k}_\upsilon, \widehat{k}_\upsilon \right\rangle + (1-\xi) \left\langle |S|^{2\alpha r}\, \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle |S^*|^{2\alpha(1-r)}\, \widehat{k}_\upsilon, \widehat{k}_\upsilon \right\rangle \\ &= \left\langle |S|^{2\alpha r}\, \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle |S^*|^{2\alpha(1-r)}\, \widehat{k}_\upsilon, \widehat{k}_\upsilon \right\rangle. \end{aligned} \tag{2.13}$$

Combining (2.12) and (2.13), we deduce that

$$\begin{aligned}\left|\left\langle S\widehat{k}_{\tau},\widehat{k}_{\upsilon}\right\rangle\right|^{2\alpha} &\leq \xi\left\langle |S|^{2\alpha r}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\left\langle |S^{*}|^{2\alpha(1-r)}\widehat{k}_{\upsilon},\widehat{k}_{\upsilon}\right\rangle \\ &+(1-\xi)\left|\left\langle S\widehat{k}_{\tau},\widehat{k}_{\upsilon}\right\rangle\right|^{\alpha}\sqrt{\left\langle |S|^{2\alpha r}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\left\langle |S^{*}|^{2\alpha(1-r)}\widehat{k}_{\upsilon},\widehat{k}_{\upsilon}\right\rangle} \\ &\leq \left\langle |S|^{2\alpha r}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\left\langle |S^{*}|^{2\alpha(1-r)}\widehat{k}_{\upsilon},\widehat{k}_{\upsilon}\right\rangle.\end{aligned}$$

■

3. Main Results

Now, our refined Berezin radius inequality could be presented like this:

Theorem 3.1. *If $X,Y\in\mathbb{L}(\mathcal{H}(\Theta))$, $0\leq\xi\leq 1$ and $\alpha\geq 1$, then we have*

$$\begin{aligned}\operatorname{ber}^{2\alpha}(Y^{*}X) &\leq (1-\xi)\operatorname{ber}^{\alpha}(Y^{*}X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\text{ber}}+\frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\text{ber}} \\ &\leq \frac{1}{2}\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\text{ber}}.\end{aligned} \tag{3.1}$$

Proof. Assume that $\widehat{k}_{\tau}\in\mathcal{H}$ is a normalized reproducing kernel. If we take $u=X\widehat{k}_{\tau}$ and $v=Y\widehat{k}_{\tau}$ in the inequality in (2.9), then we have

$$\begin{aligned}\left|\left\langle X\widehat{k}_{\tau},Y\widehat{k}_{\tau}\right\rangle\right|^{2} &\leq \left|\left\langle Y^{*}X\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\right|^{2} \\ &\leq (1-\xi)\left|\left\langle X\widehat{k}_{\tau},Y\widehat{k}_{\tau}\right\rangle\right|\left\|X\widehat{k}_{\tau}\right\|\left\|Y\widehat{k}_{\tau}\right\|+\xi\left\|X\widehat{k}_{\tau}\right\|^{2}\left\|Y\widehat{k}_{\tau}\right\|^{2} \\ &= (1-\xi)\left|\left\langle Y^{*}X\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\right|\left\langle |X|^{2}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle^{\frac{1}{2}}\left\langle |Y|^{2}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle^{\frac{1}{2}} \\ &+\xi\left\langle |X|^{2}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\left\langle |Y|^{2}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle.\end{aligned}$$

Employing the PM inequality (2.6), we get

$$\begin{aligned}\left|\left\langle Y^{*}X\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\right|^{2} \leq \Bigg((1-\xi)\left|\left\langle Y^{*}X\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\right|^{\alpha}\left\langle |X|^{2}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle^{\frac{\alpha}{2}}\left\langle |Y|^{2}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle^{\frac{\alpha}{2}} \\ +\xi\left\langle |X|^{2}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle^{\alpha}\left\langle |Y|^{2}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle^{\alpha}\Bigg)^{\frac{1}{\alpha}},\end{aligned}$$

which implies that

$$\begin{aligned}&\left|\left\langle Y^{*}X\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\right|^{2\alpha} \\ &\leq (1-\xi)\left|\left\langle Y^{*}X\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\right|^{\alpha}\left\langle |X|^{2}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle^{\frac{\alpha}{2}}\left\langle |Y|^{2}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle^{\frac{\alpha}{2}}+\xi\left\langle |X|^{2}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle^{\alpha}\left\langle |Y|^{2}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle^{\alpha} \\ &\leq (1-\xi)\left|\left\langle Y^{*}X\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\right|^{\alpha}\left\langle |X|^{2\alpha}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle^{\frac{1}{2}}\left\langle |Y|^{2\alpha}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle^{\frac{1}{2}}+\xi\left\langle |X|^{2\alpha}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\left\langle |Y|^{2\alpha}\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\end{aligned}$$

(by the inequality (2.7))

$$\leq \frac{1}{2}(1-\xi)\left|\left\langle Y^*X\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|^\alpha\left(\left\langle |X|^{2\alpha}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle+\left\langle |Y|^{2\alpha}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right)$$
$$+\frac{1}{2}\xi\left(\left\langle |X|^{4\alpha}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle+\left\langle |Y|^{4\alpha}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right)$$

(by the inequality (2.6))

$$=\frac{1}{2}(1-\xi)\left|\left\langle Y^*X\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|^\alpha\left\langle\left(|X|^{2\alpha}+|Y|^{2\alpha}\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle+\frac{1}{2}\xi\left\langle\left(|X|^{4\alpha}+|Y|^{4\alpha}\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle,$$

and

$$\sup_{\tau\in\Theta}\left|\left\langle Y^*X\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|^{2\alpha}\leq\frac{1}{2}(1-\xi)\sup_{\tau\in\Theta}\left\{\left|\left\langle Y^*X\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|^\alpha\left\langle\left(|X|^{2\alpha}+|Y|^{2\alpha}\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right\}$$
$$+\frac{1}{2}\xi\sup_{\tau\in\Theta}\left\langle\left(|X|^{4\alpha}+|Y|^{4\alpha}\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle.$$

Therefore, we have

$$\mathrm{ber}^{2\alpha}(Y^*X)\leq(1-\xi)\,\mathrm{ber}^\alpha(Y^*X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}+\frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}.$$

The desired first inequality is therefore obtained in (3.1). Nonetheless, from the inequalities (1.5) and (2.8), we get

$$\begin{aligned}\mathrm{ber}^{2\alpha}(Y^*X)&\leq\frac{1}{2}(1-\xi)\,\mathrm{ber}^\alpha(Y^*X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}+\frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}\\
&\leq\frac{1}{2}(1-\xi)\left(\frac{1}{2}\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}\right)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}\\
&+\frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}\\
&=\frac{1}{4}(1-\xi)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}^2+\frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}\\
&\leq\frac{1}{4}(1-\xi)\left\|\left(\frac{|X|^{2\alpha}+|Y|^{2\alpha}}{2}\right)^2\right\|_{\mathrm{ber}}+\frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}\\
&\leq\frac{1}{4}(1-\xi)\left\|\left(\frac{\left(2|X|^{2\alpha}\right)^2+\left(2|Y|^{2\alpha}\right)^2}{2}\right)\right\|_{\mathrm{ber}}+\frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}\\
&\leq\frac{1}{2}\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}},\end{aligned}$$

which demonstrates the second inequality in (3.1). ■

The next outcome is much better than the inequalities (3.1).

Theorem 3.2. *If $X,Y\in\mathbb{L}(\mathcal{H}(\Theta))$, $\alpha\geq 1$ and $\xi\in[0,1]$, then we get*

$$\begin{aligned}\mathrm{ber}^{2r}(Y^*X)&\leq\frac{1}{4}\xi\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}^2+\frac{1}{2}(1-\xi)\,\mathrm{ber}^\alpha(X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}\\
&\leq\frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}^2+\frac{1}{2}(1-\xi)\,\mathrm{ber}^\alpha(X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}\\
&\leq\frac{1}{2}\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}.\end{aligned}\tag{3.2}$$

Proof. Let $\tau \in \Theta$ be an arbitrary. Then for all $\xi \in [0,1]$, we have

$$\begin{aligned}\mathrm{ber}^{2\alpha}(Y^*X) &\le \xi \mathrm{ber}^{2\alpha}(Y^*X) + (1-\xi)\,\mathrm{ber}^{2\alpha}(Y^*X)\\ &= \xi \mathrm{ber}^{2\alpha}(Y^*X) + (1-\xi)\,\mathrm{ber}^{\alpha}(Y^*X)\,\mathrm{ber}^{\alpha}(Y^*X)\\ &\le \frac{1}{4}\xi \left\| |X|^{2\alpha} + |Y|^{2\alpha} \right\|^2 + \frac{1}{2}(1-\xi)\,\mathrm{ber}^{\alpha}(X) \left\| |X|^{2\alpha} + |Y|^{2\alpha} \right\|\\ &\text{(by the inequalities (1.5)),}\end{aligned}$$

which proves the first inequality in (3.2). From the inequalities (2.8),

$$\begin{aligned}\mathrm{ber}^{2\alpha}(X) &\le \frac{1}{4}\xi \left\| |X|^{2\alpha} + |Y|^{2\alpha} \right\|_{\mathrm{ber}}^2 + \frac{1}{2}(1-\xi)\,\mathrm{ber}^{\alpha}(X) \left\| |X|^{2\alpha} + |Y|^{2\alpha} \right\|_{\mathrm{ber}}\\ &= \frac{1}{4}\xi \left\| \left(\frac{\left(2|X|^{2\alpha}\right) + \left(2|Y|^{2\alpha}\right)}{2} \right)^2 \right\|_{\mathrm{ber}} + \frac{1}{2}(1-\xi)\,\mathrm{ber}^{\alpha}(X) \left\| |X|^{2\alpha} + |Y|^{2\alpha} \right\|_{\mathrm{ber}}\\ &\le \frac{1}{4}\xi \left\| \frac{\left(2|X|^{2\alpha}\right)^2 + \left(2|Y|^{2\alpha}\right)^2}{2} \right\|_{\mathrm{ber}} + \frac{1}{2}(1-\xi)\,\mathrm{ber}^{\alpha}(X) \left\| |X|^{2\alpha} + |Y|^{2\alpha} \right\|_{\mathrm{ber}}\\ &= \frac{1}{2}\xi \left\| |X|^{4\alpha} + |Y|^{4\alpha} \right\|_{\mathrm{ber}} + \frac{1}{2}(1-\xi)\,\mathrm{ber}^{\alpha}(X) \left\| |X|^{2\alpha} + |Y|^{2\alpha} \right\|_{\mathrm{ber}}\end{aligned}$$

provides the second inequality in (3.2). The third disparity in comes as a result of (3.1). ■

By taking $\alpha = 1$ and $\xi = \frac{1}{3}$ in (3.2), the outcome is as follows.

Corollary 3.3. *If $X, Y \in \mathbb{L}(\mathcal{H}(\Theta))$, then we have*

$$\begin{aligned}\mathrm{ber}^2(Y^*X) &\le \frac{1}{12} \left\| |X|^2 + |Y|^2 \right\|_{\mathrm{ber}}^2 + \frac{1}{3}\mathrm{ber}(X) \left\| |X|^2 + |Y|^2 \right\|_{\mathrm{ber}}\\ &\le \frac{1}{6} \left\| |X|^4 + |Y|^4 \right\|_{\mathrm{ber}}^2 + \frac{1}{3}\mathrm{ber}(X) \left\| |X|^2 + |Y|^2 \right\|_{\mathrm{ber}}\\ &\le \frac{1}{2} \left\| |X|^4 + |Y|^4 \right\|_{\mathrm{ber}}.\end{aligned}$$

Theorem 3.4. *If $X, Y \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \le \xi \le 1$ and $\alpha \ge 1$, then we have*

$$\begin{aligned}\mathrm{ber}^{\alpha}(Y^*X) &\le \frac{1}{\sqrt{2}}(1-\xi)\,\mathrm{ber}^{\frac{\alpha}{2}}(Y^*X) \left\| |X|^{2\alpha} + |Y|^{2\alpha} \right\|_{\mathrm{ber}}^{\frac{1}{2}} + \frac{1}{2}\xi \left\| |X|^{2\alpha} + |Y|^{2\alpha} \right\|_{\mathrm{ber}}\\ &\le \frac{1}{2} \left\| |X|^{2\alpha} + |Y|^{2\alpha} \right\|_{\mathrm{ber}}.\end{aligned}$$

Proof. Assume that $\widehat{k}_\tau \in \mathcal{H}$ is a normalized reproducing kernel. We determine the desired inequality by entering $u = X\widehat{k}_\tau$ and $v = Y\widehat{k}_\tau$ in (2.10) and continuing as in the argument of Theorem 3.1. ■

Theorem 3.5. *If $X \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \le r, \xi \le 1$ and $\varsigma \ge 1$, then we have*

$$\begin{aligned}\mathrm{ber}^{2\varsigma}(X) &\le \xi \left\| r|X|^{2\varsigma} + (1-r)|X^*|^{2\varsigma} \right\|_{\mathrm{ber}}\\ &\quad + \frac{1}{2}(1-\xi)\,\mathrm{ber}^{\varsigma}(X) \left\| |X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma} \right\|_{\mathrm{ber}}.\end{aligned} \tag{3.3}$$

Proof. Assume.that $\tau \in \Theta$ is an arbitrary. If we take $\tau = \upsilon$ in the inequality (2.11), then we get

$$\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{2\varsigma} \leq \xi \left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle$$
$$+ (1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma} \sqrt{\left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle}$$
$$\leq \xi \left\langle |X|^{2\varsigma} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{r} \left\langle |X^*|^{2\varsigma} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{(1-r)}$$
(by the inequality (2.7))
$$+ (1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma} . \left(\frac{1}{2}\left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle + \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right)$$
(bythe inequality (2.7))
$$\leq \xi \left[r\left\langle |X|^{2\varsigma} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle + (1-r)\left\langle |X^*|^{2\varsigma} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right]$$
(by the inequality (2.6))
$$+ \frac{1}{2}(1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma} . \left(\left\langle\left(|X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right) \widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right)$$
$$\leq \xi \left\langle\left(r|X|^{2\varsigma} + (1-r)|X^*|^{2\varsigma}\right) \widehat{k}_\tau, \widehat{k}_\tau\right\rangle$$
$$+ \frac{1}{2}(1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma} . \left(\left\langle\left(|X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right) \widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right),$$

and

$$\sup_{\tau\in\Theta}\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{2\varsigma} \leq \xi \sup_{\tau\in\Theta}\left\langle\left(r|X|^{2\varsigma} + (1-r)|X^*|^{2\varsigma}\right) \widehat{k}_\tau, \widehat{k}_\tau\right\rangle$$
$$+ \frac{1}{2}(1-\xi)\sup_{\tau\in\Theta}\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma}\left\langle\left(|X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right) \widehat{k}_\tau, \widehat{k}_\tau\right\rangle.$$

So, we obtain

$$\operatorname{ber}^{2\varsigma}(X) \leq \xi \left\|r|X|^{2\varsigma} + (1-r)|X^*|^{2\varsigma}\right\|_{\text{ber}}$$
$$+ \frac{1}{2}(1-\xi)\operatorname{ber}^{\varsigma}(X)\left\||X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma}\right\|_{\text{ber}}$$

which the required result. ■

In [24, Th. 3.3], it is proved that

$$\operatorname{ber}^{2\varsigma}(X) \leq \frac{1}{2}\left\|\xi|X|^{2\xi\varsigma} + (1-\xi)|X^*|^{2\varsigma}\right\|_{\text{ber}}, 0<\xi<1, \varsigma \geq 1. \tag{3.4}$$

The next finding is stronger than the disparity (3.4).

Theorem 3.6. *If $X \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \leq r, \xi \leq 1$ and $\varsigma \geq 1$, then we have*

$$\operatorname{ber}^{2\varsigma}(X) \leq \xi \left\|r|X|^{2\varsigma} + (1-r)|X^*|^{2\varsigma}\right\|_{\text{ber}}$$
$$+ (1-\xi)\operatorname{ber}^{\varsigma}(X)\sqrt{\left\|r|X|^{2\varsigma} + (1-r)|X^*|^{2\varsigma}\right\|_{\text{ber}}} \tag{3.5}$$
$$\leq \left\|r|X|^{2\varsigma} + (1-r)|X^*|^{2\varsigma}\right\|_{\text{ber}}.$$

Proof. Assume.that $\tau \in \Theta$ is an arbitrary. If we take $\tau = \upsilon$ in the inequality (2.11), then we get

$$\begin{aligned}\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{2\varsigma} &\leq \xi \left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\ &+ (1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma} \sqrt{\left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle} \\ &\leq \xi \left\langle |X|^{2\varsigma} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{r} \left\langle |X^*|^{2\varsigma} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{(1-r)} \\ &\text{(by the inequality (2.7))} \\ &+ (1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma} . \frac{1}{2}\left(\left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle + \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right) \\ &\leq \xi \left\langle \left(r|X|^{2\varsigma} + (1-r)|X^*|^{2\varsigma}\right) \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\ &+ (1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma} \sqrt{\left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle} \\ &\text{(by the inequality (2.6))}\end{aligned}$$

and

$$\begin{aligned}\sup_{\tau\in\Theta}\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{2\varsigma} &\leq \xi \sup_{\tau\in\Theta}\left\langle \left(r|X|^{2\varsigma} + (1-r)|X^*|^{2\varsigma}\right) \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\ &+ (1-\xi)\sup_{\tau\in\Theta}\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma} \sqrt{\left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle}.\end{aligned}$$

So, we deduce

$$\begin{aligned}\mathrm{ber}^{2\varsigma}(X) &\leq \xi \left\| r|X|^{2\varsigma} + (1-r)|X^*|^{2\varsigma}\right\|_{\mathrm{ber}} \\ &+ (1-\xi)\,\mathrm{ber}^{\varsigma}(X)\sqrt{\left\||rX|^{2\varsigma} + (1-r)|X^*|^{2\varsigma}\right\|_{\mathrm{ber}}}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{ber}^{2\varsigma}(X) &\leq \xi \left\| r|X|^{2\varsigma} + (1-r)|X^*|^{2\varsigma}\right\|_{\mathrm{ber}} \\ &+ (1-\xi)\,\mathrm{ber}^{\varsigma}(X)\sqrt{\left\||rX|^{2\varsigma} + (1-r)|X^*|^{2\varsigma}\right\|_{\mathrm{ber}}} \\ &\leq \left\| r|X|^{2\varsigma} + (1-r)|X^*|^{2\varsigma}\right\|_{\mathrm{ber}} \\ &\text{(by the inequality (3.4))}\end{aligned}$$

allows us to deduce the second inequality from the first inequality, demonstrating the required result. ■

Theorem 3.7. *If $X \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \leq r, \xi \leq 1$ and $\varsigma \geq 1$, then we have*

$$\mathrm{ber}^{\varsigma}(X) \leq \frac{1}{2}\xi \left\| |X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right\|_{\mathrm{ber}} + \frac{1}{\sqrt{2}}(1-\xi)\,\mathrm{ber}^{\frac{\varsigma}{2}}(X)\left\| |X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right\|_{\mathrm{ber}}^{1/2}. \tag{3.6}$$

Proof. Suppose that $\tau, \upsilon \in \Theta$ is an arbitrary. One may see from the inequality (2.12) and (2.13) that

$$\begin{aligned}\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|^{\varsigma} &\leq \xi \left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{1}{2}} \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\upsilon, \widehat{k}_\upsilon\right\rangle^{\frac{1}{2}} \\ &+ (1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|^{\frac{\varsigma}{2}} \sqrt{\left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{1}{2}} \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\upsilon, \widehat{k}_\upsilon\right\rangle^{\frac{1}{2}}} \\ &\leq \left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{1}{2}} \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\upsilon, \widehat{k}_\upsilon\right\rangle^{\frac{1}{2}}\end{aligned} \tag{3.7}$$

for every $\varsigma \geq 1$ and $0 \leq r, \xi \leq 1$. Setting $\tau = \upsilon$ in the above inequality, it follows that

$$\begin{aligned}
\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^\varsigma &\leq \xi \left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{1}{2}} \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{1}{2}} \\
&+ (1-\xi) \left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma/2} \sqrt{\left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{1}{2}} \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{1/2}} \\
&\leq \frac{1}{2}\xi \left(\left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle + \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right) \\
&+ \frac{1}{\sqrt{2}} (1-\xi) \left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma/2} \sqrt{\left(\left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle + \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right)} \\
&\text{(by the inequality (2.6))} \\
&\leq \frac{1}{2}\xi \left\langle \left(|X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right) \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&+ \frac{1}{\sqrt{2}} (1-\xi) \left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma/2} \sqrt{\left\langle \left(|X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right) \widehat{k}_\tau, \widehat{k}_\tau\right\rangle}
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\operatorname{ber}^\varsigma (X) &\leq \frac{1}{2}\xi \left\| |X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)} \right\|_{\text{ber}} \\
&+ \frac{1}{\sqrt{2}} (1-\xi) \operatorname{ber}^{\frac{\varsigma}{2}} (X) \left\| |X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)} \right\|_{\text{ber}}^{1/2}.
\end{aligned}$$

The evidence is now complete. ■

From [24, Th. 3.2], it is evident that

$$\operatorname{ber}^\varsigma (X) \leq \frac{1}{2} \left\| |X|^{2\xi\varsigma} + |X^*|^{2(1-\xi)\varsigma} \right\|_{\text{ber}} \tag{3.8}$$

if $X \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 < \xi < 1$and $\varsigma \geq 1$.

The implication that follows demonstrates that our finding (3.6) is more powerful than the inequality (3.8).

Corollary 3.8. *If $X \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \leq r, \xi \leq 1$ and $\varsigma \geq 1$, then we have*

$$\begin{aligned}
\operatorname{ber}^\varsigma (X) &\leq \frac{1}{2}\xi \left\| |X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma} \right\|_{\text{ber}} \\
&+ \frac{1}{\sqrt{2}} (1-\xi) \operatorname{ber}^{\frac{\varsigma}{2}} (X) \left\| |X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma} \right\|_{\text{ber}}^{1/2} \\
&\leq \frac{1}{2} \left\| |X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma} \right\|_{\text{ber}}.
\end{aligned}$$

Proof. Assume that $\tau, \upsilon \in \Theta$ is an arbitrary. From (3.6), we get

$$\begin{aligned}
\operatorname{ber}^\varsigma (X) &\leq \frac{1}{2}\xi \left\| |X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma} \right\|_{\text{ber}} \\
&+ \frac{1}{\sqrt{2}} (1-\xi) \operatorname{ber}^{\frac{\varsigma}{2}} (X) \left\| |X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma} \right\|_{\text{ber}}^{1/2} \\
&\leq \frac{1}{2}\xi \left\| |X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma} \right\|_{\text{ber}} + \frac{1}{2} (1-\xi) \left\| |rX|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma} \right\|_{\text{ber}} \\
&\text{(by the inequality (3.8))} \\
&= \frac{1}{2} \left\| |X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma} \right\|_{\text{ber}},
\end{aligned}$$

as required. ■

Theorem 3.9. *If $X \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \leq r, \xi \leq 1$ and $\varsigma \geq 1$, then we have*

$$\operatorname{ber}^{2\varsigma}(X) \leq \frac{1}{2}(1-\xi)\operatorname{ber}^{\varsigma}(X)\left\||X|^{2\varsigma r}+|X^*|^{2\varsigma(1-r)}\right\|_{\operatorname{ber}}+\frac{1}{2}\xi\left\||X|^{4\varsigma r}+|X^*|^{4\varsigma(1-r)}\right\|_{\operatorname{ber}}. \tag{3.9}$$

Proof. Assume that $\widehat{k}_\tau \in \mathcal{H}$ is a normalized reproducing kernel. If we take $\tau = \upsilon$ in the inequality (2.11), then we get

$$\begin{aligned}
\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{2\varsigma} &\leq \xi\left\langle |X|^{2\varsigma r}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\left\langle |X^*|^{2\varsigma(1-r)}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&\quad + (1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma}\sqrt{\left\langle |X|^{2\varsigma r}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\left\langle |X^*|^{2\varsigma(1-r)}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle} \\
&\leq \frac{1}{2}\xi\left(\left\langle |X|^{2\varsigma r}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^2+\left\langle |X^*|^{2\varsigma(1-r)}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^2\right) \\
&\quad \text{(by the inequality (2.6))} \\
&\quad + \frac{1}{2}(1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma}.\left\langle\left(|X|^{2\varsigma r}+|X^*|^{2\varsigma(1-r)}\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&\quad \text{(by the inequality (2.6))} \\
&= \frac{1}{2}\xi\left(\left\langle |X|^{4\varsigma r}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle+\left\langle |X^*|^{4\varsigma(1-r)}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right) \\
&\quad \text{(by the inequality (2.7))} \\
&\quad + \frac{1}{2}(1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma}.\left\langle\left(|X|^{2\varsigma r}+|X^*|^{2\varsigma(1-r)}\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&= \frac{1}{2}\xi\left\langle\left(|X|^{4\varsigma r}+|X^*|^{4\varsigma(1-r)}\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&\quad + \frac{1}{2}(1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma}\left\langle\left(|X|^{2\varsigma r}+|X^*|^{2\varsigma(1-r)}\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\tau\in\Theta}\left|\widetilde{X}(\tau)\right|^{2\varsigma} &\leq \frac{1}{2}\xi\sup_{\tau\in\Theta}\left\langle\left(|X|^{4\varsigma r}+|X^*|^{4\varsigma(1-r)}\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&\quad + \frac{1}{2}(1-\xi)\sup_{\tau\in\Theta}\left|\widetilde{X}(\tau)\right|^{\varsigma}\left\langle\left(|X|^{2\varsigma r}+|X^*|^{2\varsigma(1-r)}\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle.
\end{aligned}$$

Hence we get

$$\begin{aligned}
\operatorname{ber}^{2\varsigma}(X) &\leq \frac{1}{2}\xi\left\||X|^{4\varsigma r}+|X^*|^{4\varsigma(1-r)}\right\|_{\operatorname{ber}} \\
&\quad + \frac{1}{2}(1-\xi)\operatorname{ber}^{\varsigma}(X)\left\||X|^{2\varsigma r}+|X^*|^{2\varsigma(1-r)}\right\|_{\operatorname{ber}},
\end{aligned}$$

and the proof is complete. ∎

Corollary 3.10. *If $X \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \leq r, \xi \leq 1$ and $\varsigma \geq 1$, then we have*

$$\begin{aligned}
\operatorname{ber}^{2\varsigma}(X) &\leq \frac{1}{2}\xi\left\||X|^{4\varsigma r}+|X^*|^{4\varsigma(1-r)}\right\|_{\operatorname{ber}} \\
&\quad + \frac{1}{2}(1-\xi)\operatorname{ber}^{\varsigma}(X)\left\||X|^{2\varsigma r}+|X^*|^{2\varsigma(1-r)}\right\|_{\operatorname{ber}} \\
&\leq \frac{1}{2}\left\||X|^{4\varsigma r}+|X^*|^{4\varsigma(1-r)}\right\|_{\operatorname{ber}}.
\end{aligned} \tag{3.10}$$

Proof. Assume that $\tau \in \Theta$ is arbitrary. From (3.9), we get

$$\begin{aligned}
\operatorname{ber}^{2\varsigma}(X) &\leq \frac{1}{2}\xi \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\text{ber}} + \frac{1}{2}(1-\xi)\operatorname{ber}^{\varsigma}(X) \left\| |X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)} \right\|_{\text{ber}} \\
&\leq \frac{1}{2}\xi \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\text{ber}} + \frac{1}{4}(1-\xi) \left\| |X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)} \right\|_{\text{ber}}^2 \\
&\text{(by the inequality (3.8))} \\
&\leq \frac{1}{2}\xi \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\text{ber}} + \frac{1}{4}(1-\xi) \left\| \left(\frac{2|X|^{2\varsigma r} + 2|X^*|^{2\varsigma(1-r)}}{2} \right)^2 \right\|_{\text{ber}} \\
&\leq \frac{1}{2}\xi \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\text{ber}} + \frac{1}{8}(1-\xi) \left\| \left(2|X|^{2\varsigma r}\right)^2 + \left(2|X^*|^{2\varsigma(1-r)}\right)^2 \right\|_{\text{ber}} \\
&\text{(by the inequality (2.8))} \\
&\leq \frac{1}{2}\xi \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\text{ber}} + \frac{1}{2}(1-\xi) \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\text{ber}} \\
&\leq \frac{1}{2} \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\text{ber}}.
\end{aligned}$$

We determine the desired disparity (3.10). ■

We utilize the inequalities (3.4) and (3.8) for every $X \in \mathcal{B}(\mathcal{H})$, $0 \leq r, \xi \leq 1$ and $\varsigma \geq 1$. In fact, after applying (2.8), we obtain

$$\begin{aligned}
\operatorname{ber}^{2\varsigma}(X) &= \xi \operatorname{ber}^{2\varsigma}(X) + (1-\xi)\operatorname{ber}^{2\varsigma}(X) \\
&= \xi \operatorname{ber}^{2\varsigma}(X) + (1-\xi)\operatorname{ber}^{\varsigma}(X)\operatorname{ber}^{\varsigma}(X) \\
&= \frac{1}{4}\xi \left\| |X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)} \right\|_{\text{ber}} \\
&\quad + \frac{1}{2}(1-\xi)\operatorname{ber}^{\varsigma}(X) \left\| |X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)} \right\|_{\text{ber}},
\end{aligned}$$

which of course refines (3.9). In instance, we obtain

$$\operatorname{ber}^2(X) \leq \frac{1}{12} \left\| |X| + |X^*| \right\|_{\text{ber}}^2 + \frac{1}{3}\operatorname{ber}(X) \left\| |X| + |X^*| \right\|_{\text{ber}}$$

for $\varsigma = 1$, $r = \frac{1}{2}$ and $\xi = \frac{1}{3}$. It follows from Theorem 3.1 in [24] that if $X \in \mathbb{L}(\mathcal{H}(\Theta))$ then we have

$$\operatorname{ber}(X) \leq \frac{1}{2} \left\| |X| + |X^*| \right\|_{\text{ber}} \leq \frac{1}{2}\left(\|X\|_{\text{ber}} + \left\|X^2\right\|_{\text{ber}}^{1/2} \right). \tag{3.11}$$

So, from (3.11), we can deduce the inequality

$$\begin{aligned}
\operatorname{ber}^2(X) &\leq \frac{1}{12} \left\| |X| + |X^*| \right\|_{\text{ber}}^2 + \frac{1}{3}\operatorname{ber}(X) \left\| |X| + |X^*| \right\|_{\text{ber}} \\
&= \frac{1}{12} \left\| |X| + |X^*| \right\|_{\text{ber}}^2 + \frac{1}{3}\left(\frac{1}{2} \left\| |X| + |X^*| \right\|_{\text{ber}} \right) \left\| |X| + |X^*| \right\|_{\text{ber}} \\
&= \frac{1}{12} \left\| |X| + |X^*| \right\|_{\text{ber}}^2 + \frac{1}{6} \left\| |X| + |X^*| \right\|_{\text{ber}}^2 \\
&= \frac{1}{4} \left\| |X| + |X^*| \right\|_{\text{ber}}^2,
\end{aligned}$$

which indeed refines (3.11). Thus, we have

$$\begin{aligned}
\mathrm{ber}^2(X) &\le \frac{1}{12}\left\| |X| + |X^*| \right\|_{\mathrm{ber}}^2 + \frac{1}{3}\mathrm{ber}(X)\left\| |X| + |X^*| \right\|_{\mathrm{ber}} \\
&= \frac{1}{12}\left\| \left(\frac{2|X| + 2|X^*|}{2} \right)^2 \right\|_{\mathrm{ber}} + \frac{1}{3}\mathrm{ber}(X)\left\| |X| + |X^*| \right\|_{\mathrm{ber}} \\
&\le \frac{1}{24}\left\| (2|X|)^2 + (2|X^*|)^2 \right\|_{\mathrm{ber}} + \frac{1}{3}\mathrm{ber}(X)\left\| |X| + |X^*| \right\|_{\mathrm{ber}} \\
&\quad \text{(by the inequality (2.8))} \\
&= \frac{1}{6}\left\| |X| + |X^*| \right\|_{\mathrm{ber}}^2 + \frac{1}{3}\mathrm{ber}(X)\left\| |X| + |X^*| \right\|_{\mathrm{ber}},
\end{aligned}$$

which the inequality in (1.6), as required.

We recommend [8, 16–19, 22–24] for more recent findings on Berezin radius inequalities for operators and related findings.

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Permuting Tri-derivations in MV-algebras

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Abstract. An MV-algebra is an algebraic structure with a binary operation $\oplus$, a unary operation $'$ and the constant 0 satisfying certain axioms. MV-algebras are the algebraic semantics of Lukasiewicz logic. This work includes a type of derivation research on MV-algebras. Our aim is to introduce the concept of permuting tri-derivation on MV-algebras and to discuss some results.
AMS Subject Classifications: 06D35, 03G25, 13N15.

Keywords: MV-algebra, permuting tri-derivation, fixed set, Boolean algebra, isotone.

Contents

1. Introduction

The concept of derivation has an important place in the research of the structure and properties of algebraic systems. In prime rings, the notion of derivation was introduced by Posner [16]. In [17], Szasz applied the derivation concept to lattices. Xin et al. developed derivation for a lattice and they offered some equivalent conditions under which a derivation is isotone for lattices with a greatest element, modular lattices and distributive lattices, in [18] and [19]. Later, different derivations and properties in lattices were examined, for example [5], [6]. In [15], Öztürk achieved some results by introducing the idea of permuting tri-derivations in rings. After, Öztürk et al. studied the permuting tri-derivations in lattices [14]. Further, permuting skew 3-derivations, permuting skew n-derivations in rings have studied and commutativity of a ring satisfying certain identities involving the trace of permuting n-derivations (see [3], [9], [10]).

When dealing with information and uncertainty, non-classical logic is useful in terms of uncertain and fuzzy information in computer science. MV-algebras as the algebraic counterpart of many-valued prepositional calculus were proposed by Chang [7]. Classical two-valued logic makes it meaningful to study Boolean algebras, and while every Boolean algebra is an MV-algebra, the reverse is not true. MV-algebras have many applications as they are generalization of Boolean algebras. Also, MV-algebras are categorically equivalent to some mathematical structures. For example, perfect MV-algebras categorically equivalent to abelian

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lattice-groups with strong unit and to bounded commutative BCK-algebras (see [12], [13]). In [1], Alshehri presented the concept of derivation in MV-algebras and examined some properties of the derivation in MV-algebras with the help of isotone derivations. Recently, several authors studied different derivations in MV-algebras, for example [2], [11], [20].

In this paper, we introduce the notion of permuting tri-derivations in MV-algebras. This article is organized as follows: In the next section, some results and basic concepts about MV-algebras are reminded. In section 3, permuting tri-derivation structure in MV-algebras is characterized and some results are obtained. Also, fixed point set structure of isotone permutig tri-derivations is established.

2. Preliminaries

Definition 2.1. *[7] Let us define $\oplus$ binary operation, $'$ a unary operation on the set Δ and 0 be a constant in Δ. If the following axioms are satisfied, then we say $(\Delta, \oplus, ', 0)$ is MV-algebra:*

(i) $(\Delta, \oplus, 0)$ is a commutative monoid,
(ii) $(\delta^{'})^{'} = \delta$,
(iii) $0^{'} \oplus \delta = 0^{'}$,
(iv) $(\delta^{'} \oplus \eta)^{'} \oplus \eta = (\eta^{'} \oplus \delta)^{'} \oplus \delta$ for all $\delta, \eta \in \Delta$.

In the remainder of the article, we denote an MV-algebra $(\Delta, \oplus, ', 0)$ by Δ.

Define the operations $\odot$ and $\ominus$ and the constant 1 as follows: $1 = 0^{'}$, $\delta \odot \eta = (\delta^{'} \oplus \eta^{'})^{'}$, $\delta \ominus \eta = \delta \odot \eta^{'}$. If we define $\delta \leq \eta$ if and only if $\delta^{'} \oplus \eta = 1$, then "$\leq$" is a partial order which called the natural order of Δ. This order determines a bounded distributive lattice structure. For the elements δ and η, the join $\delta \vee \eta$ and the meet $\delta \wedge \eta$ defined by: $\delta \vee \eta = (\delta \odot \eta^{'}) \oplus \eta = (\delta \ominus \eta) \oplus \eta$ and $\delta \wedge \eta = \delta \odot (\delta^{'} \oplus \eta) = \delta \ominus (\delta \ominus \eta) = (\delta^{'} \vee \eta^{'})^{'}$. Also, Δ is called linearly ordered, if the order relation "$\leq$" is total.

Example 2.2. *[8] Let $\Delta = [0, 1]$ be the real unit interval. For all $\delta, \eta \in \Delta$, if we define $\delta \oplus \eta = \min\{1, \delta + \eta\}$, $\delta \odot \eta = \max\{0, \delta + \eta - 1\}$ and $\delta^{'} = 1 - \delta$, then $(\Delta, \oplus, ', 0)$ is an MV-algebra. For each integer $n \geq 2$, the n-element set $\Delta_n = \left\{0, \frac{1}{n-1}, ..., \frac{n-2}{n-1}, 1\right\}$ is a linearly ordered MV-algebra which called MV-chain.*

Proposition 2.3. *[4, 8] Suppose that Δ is an MV-algebra and $\delta, \eta, \sigma \in \Delta$. Thus the followings hold:*

(1) $\delta \oplus \delta^{'} = 1$, $\delta \odot \delta^{'} = 0$, $\delta \oplus 1 = 1$,
(2) Provided that $\delta \oplus \eta = 0$, then $\delta = \eta = 0$,
(3) Provided that $\delta \odot \eta = 1$, then $\delta = \eta = 1$,
(4) If $\delta \leq \eta$, then $\delta \oplus \sigma \leq \eta \oplus \sigma$ and $\delta \odot \sigma \leq \eta \odot \sigma$,
(5) $\delta \odot \eta \leq \delta \wedge \eta \leq \delta, \eta \leq \delta \vee \eta \leq \delta \oplus \eta$,
(6) $\delta \leq \eta$ iff $\eta^{'} \leq \delta^{'}$,
(7) $\delta \oplus \eta = \eta$ iff $\delta \odot \eta = \delta$,
(8) $\delta \odot (\eta \vee \sigma) = (\delta \odot \eta) \vee (\delta \odot \sigma)$,
(9) $\delta \oplus (\eta \wedge \sigma) = (\delta \oplus \eta) \wedge (\delta \oplus \sigma)$,
(10) $\delta \oplus \eta = \eta$ iff $\delta \wedge \eta^{'} = 0$,
(11) If $\delta \odot \eta = \delta \odot \sigma$ and $\delta \oplus \eta = \delta \oplus \sigma$, then $\eta = \sigma$.

MV-algebras that do not satisfy idempotent conditions are generalizations of Boolean algebras. For any MV-algebra Δ, if we define $B(\Delta) = \{\delta \in \Delta | \, \delta \odot \delta = \delta\} = \{\delta \in \Delta | \, \delta \oplus \delta = \delta\}$, then $(B(\Delta), \oplus, ', 0)$ is a largest subalgebra of Δ, which is called Boolean center of Δ.

Theorem 2.4. *[8] Let Δ be an MV-algebra. Then for each element δ in Δ, the following conditions are equivalent:*

(1) $\delta \in B(\Delta)$,
(2) $\delta \vee \delta^{'} = 1$,
(3) $\delta \wedge \delta^{'} = 0$,
(4) $\delta \oplus \delta = \delta$,
(5) $\delta \odot \delta = \delta$,
(6) $\delta \oplus \eta = \delta \vee \eta$ for all $\eta \in \Delta$,
(7) $\delta \odot \eta = \delta \wedge \eta$ for all $\eta \in \Delta$.

Theorem 2.5. *[7] Assume that Δ is an MV-algebra. Therefore, the following expressions are equivalent:*

(i) $\delta \leq \eta$,
(ii) $\eta \oplus \delta^{'} = 1$,
(iii) $\delta \odot \eta^{'} = 0$.

Definition 2.6. *[7] Suppose Δ be an MV-algebra and $\emptyset \neq I \subseteq \Delta$. If the following situations are satisfied,*

(1) $0 \in I$,
(2) Provided that $\delta, \eta \in I$, then $\delta \oplus \eta \in I$,
(3) Provided that $\eta \in I$, $\delta \in \Delta$ and $\delta \leq \eta$, then $\delta \in I$
then I is called an ideal of Δ.

Proposition 2.7. *[7] Assume that Δ is a linearly ordered MV-algebra. Then $\delta \oplus \eta = \delta \oplus \sigma$ and $\delta \oplus \sigma \neq 1$ imply that $\eta = \sigma$.*

Definition 2.8. *[1] Assume that Δ is an MV-algebra. A mapping $D : \Delta \to \Delta$ is called a derivation on Δ if it provides*

$$D(\delta_1 \odot \delta_2) = (D(\delta_1) \odot \delta_2) \oplus (\delta_1 \odot D(\delta_2))$$

for all $\delta_1, \delta_2 \in \Delta$.

3. Permuting tri-derivations on MV-algebras

We begin with the following definition.

Definition 3.1. *Suppose that Δ is an MV-algebra. A map $\Gamma : \Delta \times \Delta \times \Delta \to \Delta$ is called permuting if $\Gamma(\delta, \eta, \sigma) = \Gamma(\delta, \sigma, \eta) = \Gamma(\eta, \delta, \sigma) = \Gamma(\eta, \sigma, \delta) = \Gamma(\sigma, \delta, \eta) = \Gamma(\sigma, \eta, \delta)$ holds for all $\delta, \eta, \sigma \in \Delta$.*

A mapping $\gamma : \Delta \to \Delta$ defined by $\gamma(\delta) = \Gamma(\delta, \delta, \delta)$ is called the trace of Γ, where $\Gamma : \Delta \times \Delta \times \Delta \to \Delta$ is a permuting mapping. In that follows, we often abbreviate $\gamma(\delta)$ to $\gamma\delta$.

Definition 3.2. *Suppose that Δ is an MV-algebra and $\Gamma : \Delta \times \Delta \times \Delta \to \Delta$ is a permuting mapping. If Γ satisfies the following*

$$\Gamma(\delta \odot \rho, \eta, \sigma) = (\Gamma(\delta, \eta, \sigma) \odot \rho) \oplus (\delta \odot \Gamma(\rho, \eta, \sigma))$$

for all $\delta, \eta, \sigma, \rho \in \Delta$, then Γ is called a permuting tri-derivation. Clearly, if Γ is a permuting tri-derivation on Δ, then the relations hold: for all $\delta, \eta, \sigma, \rho \in \Delta$,

$$\Gamma(\delta, \eta \odot \rho, \sigma) = (\Gamma(\delta, \eta, \sigma) \odot \rho) \oplus (\eta \odot \Gamma(\delta, \rho, \sigma))$$

and

$$\Gamma(\delta, \eta, \sigma \odot \rho) = (\Gamma(\delta, \eta, \sigma) \odot \rho) \oplus (\sigma \odot \Gamma(\delta, \eta, \rho)).$$

Example 3.3. *Let $\Delta = \{0, \delta, \eta, 1\}$. Consider the tables given below:*

$\oplus$	0	δ	η	1
0	0	δ	η	1
δ	δ	δ	1	1
η	η	1	η	1
1	1	1	1	1

$'$	0	δ	η	1
	1	η	δ	0

Then $(\Delta, \oplus, ', 0)$ is an MV- algebra. Define a mapping $\Gamma : \Delta \times \Delta \times \Delta \to \Delta$ by $\Gamma(x_1, x_2, x_3) = \begin{cases} \delta, \ x_1, x_2, x_3 \in \{1, \delta\} \\ 0, \ otherwise \end{cases}$.

It appears that Γ is a permuting tri-derivation on Δ.

Proposition 3.4. *Suppose that Δ is an MV-algebra, Γ is a permuting tri-derivation on Δ and γ is the trace of Γ. For all $\delta \in \Delta$, we have*

(1) $\gamma 0 = 0$,
(2) $\gamma\delta \odot \delta' = \delta \odot \gamma\delta' = 0$,
(3) $\gamma\delta = \gamma\delta \oplus (\delta \odot \Gamma(\delta, \delta, 1))$,
(4) $\gamma\delta \leq \delta$,
(5) If I is an ideal of Δ, then $\gamma(I) \subseteq I$.

Proof. (1) We can write

$$\begin{aligned} \gamma 0 &= \Gamma(0,0,0) = \Gamma(0 \odot 0, 0, 0) \\ &= (\Gamma(0,0,0) \odot 0) \oplus (0 \odot \Gamma(0,0,0)) \\ &= 0 \oplus 0 = 0. \end{aligned}$$

(2) For all $\delta \in \Delta$,

$$\begin{aligned} \Gamma(\delta,\delta,0) &= \Gamma(\delta,\delta,0 \odot 0) \\ &= (\Gamma(\delta,\delta,0) \odot 0) \oplus (0 \odot \Gamma(\delta,\delta,0)) \\ &= 0 \oplus 0 = 0. \end{aligned}$$

Then, we get

$$\begin{aligned} 0 &= \Gamma(\delta,\delta,0) = \Gamma(\delta,\delta,\delta \odot \delta') \\ &= (\Gamma(\delta,\delta,\delta) \odot \delta') \oplus (\delta \odot \Gamma(\delta,\delta,\delta')). \end{aligned}$$

By the property (2) of Proposition 2.3, $\gamma\delta \odot \delta' = 0$ and $\delta \odot \Gamma(\delta,\delta,\delta') = 0$. We can see that $\delta \odot \gamma\delta' = 0$ for all $\delta \in \Delta$, similarly.

(3) For all $\delta \in \Delta$,

$$\begin{aligned} \gamma\delta &= \Gamma(\delta,\delta,\delta) = \Gamma(\delta,\delta,\delta \odot 1) \\ &= (\Gamma(\delta,\delta,\delta) \odot 1) \oplus (\delta \odot \Gamma(\delta,\delta,1)) \\ &= \gamma\delta \oplus (\delta \odot \Gamma(\delta,\delta,1)). \end{aligned}$$

(4) For all $\delta \in \Delta$,

$$1 = 0' = (\gamma\delta \odot \delta')' = [((\gamma\delta)' \oplus (\delta')')']' = (\gamma\delta)' \oplus \delta.$$

Then, by Theorem 2.5, we have $\gamma\delta \leq \delta$ for all $\delta \in \Delta$.

(5) If $\eta \in \gamma(I)$, then $\eta = \gamma(\delta)$ for some $\delta \in I$. From (4), we have $\gamma(\delta) \leq \delta$. Since I is an ideal of Δ, we get $\eta \in I$ and so $\gamma(I) \subseteq I$. ■

Remark 3.5. *We have* $\delta \odot \Gamma(\delta, \delta, \delta^{'}) = 0$ *for all* $\delta \in \Delta$. *Thus,* $\Gamma(\delta, \delta, \delta^{'}) \leq \delta^{'}$ *and* $\delta \leq (\Gamma(\delta, \delta, \delta^{'}))^{'}$. *For all* $\delta, \eta, \sigma \in \Delta$,

$$0 = \Gamma(\delta \odot \delta^{'}, \eta, \sigma) = (\Gamma(\delta, \eta, \sigma) \odot \delta^{'}) \oplus (\delta \odot \Gamma(\delta^{'}, \eta, \sigma))$$

and hence $\Gamma(\delta, \eta, \sigma) \leq \delta$ *and* $\Gamma(\delta^{'}, \eta, \sigma) \leq \delta^{'}$.

Proposition 3.6. *Let* Δ *be an MV-algebra,* Γ *be a permuting tri-derivation on* Δ *and* γ *be the trace of* Γ. *For* $\delta, \eta \in \Delta$, *if* $\delta \leq \eta$ *then*

(1) $\gamma(\delta \odot \eta^{'}) = 0$,
(2) $\gamma\eta^{'} \leq \delta^{'}$,
(3) $\gamma\delta \odot \gamma\eta^{'} = 0$.

Proof. (1) We assume $\delta \leq \eta$ for $\delta, \eta \in \Delta$. By the property (4) of Proposition 2.3, we have $\delta \odot \eta^{'} \leq \eta \odot \eta^{'} = 0$. Then, $\delta \odot \eta^{'} = 0$ and so $\gamma(\delta \odot \eta^{'}) = 0$, since $\gamma 0 = 0$.

(2) We have $\delta \odot \gamma\eta^{'} \leq \eta \odot \gamma\eta^{'} \leq \eta \odot \eta^{'} = 0$ since $\delta \leq \eta$. From here we obtain that $\delta \odot \gamma\eta^{'} = 0$ and $\gamma\eta^{'} \leq \delta^{'}$.

(3) We have $\gamma\delta \leq \eta$ since $\delta \leq \eta$. Hence $\gamma\delta \odot \gamma\eta^{'} \leq \eta \odot \gamma\eta^{'} \leq \eta \odot \eta^{'} = 0$ and so $\gamma\delta \odot \gamma\eta^{'} = 0$. ■

Proposition 3.7. *Suppoese that* Δ *is an MV-algebra,* Γ *is a permuting tri-derivation on* Δ *and* γ *is the trace of* Γ. *Then,*

(1) $\gamma\delta \odot \gamma\delta^{'} = 0$,
(2) $\gamma\delta^{'} = (\gamma\delta)^{'}$ iff γ is the identity on Δ.

Proof. (1) From Proposition 3.6(3), $\gamma\delta \odot \gamma\eta^{'} = 0$. Taking η by δ, we have $\gamma\delta \odot \gamma\delta^{'} = 0$.

(2) Since $\delta \odot \gamma\delta^{'} = 0$, we get $\delta \odot (\gamma\delta)^{'} = 0$. Then, $\gamma\delta \leq \delta$ and $\delta \leq \gamma\delta$ i.e., $\gamma\delta = \delta$. Thus, γ is identity on Δ. Conversely, if γ is identity on Δ, then $\gamma\delta^{'} = (\gamma\delta)^{'}$, $\forall \delta \in \Delta$. ■

Definition 3.8. *Suppose that* Δ *is an MV-algebra and* Γ *is a permuting tri-derivation on* Δ. *If* $\delta \leq \rho$ *implies* $\Gamma(\delta, \eta, \sigma) \leq \Gamma(\rho, \eta, \sigma)$ *for all* $\delta, \eta, \sigma, \rho \in \Delta$, *then* Γ *is called an isotone. If* γ *is the trace of* Γ *and* Γ *is an isotone, then* $\delta \leq \eta$ *implies* $\gamma\delta \leq \gamma\eta$ *for all* $\delta, \eta \in \Delta$.

Example 3.9. *Let* $\Delta = \{0, \delta_1, \delta_2, \delta_3, \delta_4, 1\}$. *Consider the following tables:*

$\oplus$	0	δ_1	δ_2	δ_3	δ_4	1
0	0	δ_1	δ_2	δ_3	δ_4	1
δ_1	δ_1	δ_3	δ_4	δ_3	1	1
δ_2	δ_2	δ_4	δ_2	1	δ_4	1
δ_3	δ_3	δ_3	1	δ_3	1	1
δ_4	δ_4	1	δ_4	1	1	1
1	1	1	1	1	1	1

$'$	0	δ_1	δ_2	δ_3	δ_4	1
	1	δ_4	δ_3	δ_2	δ_1	0

Then $(\Delta, \oplus, ', 0)$ is an MV-algebra. Let us define a map $\Gamma : \Delta \times \Delta \times \Delta \rightarrow \Delta$ by $\Gamma(x_1, x_2, x_3) = \begin{cases} \delta_2, & x_1, x_2, x_3 \in \{\delta_2, \delta_4, 1\} \\ 0, & otherwise \end{cases}$. We can see that Γ is an isotone permuting tri-derivation on Δ.

Example 3.10. *Consider* $\Delta_4 = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ *as in Example 2.2 and define* $\Gamma : \Delta_4 \times \Delta_4 \times \Delta_4 \rightarrow \Delta_4$ *by* $\Gamma(x_1, x_2, x_3) = \begin{cases} \frac{1}{3}, & (x_1, x_2, x_3) \in \{(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})\} \\ 0, & otherwise \end{cases}$.

Then Γ is a permuting tri-derivation on Δ_4, but Γ is not isotone, because $\Gamma(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \nleq \Gamma(1, \frac{2}{3}, \frac{2}{3})$.

Proposition 3.11. *Suppose that Δ is an MV-algebra, Γ is a permuting tri-derivation on Δ and γ is the trace of Γ. If $\gamma\delta^{'} = \gamma\delta$ for all $\delta \in \Delta$, then the followings hold:*

(1) $\gamma 1 = 0$,
(2) $\gamma\delta \odot \gamma\delta = 0$,
(3) Provided that Γ is an isotone on Δ, then $\gamma = 0$.

Proof. (1) Replacing δ by 0 in hypothesis, we have $\gamma 1 = 0$.
(2) For all $\delta \in \Delta$, $\gamma\delta \odot \gamma\delta = \gamma\delta \odot \gamma\delta^{'} = 0$, by Proposition 3.7.
(3) Suppose that Γ is an isotone on Δ. For $\delta \in \Delta$, since $\gamma\delta \leq \gamma 1 = 0$, we have $\gamma\delta = 0$ and so $\gamma = 0$. ■

Proposition 3.12. *Suppose that Δ is an MV-algebra, Γ is a permuting tri-derivation on Δ and γ is the trace of Γ and $\delta \in B(\Delta)$. Then the followings hold:*

(1) If $\delta \leq \Gamma(1, \eta, \sigma)$ for all $\eta, \sigma \in \Delta$, then $\Gamma(\delta, \eta, \sigma) = \delta$,
(2) $\delta \wedge \Gamma(\delta, \delta, 1) \wedge (\gamma\delta)^{'} = 0$,
(3) If $\delta \leq \Gamma(\delta, \delta, 1)$, then $\gamma\delta = \delta$.

Proof. (1) We have

$$\begin{aligned}\Gamma(\delta, \eta, \sigma) &= \Gamma(\delta \odot 1, \eta, \sigma)\\ &= (\Gamma(\delta, \eta, \sigma) \odot 1) \oplus (\delta \odot \Gamma(1, \eta, \sigma))\\ &= \Gamma(\delta, \eta, \sigma) \oplus \delta = \delta.\end{aligned}$$

(2) Since $\gamma\delta = \gamma\delta \oplus (\delta \odot \Gamma(\delta, \delta, 1))$, it follows that $(\delta \odot \Gamma(\delta, \delta, 1)) \wedge (\gamma\delta)^{'} = 0$. Then, by Theorem 2.4, we obtain $\delta \wedge \Gamma(\delta, \delta, 1) \wedge (\gamma\delta)^{'} = 0$.
(3) Let $\delta \leq \Gamma(\delta, \delta, 1)$. Then, we get $\delta \odot (\gamma\delta)^{'} = 0$ by (2). Thus, $\delta \leq \gamma\delta \leq \delta$ and so $\gamma\delta = \delta$. ■

Theorem 3.13. *Suppose that Δ is an MV-algebra. We define a map by $\Gamma(\delta, \eta, \sigma) = \delta \odot \eta \odot \sigma$ for all $\delta, \eta, \sigma \in \Delta$. Then Γ is a permuting tri-derivation on $B(\Delta)$.*

Proof. We have

$$\Gamma(\delta \odot \rho, \eta, \sigma) = (\delta \odot \rho) \odot \eta \odot \sigma$$

for all $\delta, \eta, \sigma, \rho \in B(\Delta)$. Moreover,

$$\begin{aligned}(\Gamma(\delta, \eta, \sigma) \odot \rho) \oplus (\delta \odot \Gamma(\rho, \eta, \sigma)) &= ((\delta \odot \eta \odot \sigma) \odot \rho) \oplus (\delta \odot (\rho \odot \eta \odot \sigma))\\ &= (\delta \odot \rho) \odot \eta \odot \sigma.\end{aligned}$$

Thus, Γ is a permuting tri-derivation on $B(\Delta)$. ■

Definition 3.14. *Suppose that Δ is an MV-algebra, Γ is a permuting mapping on Δ. If $\Gamma(\delta \oplus \rho, \eta, \sigma) = \Gamma(\delta, \eta, \sigma) \oplus \Gamma(\rho, \eta, \sigma)$ for all $\delta, \eta, \sigma, \rho \in \Delta$, then Γ is said to be tri-additive mapping.*

Theorem 3.15. *Suppose that Δ is an MV-algebra, Γ is a tri-additive mapping on Δ and γ is the trace of Γ. Thus, $\gamma(B(\Delta)) \subseteq B(\Delta)$.*

Proof. Let $\delta \in \gamma(B(\Delta))$. Then, $\delta = \gamma(\eta)$ for some $\eta \in B(\Delta)$. Hence, $\delta \oplus \delta = \gamma\eta \oplus \gamma\eta = \Gamma(\eta \oplus \eta, \eta, \eta) = \gamma\eta = \delta$. Therefore, $\delta \in B(\Delta)$ i.e., $\gamma(B(\Delta)) \subseteq B(\Delta)$. ■

Theorem 3.16. *Suppose that Δ is a linearly ordered MV-algebra, Γ is a tri-additive permuting tri-derivation on Δ and γ is the trace of Γ. Then $\gamma = 0$ or $\gamma 1 = 1$.*

Proof. For all $\delta \in \Delta$, we have $\delta \oplus \delta^{'} = 1$ and $\delta \oplus 1 = 1$. Thus,

$$\gamma 1 = \Gamma(1,1,1) = \Gamma(\delta \oplus \delta^{'},1,1) = \Gamma(\delta,1,1) \oplus \Gamma(\delta^{'},1,1)$$

and

$$\gamma 1 = \Gamma(1,1,1) = \Gamma(\delta \oplus 1,1,1) = \Gamma(\delta,1,1) \oplus \gamma 1.$$

If $\gamma 1 \neq 1$, then we have $\gamma 1 = \Gamma(\delta^{'},1,1)$ by Proposition 2.7. Replacing δ by 1, we have $\gamma 1 = 0$. For all $\delta \in \Delta$,

$$0 = \gamma 1 = \Gamma(\delta,1,1) \oplus \gamma 1 = \Gamma(\delta,1,1)$$

and

$$\begin{aligned}\Gamma(\delta,1,1) &= \Gamma(\delta,1,\delta \oplus 1) = \Gamma(\delta,1,\delta) = \Gamma(\delta,\delta \oplus 1,\delta)\\ &= \gamma\delta \oplus \Gamma(\delta,1,\delta) = \gamma\delta.\end{aligned}$$

Therefore, $\gamma\delta = 0$ for all $\delta \in \Delta$. In this case, we have $\gamma = 0$. ∎

Proposition 3.17. *Suppose that Δ is an MV-algebra, Γ is a tri-additive permuting tri-derivation on Δ. Then,*

(1) Γ is an isotone,
(2) If γ is trace of Γ, then $\gamma\delta = \delta \odot \Gamma(\delta,\delta,1)$ for all $\delta \in B(\Delta)$.

Proof. (1) Let $\delta \leq \rho$. Then,

$$\begin{aligned}\Gamma(\rho,\eta,\sigma) &= \Gamma(\rho \vee \delta,\eta,\sigma) = \Gamma((\rho \odot \delta^{'}) \oplus \delta,\eta,\sigma)\\ &= \Gamma(\rho \odot \delta^{'},\eta,\sigma) \oplus \Gamma(\delta,\eta,\sigma) \geq \Gamma(\delta,\eta,\sigma)\end{aligned}$$

for all $\delta,\eta,\sigma,\rho \in \Delta$.

(2) Since Γ is an isotone, we have $\gamma\delta \leq \Gamma(\delta,\delta,1)$. Thus

$$\delta \odot \gamma\delta \leq \delta \odot \Gamma(\delta,\delta,1) \leq \gamma\delta \oplus (\delta \odot \Gamma(\delta,\delta,1)) = \gamma\delta.$$

Also, $\delta \in B(\Delta)$ implies that $\delta \odot \gamma\delta = \delta \wedge \gamma\delta = \gamma\delta$. Hence $\gamma\delta = \delta \odot \Gamma(\delta,\delta,1)$. ∎

Remark 3.18. *Suppose that Δ is an MV-algebra, Γ is a tri-additive permuting tri-derivation on Δ and γ is the trace of Γ. If $\gamma\delta = 0$ for all $\delta \in \Delta$, then $\Gamma(\delta,\delta,\eta) = 0$ for all $\eta \in \Delta$. Indeed, we have*

$$0 = \gamma\delta = \Gamma(\delta,\delta,\delta) = \gamma\delta \oplus \Gamma(1,\delta,\delta) = \Gamma(1,\delta,\delta)$$

and so

$$0 = \Gamma(1,\delta,\delta) = \Gamma(\eta \oplus 1,\delta,\delta) = \Gamma(\eta,\delta,\delta).$$

Theorem 3.19. *Suppose that Δ is an MV-algebra, Γ is a tri-additive permuting tri-derivation on Δ and γ is the trace of Γ. Then,*

$$\ker\gamma = \gamma^{-1}(0) = \{\delta \in \Delta |\, \gamma\delta = 0\}$$

is an ideal of Δ.

Proof. We have $\gamma 0 = 0$, by Proposition 3.4(1). This yields that $0 \in \gamma^{-1}(0)$. Assume that $\delta,\eta \in \gamma^{-1}(0)$. Then,

$$\begin{aligned}\gamma(\delta \oplus \eta) &= \Gamma(\delta \oplus \eta,\delta \oplus \eta,\delta \oplus \eta)\\ &= \gamma\delta \oplus \Gamma(\delta,\delta,\eta) \oplus \Gamma(\delta,\eta,\delta) \oplus \Gamma(\delta,\eta,\eta)\\ &\oplus \Gamma(\eta,\delta,\delta) \oplus \Gamma(\eta,\delta,\eta) \oplus \Gamma(\eta,\eta,\delta) \oplus \gamma\eta.\end{aligned}$$

Using Remark 3.8, $\gamma(\delta \oplus \eta) = 0$ which ensures that $\delta \oplus \eta \in \gamma^{-1}(0)$. Suppose $\delta \in \gamma^{-1}(0)$ and $\eta \leq \delta$. Since Γ is an isotone, we get $\gamma\eta \leq \gamma\delta = 0$. Thus $\gamma\eta = 0$ and so $\eta \in \gamma^{-1}(0)$. ∎

Now, we discuss the structures and some properties of fixed points set of isotone permuting tri-derivations. Let Γ be an isotone permuting tri-derivation on Δ. We denote by $Fix_{\gamma}(\Delta)$ the set of all fixed points of Δ for γ. That is,

$$Fix_{\gamma}(\Delta) = \{\delta \in \Delta | \gamma\delta = \delta\}.$$

Theorem 3.20. *Suppose that Δ is an MV-algebra and Γ is a tri-additive permuting tri-derivation on Δ and γ is the trace of Γ. Then,*

(1) $\gamma\delta = \gamma 1 \odot \delta$ for any $\delta \in Fix_{\gamma}(\Delta)$,
(2) $\gamma^2(\delta) = \gamma(\delta)$ for any $\delta \in Fix_{\gamma}(\Delta)$; where $\gamma^2(\delta) = \gamma(\gamma(\delta))$,
(3) $Fix_{\gamma}(\Delta) = \gamma(Fix_{\gamma}(\Delta))$.

Proof. (1) We have $\gamma\delta = \delta$. Thus, $\gamma 1 \odot \gamma\delta = \gamma 1 \wedge \gamma\delta = \gamma\delta$ implies that $\gamma\delta = \gamma 1 \odot \delta$ for all $\delta \in Fix_{\gamma}(\Delta)$.

(2) If δ is a fixed point of γ, then $\gamma(\delta) = \delta$, so $\gamma(\gamma(\delta)) = \gamma(\delta) = \delta$.

(3) If $\delta \in Fix_{\gamma}(\Delta)$, then $\delta \in \gamma(Fix_{\gamma}(\Delta))$. If $\delta \in \gamma(Fix_{\gamma}(\Delta))$, then for some $\eta \in Fix_{\gamma}(\Delta)$, $\delta = \gamma\eta = \eta$. Thus, we have $\delta \in Fix_{\gamma}(\Delta)$. Therefore, $Fix_{\gamma}(\Delta) = \gamma(Fix_{\gamma}(\Delta))$. ■

Example 3.21. *In Example 2.2, considering $\Delta_3 = \left\{0, \frac{1}{2}, 1\right\}$ and defining $\Gamma : \Delta_3 \times \Delta_3 \times \Delta_3 \to \Delta_3$ by $\Gamma(x_1, x_2, x_3) = \begin{cases} \frac{1}{2}, & x_1 = x_2 = x_3 = \frac{1}{2} \\ 0, & otherwise \end{cases}$. One can check that Γ is a permuting tri-derivation on Δ_3 and $Fix_{\gamma}(\Delta_3) = \left\{0, \frac{1}{2}\right\}$. Since $\frac{1}{2} \oplus \frac{1}{2} = 1 \notin Fix_{\gamma}(\Delta_3)$. Then, we have $Fix_{\gamma}(\Delta_3)$ is not an ideal of Δ_3.*

As can be seen from the example above, we encounter the following open problem:
For any ideal I of a MV-algebra Δ, whether there is a permuting tri-derivation Γ such that $Fix_{\gamma}(\Delta_3) = I$.

4. Conclusion

The concept of derivation was presented by Posner in 1957. In the following years, many mathematicians used derivations to examine the properties of algebraic structures. In their studies, on different derivations, the conditions for the ring to be commutative are examined. Some characterizations of algebraic structures are determined by the trace of permuting tri-additive mappings. With the help of the trace of permuting tri-derivation, the commutativity conditions of rings and how the elements are ordered in some structures such as lattices are investigated. The derivation type used in this article was put forward by Öztürk in rings. In this study, we obtained some results by presenting permuting tri-derivations on MV-algebras. The first aim of this study is to give the notion of permuting tri-derivation on this algebraic structure. Then some features provided by this derivation are listed. Fixed set structure has been investigated for such derivations by defining isotone permuting tri-derivations. After this study, permuting tri-f-derivations and permuting tri-(f, g)-derivations can be studied on MV-algebras. Also, since MV-algebras are BL-algebras that provide double negation property, permuting tri-derivation structure can be examined in BL-algebras.

applicable.

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On $\mathcal{I}$ and $\mathcal{I}^*$-equal convergence in linear 2-normed spaces

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Abstract. In this paper we study the notion of $\mathcal{I}$ and $\mathcal{I}^*$-equal convergence in linear 2-normed spaces and some of their properties. We also establish the relationship between them.

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Keywords: Ideal, linear 2-normed spaces, $\mathcal{I}$-equal convergence, condition (AP), $\mathcal{I}^*$-equal convergence.

Contents

1. Introduction

The idea of usual convergence of a real sequence was extended to statistical convergence independently by Fast [11] and Steinhaus [21] in the year 1951. Lot of developments were made on this notion of convergence after the pioneering works of Šalát [22] and Fridy [12]. After long fifty years, the concept of statistical convergence was extended to the idea of $\mathcal{I}$-convergence depending on the structure of ideals $\mathcal{I}$ of $\mathbb{N}$, the set of natural numbers, by Kostyrko et al. [17]. Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers respectively. $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and $B \in \mathcal{I}$ whenever $B \subset A \in \mathcal{I}$. $\mathcal{I}$ is called an admissible ideal of $\mathbb{N}$ if $\{x\} \in \mathcal{I}$ for each $x \in \mathbb{N}$. $\mathcal{I} \subset 2^{\mathbb{N}}$ is called non-trivial ideal if $\mathcal{I} \neq \{\phi\}$ and $\mathbb{N} \notin \mathcal{I}$. If $\mathcal{I}$ is a non-trivial proper ideal of $\mathbb{N}$ then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$ is a filter on $\mathbb{N}$, called the filter associated with the ideal $\mathcal{I}$. Indeed, the concept of $\mathcal{I}$- convergence of real sequences is a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subsets of $\mathbb{N}$. $\mathcal{I}$-convergence of real sequences coincides with the ordinary convergence if $\mathcal{I}$ is the ideal of all finite subsets of $\mathbb{N}$ and with the statistical convergence if $\mathcal{I}$ is the ideal of $\mathbb{N}$ of natural density zero. In [17] the concept of $\mathcal{I}^*$-convergence was also introduced. Last few years several works on $\mathcal{I}$-convergence and its related areas were carried out in different directions in different spaces viz. metric spaces, normed linear spaces, probabilistic metric spaces, S-metric spaces, linear 2-normed spaces, cone metric spaces, topological spaces etc. (see [3, 4, 6, 18] and many more references therein). Ordinary convergence always implies statistical convergence and when $\mathcal{I}$ is admissible ideal, $\mathcal{I}^*$-convergence implies $\mathcal{I}$-convergence. But the reverse implication

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does not hold in general. But when $\mathcal{I}$ satisfies the condition (AP), $\mathcal{I}$-convergence implies $\mathcal{I}^*$-convergence. A remarkable observation is that a statistically convergent sequence and $\mathcal{I}$ and $\mathcal{I}^*$-convergent sequence need not even be bounded.

Recently some significant investigations have been done on sequences of real valued functions by using the idea of statistical and $\mathcal{I}$-convergence [8, 10, 15, 19]. The interesting notion of equal convergence was introduced by Császár and Laczkovich [7] for sequences of real valued functions (also known as quasinormal convergence [2]). It is known that equal convergence is weaker than uniform convergence and stronger than pointwise converegence for the sequences of real valued functions. A detailed investigation was carried out by Császár and Laczkovich in [7] on such type of convergence. In [9, 10, 13] the concept of equal convergence of sequences of real functions was generalized to the ideas of $\mathcal{I}$ and $\mathcal{I}^*$-equal convergence using ideals of $\mathbb{N}$ and the relationship between them were investigated. $\mathcal{I}$-equal convergence is weaker than $\mathcal{I}$-uniform convergence and stronger than $\mathcal{I}$-pointwise convergence [10].

The notion of linear 2-normed spaces was initially introduced by Gähler [14] and since then the concept has been studied by many authors. In [24] some significant investigations on $\mathcal{I}$-uniform and $\mathcal{I}$-pointwise convergence have been studied in this space.

2. Preliminaries

Throughout the paper $\mathcal{I} \subset 2^{\mathbb{N}}$ will stand for an admissible ideal. Now we recall some basic definitions and notations.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent to $x \in \mathbb{R}$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}^*$-convergent to $x \in \mathbb{R}$ if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that x is the limit of the subsequence $\{x_{m_k}\}_{k\in\mathbb{N}}$ [17].

Let f, f_n be real valued functions defined on a non empty set X. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be equally convergent ([7]) to f if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_{n\to\infty} \varepsilon_n = 0$ such that for every $x \in X$ there is $m = m(x) \in \mathbb{N}$ with $|f_n(x) - f(x)| < \varepsilon_n$ for $n \geq m$. In this case we write $f_n \xrightarrow{e} f$.

Now we see the key ideas of $\mathcal{I}$-uniform convergent [5] and $\mathcal{I}$ and $\mathcal{I}^*$-equal convergent [10] sequences of real valued functions which will be needed for generalizations into linear 2-normed spaces.

A sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-uniformly convergent to f if for each $\varepsilon > 0$ there exists a set $B \in \mathcal{I}$ such that for all $n \in B^c$ and for all $x \in X$, $|f_n(x) - f(x)| < \varepsilon$. In this case we write $f_n \xrightarrow{\mathcal{I}-u} f$. f is called $\mathcal{I}$-equal limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$ such that for any $x \in X$, the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$. In this case we write $f_n \xrightarrow{\mathcal{I}-e} f$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}^*$-equal convergent to f if there exists a set $M = \{m_1 < m_2 < \cdots < m_k \cdots\} \in \mathcal{F}(\mathcal{I})$ such that f is the equal limit of the subsequence $\{f_{m_k}\}_{k\in\mathbb{N}}$. In this case we write $f_n \xrightarrow{\mathcal{I}^*-e} f$.

Now we recall the following two important notions which are basically equivalent to each other (due to Lemma 3.9. and Definition 3.10. in [20]). Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. $\mathcal{I}$ is called P-ideal if for every sequence of mutually disjoint sets $\{A_1, A_2, \cdots\}$ belonging to $\mathcal{I}$ there exists a sequence $\{B_1, B_2, \cdots\}$ of sets belonging to $\mathcal{I}$ such that $A_j \triangle B_j$ is finite for $j \in \mathbb{N}$ and $B = \bigcup_{j\in\mathbb{N}} B_j \in \mathcal{I}$. This notion is also called condition (AP) while in [20] it is denoted as $AP(\mathcal{I}, Fin)$. An ideal $\mathcal{I}$ is a P-ideal if for any sets $A_1, A_2, \cdots$ belonging to $\mathcal{I}$ there exists a set $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for $n \in \mathbb{N}$.

Now we state some results from [16] for the sequences of real numbers.

Theorem 2.1. *Suppose that* $\{x_n\}_{n\in\mathbb{N}}$ *is a sequence of real numbers and* $\mathcal{I}$ *is an admissible ideal in* $\mathbb{N}$. *If* $\mathcal{I}^*\text{-}\lim_{n\to\infty} x_n = \xi$ *then* $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = \xi$.

Theorem 2.2. $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = \xi$ *implies* $\mathcal{I}^*\text{-}\lim_{n\to\infty} x_n = \xi$ *if and only if* $\mathcal{I}$ *satisfies the condition* (AP).

We will now recall the definition of linear 2-normed spaces which will play very important role throughout the paper.

Definition 2.3. *([14]) Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to \mathbb{R}$ which satisfies the following conditions:*
(C1) $\|x, y\| = 0$ if and only if x and y are linearly dependent in X;
(C2) $\|x, y\| = \|y, x\|$ for all x, y in X;
(C3) $\|\alpha x, y\| = |\alpha|\,\|x, y\|$ for all α in $\mathbb{R}$ and for all x, y in X;
(C4) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all x, y, z in X.

The pair $(X, \|.,.\|)$ is called a linear 2-normed space. A simple example ([24]) of a linear 2-normed space is $(\mathbb{R}^2, \|.,.\|)$ where the equipped 2-norm is given by $\|x, y\| = |x_1y_2 - x_2y_1|,\ x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

Let X be a 2-normed space of dimension d, $2 \leq d < \infty$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be convergent ([1]) to $\xi \in X$ if $\lim_{n\to\infty} \|x_n - \xi, z\| = 0$, for every $z \in X$. In such a case ξ is called limit of $\{x_n\}_{n\in\mathbb{N}}$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent ([23]) to $\xi \in X$ if for each $\varepsilon > 0$ and $z \in X$, the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - \xi, z\| \geq \varepsilon\} \in \mathcal{I}$. The number ξ is called $\mathcal{I}$- limit of $\{x_n\}_{n\in\mathbb{N}}$.

3. Main Results

In this paper we study the concepts of $\mathcal{I}$ and $\mathcal{I}^*$-equal convergence of sequences of functions and investigate relationship between them in linear 2-normed spaces. Throughout the paper we propose X as a non empty set and Y as a linear 2-normed space having dimension d with $2 \leq d < \infty$.

Definition 3.1. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be equally convergent to f if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_{n\to\infty} \varepsilon_n = 0$ such that for every $x \in X$ there is $m = m(x) \in \mathbb{N}$ with $\|f_n(x) - f(x), z\| < \varepsilon_n$ for $n \geq m$ and for every $z \in Y$. In this case we write $f_n \xrightarrow{e} f$.*

Definition 3.2. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-uniformly convergent to f if for any $\varepsilon > 0$ there exists a set $A \in \mathcal{I}$ such that for all $n \in A^c$ and for all $x \in X, z \in Y$, $\|f_n(x) - f(x), z\| < \varepsilon$. In this case we write $f_n \xrightarrow{\mathcal{I}-u} f$.*

Definition 3.3. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. Then the the sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-equal convergent to f if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$ such that for any $x \in X$ and for any $z \in Y$, the set $\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \varepsilon_n\} \in \mathcal{I}$. In this case f is called $\mathcal{I}$-equal limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ and we write $f_n \xrightarrow{\mathcal{I}-e} f$.*

Example 3.4. *Let $\mathcal{I}$ be a non trivial proper admissible ideal. Let $X = \mathbb{R}^2$ and $Y = \{(a, 0) : a \in \mathbb{R}\}$. Define $f_n(x_1, x_2) = (\frac{1}{n+1}, 0)$ and $f(x_1, x_2) = (0, 0)$ for all $(x_1, x_2) \in \mathbb{R}^2$. Suppose $\varepsilon_n = \frac{1}{n}$. Then $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$. Here we use the 2-norm on $\mathbb{R}^2$ by $\|x, y\| = |x_1y_2 - x_2y_1|,\ x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Now we consider the set $A = \{n \in \mathbb{N} : \|f_n(x_1, x_2) - f(x_1, x_2), z\| \geq \varepsilon_n\}$ for all $z = (y_1, y_2) \in Y$. Then $A = \{n \in \mathbb{N} : \left\|(\frac{1}{n+1}, 0) - (0, 0), (y_1, y_2)\right\| \geq \frac{1}{n}\} = \{n \in \mathbb{N} : \frac{y_2}{n+1} \geq \frac{1}{n}\} = \{n \in \mathbb{N} : 0 \geq \frac{1}{n}\} = \phi \in \mathcal{I}$, since $y_2 = 0$. Therefore $f_n \xrightarrow{\mathcal{I}-e} f$.*

Now we investigate some arithmetical properties of $\mathcal{I}$-equal convergent sequences of functions.

Theorem 3.5. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. If $f_n \xrightarrow{\mathcal{I}-e} f$ then f is unique.*

Proof. If possible let f and g be two distinct $\mathcal{I}$-equal limit of $\{f_n\}_{n\in\mathbb{N}}$. Then there are two sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$ and $\mathcal{I}\text{-}\lim_{n\to\infty} \gamma_n = 0$ and for any $x \in X$ and for any $z \in Y$, the sets $K_1 = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \varepsilon_n\}$, $K_2 = \{n \in \mathbb{N} : \|f_n(x) - g(x), z\| \geq \gamma_n\} \in \mathcal{I}$. Therefore $K_1^c = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| < \varepsilon_n\}$, $K_2^c = \{n \in \mathbb{N} : \|f_n(x) - g(x), z\| < \gamma_n\} \in \mathcal{F}(\mathcal{I})$. Let $z \in Y$ be linearly independent with $f(x) - g(x)$. Put $\varepsilon = \frac{1}{2}\|f(x) - g(x), z\| > 0$. As $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$ and $\mathcal{I}\text{-}\lim_{n\to\infty} \gamma_n = 0$, the sets $K_3^c = \{n \in \mathbb{N} : \varepsilon_n < \varepsilon\}$, $K_4^c = \{n \in \mathbb{N} : \gamma_n <$

$\varepsilon\} \in \mathcal{F}(\mathcal{I})$. As $\phi \notin \mathcal{F}(\mathcal{I})$, $K_1^c \cap K_2^c \cap K_3^c \cap K_4^c \neq \phi$. Then there exists $m \in \mathbb{N}$ such that $m \in K_1^c \cap K_2^c \cap K_3^c \cap K_4^c$. Then $\|f_m(x) - f(x), z\| < \varepsilon_m$, $\|f_m(x) - g(x), z\| < \gamma_m$, $\varepsilon_m < \varepsilon$ and $\gamma_m < \varepsilon$. Now $\|f(x) - g(x), z\| = \|f(x) - f_m(x) + f_m(x) - g(x), z\| \leq \|f_m(x) - f(x), z\| + \|f_m(x) - g(x), z\| < \varepsilon_m + \gamma_m < \varepsilon + \varepsilon = \frac{1}{2}\|f(x) - g(x), z\| + \frac{1}{2}\|f(x) - g(x), z\| = \|f(x) - g(x), z\|$, which is absurd. Hence $\mathcal{I}$-equal limit f of the sequence $\{f_n\}_{n\in\mathbb{N}}$ must be unique if it exists. ■

Theorem 3.6. *Let $f, f_n : X \to Y$ and $g, g_n : X \to Y$, $n \in \mathbb{N}$. If $f_n \xrightarrow{\mathcal{I}-e} f$ and $g_n \xrightarrow{\mathcal{I}-e} g$, $f_n + g_n \xrightarrow{\mathcal{I}-e} f + g$.*

Proof. Since $f_n \xrightarrow{\mathcal{I}-e} f$ and $g_n \xrightarrow{\mathcal{I}-e} g$, there exist sequences $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{\rho_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}\text{-}\lim_{n\to\infty} \xi_n = 0$ and $\mathcal{I}\text{-}\lim_{n\to\infty} \rho_n = 0$ such that for $x \in X$ and $z \in Y$, we have $A_1 = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \xi_n\}$, $A_2 = \{n \in \mathbb{N} : \|g_n(x) - g(x), z\| \geq \rho_n\} \in \mathcal{I}$. So $A_1^c = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| < \xi_n\}$, $A_2^c = \{n \in \mathbb{N} : \|g_n(x) - g(x), z\| < \rho_n\} \in \mathcal{F}(\mathcal{I})$. As $\phi \notin \mathcal{F}(\mathcal{I})$, $A_1^c \cap A_2^c \neq \phi$.

Now let $n \in A_1^c \cap A_2^c$ and consider the set $A_3^c = \{n \in \mathbb{N} : \|f_n(x) + g_n(x) - \{f(x) + g(x)\}, z\| < \xi_n + \rho_n\}$. As $\|f_n(x) + g_n(x) - \{f(x) + g(x)\}, z\| \leq \|f_n(x) - f(x), z\| + \|g_n(x) - g(x), z\| < \xi_n + \rho_n$, therefore $n \in A_3^c$ i.e. $A_1^c \cap A_2^c \subset A_3^c$. So $A_3 \subset A_1 \cup A_2$. Since $A_1 \cup A_2 \in \mathcal{I}$, $A_3 \in \mathcal{I}$. i.e. $\{n \in \mathbb{N} : \|f_n(x) + g_n(x) - \{f(x) + g(x)\}, z\| \geq \xi_n + \rho_n\} \in \mathcal{I}$. As $\mathcal{I}\text{-}\lim_{n\to\infty} \xi_n + \rho_n = 0$, $f_n + g_n \xrightarrow{\mathcal{I}-e} f + g$. This proves the theorem. ■

Theorem 3.7. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. Let $a(\neq 0) \in \mathbb{R}$. If $f_n \xrightarrow{\mathcal{I}-e} f$, $af_n \xrightarrow{\mathcal{I}-e} af$.*

Proof. Since $f_n \xrightarrow{\mathcal{I}-e} f$, there is a sequence $\{\beta_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}\text{-}\lim_{n\to\infty} \beta_n = 0$ such that for $x \in X, z \in Y$, the set $B_1 = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \frac{\beta_n}{|a|}\} \in \mathcal{I}$. Put $B_2 = \{n \in \mathbb{N} : \|af_n(x) - af(x), z\| \geq \beta_n\}$. As, $\|af_n(x) - af(x), z\| \geq \beta_n \Rightarrow \|f_n(x) - f(x), z\| \geq \frac{\beta_n}{|a|}$. Therefore $B_2 \subset B_1$. So $B_2 \in \mathcal{I}$. This proves the result. ■

In [10] it has been proved for real valued functions that $\mathcal{I}$-uniform convergence implies $\mathcal{I}$-equal convergence. Now we investigate it in linear 2-normed spaces which will be needed in the sequel. First we give an important lemma which has been stated as remark in [24].

Lemma 3.8. *(cf.[24]) Let $f, f_n : X \to Y$, $n \in \mathbb{N}$. If $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-uniformly convergent to f then $\{\sup_{x\in X} \|f_n(x) - f(x), z\|\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to zero for all $z \in Y$.*

Proof. First we assume that $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-uniformly convergent to f. Then for any $\varepsilon > 0$ there exists $M \in \mathcal{I}$ such that for all $n \in M^c$ and for $x \in X$, $z \in Y$, $\|f_n(x) - f(x), z\| < \frac{\varepsilon}{2}$. This implies

$$\sup_{x\in X} \|f_n(x) - f(x), z\| \leq \frac{\varepsilon}{2} < \varepsilon.$$

So the set $\{n \in \mathbb{N} : |\sup_{x\in X} \|f_n(x) - f(x), z\| - 0| \geq \varepsilon\} \subset M \in \mathcal{I}$, for all $z \in Y$. Therefore $\{\sup_{x\in X} \|f_n(x) - f(x), z\|\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to zero for all $z \in Y$. ■

Theorem 3.9. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. $f_n \xrightarrow{\mathcal{I}-u} f$ implies $f_n \xrightarrow{\mathcal{I}-e} f$.*

Proof. Since the sequence $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-uniformly convergent to f in Y, due to the Lemma 3.8 the sequence $\{u_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to zero where $u_n = \sup_{x\in X} \|f_n(x) - f(x), z\|$, for all $z \in Y$. Let $\varepsilon > 0$ be given. Then the set $B = \{n \in \mathbb{N} : u_n \geq \varepsilon\} \in \mathcal{I}$. Define $\xi_n = \begin{cases} \frac{1}{n}, & \text{if } n \in B \\ u_n + \frac{1}{n}, & \text{if } n \notin B \end{cases}$. We show $\{\xi_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to zero. For, let $\varepsilon_1 > 0$, we have $\{n : \xi_n \geq \varepsilon_1\} = \{n \in B : \xi_n \geq \varepsilon_1\} \cup \{n \in B^c : \xi_n \geq \varepsilon_1\} = \{n : \frac{1}{n} \geq \varepsilon_1\} \cup \{n : u_n + \frac{1}{n} \geq \varepsilon_1\} = M_1 \cup M_2$. Clearly M_1 is finite. If $n \in M_2$ then $n \in B^c$. So $u_n < \varepsilon$. Now $u_n + \frac{1}{n} \geq \varepsilon_1$ if $\frac{1}{n} \geq \varepsilon_1 - u_n$ i.e. if $\frac{1}{n} \geq \varepsilon_1 - \varepsilon$ which is for finite number values of n. Therefore M_2 is finite. As

$\mathcal{I}$ is admissible, $M_1 \cup M_2 \in \mathcal{I}$. Hence $\mathcal{I}\text{-}\lim_{n\to\infty} \xi_n = 0$. Now, for all $z \in Y$, we have $\|f_n(x) - f(x), z\| \leq \sup_{x\in X} \|f_n(x) - f(x), z\| < \sup_{x\in X} \|f_n(x) - f(x), z\| + \frac{1}{n} = u_n + \frac{1}{n} = \xi_n$ if $n \in B^c$ where $B \in \mathcal{I}$. Therefore $\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \xi_n\} \in \mathcal{I}$. As $\mathcal{I}\text{-}\lim_{n\to\infty} \xi_n = 0$, $f_n \xrightarrow{I-e} f$. Hence the theorem follows. ■

Now we intend to proceed with the notion of I^*-equal convergence in linear 2-normed spaces.

Definition 3.10. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}^*$-equal convergent to f if there exists a set $M = \{m_1 < m_2 < \cdots < m_k \cdots\} \in \mathcal{F}(\mathcal{I})$ and a sequence $\{\varepsilon_k\}_{k\in M}$ of positive reals with $\lim_{k\to\infty} \varepsilon_k = 0$ such that for every $x \in X$, there is a number $p \in \mathbb{N}$ and for every $z \in Y$, $\|f_{m_k}(x) - f(x), z\| < \varepsilon_k$ for all $k \geq p$. In this case we write $f_n \xrightarrow{\mathcal{I}^*-e} f$.*

We proceed to investigate the relationship between $\mathcal{I}$-equal and $\mathcal{I}^*$-equal convergence in linear 2-normed spaces.

Theorem 3.11. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. If $f_n \xrightarrow{\mathcal{I}^*-e} f$ then $f_n \xrightarrow{\mathcal{I}-e} f$.*

Proof. We assume $f_n \xrightarrow{\mathcal{I}^*-e} f$. Then there exist a set $M = \{m_1 < m_2 < \cdots < m_k \cdots\} \in \mathcal{F}(\mathcal{I})$ and a sequence $\{\varepsilon_k\}_{k\in M}$ of positive reals with $\lim_{k\to\infty} \varepsilon_k = 0$ such that for every $x \in X$, there is a number $p \in \mathbb{N}$ and for every $z \in Y$, $\|f_{m_k}(x) - f(x), z\| < \varepsilon_k$ for $k > p$. Then clearly $\|f_n(x) - f(x), z\| \geq \varepsilon_n$ holds for $n \in (\mathbb{N}\backslash M) \cup \{m_1, m_2, \cdots, m_p\}$. This implies $\{n : \|f_n(x) - f(x), z\| \geq \varepsilon_n\} \subset (\mathbb{N}\backslash M) \cup \{m_1, m_2, \cdots, m_p\}$. Since $\mathcal{I}$ is admissible, $\{n : \|f_n(x) - f(x), z\| \geq \varepsilon_n\} \in \mathcal{I}$. Hence $f_n \xrightarrow{\mathcal{I}-e} f$. ■

Remark 3.12. *The converse of the above theorem may not hold in general as shown by the following example.*

Example 3.13. *Consider a decomposition $\mathbb{N} = \bigcup_{i=1}^{\infty} D_i$ such that each D_i is infinite and $D_i \cap D_j = \phi$ for $i \neq j$. Let $\mathcal{I}$ be the class of all subsets of $\mathbb{N}$ which intersects only a finite number of $D_i^{'}s$. Then $\mathcal{I}$ is a non-trivial admissible ideal. Let $f, f_n : X \to Y, n \in \mathbb{N}$ such that $\{f_n\}_{n\in N}$ is uniformly convergent to f and $f_n \neq f$ for any $n \in \mathbb{N}$. Then for each $\varepsilon > 0$ there exists $p \in \mathbb{N}$ such that for all $x \in X, z \in Y$, $\|f_n(x) - f(x), z\| < \varepsilon$ for all $n > p$. Define a sequence $\{g_n\}_{n\in\mathbb{N}}$ by $g_n = f_j$ if $n \in D_j$. Then for all $x \in X, z \in Y$ the set $\{n \in \mathbb{N} : \|g_n(x) - f(x), z\| \geq \varepsilon\} \subset D_1 \cup D_2 \cup \cdots \cup D_p$. Therefore $\{n \in \mathbb{N} : \|g_n(x) - f(x), z\| \geq \varepsilon\} \in \mathcal{I}$. Hence $g_n \xrightarrow{\mathcal{I}-u} f$. By the Theorem 3.9, $g_n \xrightarrow{\mathcal{I}-e} f$.*

Now we shall show that $\{g_n\}_{n\in\mathbb{N}}$ is not $\mathcal{I}^*$-equal convergent in Y. If possible let $g_n \xrightarrow{\mathcal{I}^*-e} f$. Now, by definition, if $H \in \mathcal{I}$, then there is a $p \in \mathbb{N}$ such that $H \subset D_1 \cup D_2 \cup \cdots \cup D_p$. Then $D_{P+1} \subset \mathbb{N} \setminus H$ and so we have $g_{m_k} = f_{p+1}$ for infinitely many of $k^{'}$s. Let $z \in Y$ be linearly independent with $f_{p+1} - f(x)$. Now we have $\lim_{n\to\infty} \|g_{m_k}(x) - f(x), z\| = \|f_{p+1}(x) - f(x), z\| \neq 0$. Which shows that $\{g_n\}_{n\in\mathbb{N}}$ is not $\mathcal{I}^*$-equal convergent in Y.

Now we see, if X and Y are countable and $\mathcal{I}$ satisfies the condition (AP) then the converse of the Theorem 3.11 also holds. In the next theorem we investigate whether the two concepts $f_n \xrightarrow{\mathcal{I}-e} f$ and $f_n \xrightarrow{\mathcal{I}^*-e} f$ coincide in linear 2-normed spaces when $\mathcal{I}$ is a P-ideal.

Theorem 3.14. *Let $f, f_n : X \to Y, n \in \mathbb{N}$ and let X and Y be countable sets. Then $f_n \xrightarrow{\mathcal{I}-e} f$ implies $f_n \xrightarrow{\mathcal{I}^*-e} f$ whenever $\mathcal{I}$ is a P-ideal.*

Proof. From the given condition there exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}\text{-}\lim_{n\to\infty} \sigma_n = 0$ and for every $z \in Y$ and for each $x \in X$, there is a set $B = B(x, z) \in \mathcal{F}(\mathcal{I})$, $\|f_n(x) - f(x), z\| < \sigma_n$ for all $n \in B$. Now by Theorem 2.2, $\mathcal{I}^*\text{-}\lim_{n\to\infty} \sigma_n = 0$. So we will get a set $H \in \mathcal{F}(\mathcal{I})$ for which $\{\sigma_n\}_{n\in H}$ is convergent to zero. Since X and Y are countable sets, so $X \times Y$ is countable. So let us enumerate $X \times Y$ by $\{(x_i, z_i) : x_i \in X, z_i \in Y, i = 1, 2, \ldots\}$. So for each element $(x_i, z_i) \in X \times Y$, there is a set $B_i = B(x_i, z_i) \in$

$\mathcal{F}(\mathcal{I})$, we have $\|f_n(x_i) - f(x_i), z_i\| < \sigma_n$ for all $n \in B_i$. $\mathcal{I}$-being a P-ideal, there is a set $A \in \mathcal{F}(\mathcal{I})$ such that $A \setminus B_i$ is finite for all i. So for every $z \in Y$ and for all $n \in A \cap H$ except for finite number of values, we have $\|f_n(x) - f(x), z\| < \sigma_n$. Therefore $f_n \xrightarrow{\mathcal{I}^*-e} f$. Hence the theorem follows. ■

Theorem 3.15. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. Suppose that $f_n \xrightarrow{\mathcal{I}-e} f$ implies $f_n \xrightarrow{\mathcal{I}^*-e} f$. Then $\mathcal{I}$ satisfies the condition (AP).*

Proof. Let $f, f_n : X \to Y, n \in \mathbb{N}$ such that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent to f and $f_n \neq f$ for any $n \in \mathbb{N}$. Then for each $\varepsilon > 0$ there exists $p \in \mathbb{N}$ such that for all $x \in X, z \in Y$, $\|f_n(x) - f(x), z\| < \varepsilon$ for all $n > p$. Suppose $\{M_1, M_2, \cdots\}$ be a class of mutually disjoint non empty sets from $\mathcal{I}$. Define a sequence $\{h_n\}_{n\in\mathbb{N}}$ by $h_n = \begin{cases} f_j, & \text{if } n \in M_j \\ f, & \text{if } n \in \mathbb{N} \setminus \bigcup_j M_j \end{cases}$. First of all we shall show that $h_n \xrightarrow{\mathcal{I}-u} f$. Let $\varepsilon > 0$ be given. Observe that the set $M = M_1 \cup M_2 \cup \ldots \cup M_p \in \mathcal{I}$ and for all $x \in X, z \in Y$, we have $\|h_n(x) - f(x), z\| < \varepsilon$ for all $n \in M^c$. i.e. $\{n \in \mathbb{N} : \|h_n(x) - f(x), z\| \geq \varepsilon\} \subset M_1 \cup M_2 \cup \cdots \cup M_p \in \mathcal{I}$. Therefore $h_n \xrightarrow{\mathcal{I}-u} f$. By the Theorem 3.9 we have $h_n \xrightarrow{\mathcal{I}-e} f$. So by the given condition $h_n \xrightarrow{\mathcal{I}^*-e} f$. Therefore there is a set $B \in \mathcal{I}$ such that

$$H = \mathbb{N} \setminus B = \{a_1 < a_2 < \cdots < a_k < \cdots\} \in \mathcal{F}(\mathcal{I}) \text{ and } h_{a_k} \xrightarrow{e} f. \tag{3.1}$$

Put $B_j = M_j \cap B$ $(j = 1, 2, \cdots)$. So $\{B_1, B_2, \cdots\}$ is a class of sets belonging to $\mathcal{I}$. Now $\bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty}(M_j \cap B) = (B \cap \{\bigcup_{j=1}^{\infty} M_j\} \subset B$. Since $B \in \mathcal{I}$ it follows $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$. Now from the equation 3.1 we see that the set M_j has a finite number of elements common with the set $\mathbb{N} \setminus B$. So $M_j \triangle B_j \subset M_j \cap (\mathbb{N} \setminus B)$. Therefore $M_j \triangle B_j$ is finite. Therefore $\mathcal{I}$ satisfies the condition AP. ■

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On the rational difference equation $x_{n+1} = \frac{x_n \cdot (\overline{a} x_{n-k} + a x_{n-k+1})}{b x_{n-k+1} + c x_{n-k}}$

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Abstract. This work studies an explicit and a constructive solution for the difference equation

$$x_{n+1} = \frac{x_n \cdot (\overline{a} x_{n-k} + a x_{n-k+1})}{b x_{n-k+1} + c x_{n-k}}, \quad n = 0, 1, \ldots,$$

where $\overline{a} \geq 0, a > 0, b > 0, c > 0$ and $k \geq 1$ is an integer, with initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$. We also will determine the global behavior of this solution. For the case when $\overline{a} = 0$, the method presented here gives us the particular solution obtained by Gümüş and Abo-Zeid that establishes an inductive type of proof.

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Contents

1. Introduction

The study of rational difference equations currently represents a fruitful area of study that attracts many mathematical researchers. Many difference equations have been successfully used for modeling real phenomena [3, 5, 7].

In 2019 Abo-Zeid [1] published a study on the global behavior of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-1}}{\pm b x_{n-1} + c x_{n-2}}, \quad n = 0, 1, \ldots,$$

where a, b, c are positive real numbers, and obtained its general solution. Similarly, Abo-Zeid [2] also studied the solutions to

$$x_{n+1} = \frac{x_n x_{n-2}}{a x_{n-2} + b x_{n-3}}, \quad n = 0, 1 \ldots,$$

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for a, b positive constants. Motivated by these results, in 2020 Gümüş and Abo-Zeid [4] found an explicit solution and studied the global behavior of the equation

$$x_{n+1} = \frac{a x_n x_{n-k+1}}{b x_{n-k+1} + c x_{n-k}}, \quad n = 0, 1 \ldots,$$

where a, b, c are positive constants and $k \geq 1$ is an integer.

In this work we will generalize the result found by Gümüş and Abo-Zeid by explicitly solving

$$x_{n+1} = \frac{x_n \cdot (\overline{a} x_{n-k} + a x_{n-k+1})}{(b x_{n-k+1} + c x_{n-k})}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $\overline{a} \geq 0, a > 0, b > 0, c > 0$ and $k \geq 1$ is an integer, with the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$.

2. Preliminaries

The Riccati difference equation is defined by

$$R_n R_{n-1} + A(n) R_n + B(n) R_{n-1} = C(n). \tag{2.1}$$

Following the ideas found in the book by Mickens [6, Chapter 6], we make the change of variable

$$R_n = \frac{Q_n - B(n) Q_{n+1}}{Q_{n+1}},$$

which transforms (2.1) into a linear second order equation of the form

$$(A(n)B(n) + C(n)) Q_{n+1} + (B(n-1) - A(n)) Q_n - Q_{n-1} = 0.$$

In order to solve (1.1), the first step is to transform it into a Riccati equation. Indeed, (1.1) is equivalent to

$$b x_{n+1} x_{n-k+1} + c x_{n+1} x_{n-k} = \overline{a} x_n x_{n-k} + a x_n x_{n-k+1},$$

or

$$b \frac{x_{n+1}}{x_n} \cdot \frac{x_{n-k+1}}{x_{n-k}} + c \frac{x_{n+1}}{x_n} = \overline{a} + a \frac{x_{n-k+1}}{x_{n-k}}.$$

Upon applying the change of variable

$$y_n = \frac{x_{n+1}}{x_n}, \tag{2.2}$$

we have

$$y_n y_{n-k} + \frac{c}{b} y_n - \frac{a}{b} y_{n-k} = \frac{\overline{a}}{b}. \tag{2.3}$$

We can see here that the solution for y_n depends exclusively on what happens to y_{n-k} (that is, k steps before). Therefore, we can solve the Riccati equation

$$z_m z_{m-1} + \frac{c}{b} z_m - \frac{a}{b} z_{m-1} = \frac{\overline{a}}{b}, \tag{2.4}$$

with initial condition $z_{-1} := y_{-k+i}$, where $y_{-k+i} = \frac{x_{-k+i+1}}{x_{-k+i}}$ for some $i = 0, 1, \ldots, k-1$ fixed (z_{-1} depends on i). It is evident that the solutions to (2.3) and (2.4) are related by

$$z_m = y_{mk+i}. \tag{2.5}$$

By making the change of variable

$$z_m = \frac{w_m + (a/b)w_{m+1}}{w_{m+1}},$$

equation (2.4) transforms into the homogeneous linear second order equation with constant coefficients

$$(\overline{a}b - ac)w_{m+1} - (a+c)bw_m - b^2 w_{m-1} = 0.$$

The roots of the characteristic polynomial associated to this last equation are given by

$$r_{2,1} := \frac{(a+c)b \pm b\sqrt{(a-c)^2 + 4\overline{a}b}}{2(\overline{a}b - ac)}. \tag{2.6}$$

Hence, the general solution of (2.4) is given by

$$z_m = \frac{(C_1 r_1^m + C_2 r_2^m) + (a/b)(C_1 r_1^{m+1} + C_2 r_2^{m+1})}{C_1 r_1^{m+1} + C_2 r_2^{m+1}}.$$

Making the change of variable $\overline{C}_i := C_2/C_1$, this becomes

$$z_m = \frac{(1 + \frac{a}{b} r_1)(\frac{r_1}{r_2})^m + \overline{C}_i(1 + \frac{a}{b} r_2)}{r_1(\frac{r_1}{r_2})^m + \overline{C}_i r_2}. \tag{2.7}$$

With the initial condition z_{-1}, we obtain

$$\overline{C}_i = -\frac{r_2}{r_1} \cdot \frac{(b + ar_1 - br_1 z_{-1})}{(b + ar_2 - br_2 z_{-1})}. \tag{2.8}$$

Therefore, by means of recursive backward application of the changes of variable previously done, we obtain the explicit solution to (1.1), as shown in Theorem 3.1 below.

Remark 2.1. *In the particular case when $\overline{a} = 0$, we get $r_1 = -\frac{b}{c}$ and $r_2 = -\frac{b}{a}$, and thus we have*

$$z_m = \frac{1}{\frac{b}{a-c} + \overline{C} \cdot (\frac{c}{a})^m},$$

with $\overline{C} = \frac{c}{a}\left(\frac{a-c-bz_{-1}}{(a-c)z_{-1}}\right)$. By recursive backward application of the changes of variables previously done, we get

$$y_{mk+i} = \frac{a-c}{\left(\frac{a-c-bz_{-1}}{z_{-1}}\right)(\frac{c}{a})^{m+1} + b},$$

which implies that

$$x_{mk+i+1} = x_{mk+i} \cdot \left(\frac{a-c}{\frac{a-c-by_{-k+i}}{y_{-k+i}}(\frac{c}{a})^{m+1} + b}\right),$$

from which we can deduce the Gümüş and Abo-Zeid result in [4].

3. Solution to equation (1.1)

Since the case $\overline{a} = 0$ was already solved by Gümüs and Abo-Zeid [4], we can focus on the case $\overline{a} \neq 0$ and normalize this coefficient to obtain

$$x_{n+1} = \frac{x_n \cdot (x_{n-k} + ax_{n-k+1})}{bx_{n-k+1} + cx_{n-k}}, \quad n = 0, 1, \ldots \tag{3.1}$$

We also can assume that $b \neq ac$. Indeed, if $b = ac$, then (3.1) reduces to

$$x_{n+1} = \frac{x_n}{c},$$

which represents a simple case.

Observe that under these conditions, the roots r_1, r_2 in (2.6) are equal to

$$r_{2,1} = \frac{(a+c)b \pm b\sqrt{(a-c)^2 + 4b}}{2(b - ac)}. \tag{3.2}$$

Moreover, since $\left|(a+c) - \sqrt{(a-c)^2 + 4b}\right| < \left|(a+c) + \sqrt{(a-c)^2 + 4b}\right|$, these roots satisfy

$$\left|\frac{r_1}{r_2}\right| < 1.$$

We also note that $r_1 \neq 0, r_2 \neq 0$.

In order for the solution of (3.1) to be well defined, it is necessary to assume that the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ satisfy the following conditions:

$$(\mathrm{H}): \begin{cases} 1)\ x_{-k}, \ldots, x_{-1} \text{ are non-zero.} \\ 2)\ b + ar_2 \neq br_2\left(\frac{x_{-k+i+1}}{x_{-k+i}}\right), \text{ for every } i = 0, 1, \ldots, k-1, \text{ where } r_2 \\ \ \text{is defined as in (3.2), and } b \neq ac. \\ 3)\ \left(\frac{r_1}{r_2}\right)^{j+1} \neq -\overline{C}_i \text{ for every integer } j \geq 0 \text{ and for every } i = 0, 1, \ldots, k-1, \\ \ \text{where } \overline{C}_i \text{ is defined as in (2.8), and } z_{-1} = \frac{x_{-k+i+1}}{x_{-k+i}}. \end{cases}$$

Theorem 3.1. *Consider the difference equation*

$$x_{n+1} = \frac{x_n \cdot (x_{n-k} + ax_{n-k+1})}{bx_{n-k+1} + cx_{n-k}},$$

with $a, b, c > 0$ such that $b \neq ac$, and initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ satisfying (H). Let r_1 and r_2 be defined as in (3.2)*. Let us define the functions*

$$\beta_i(j) = \frac{(1 + \frac{a}{b}r_1)(\frac{r_1}{r_2})^j + \overline{C}_i(1 + \frac{a}{b}r_2)}{r_1(\frac{r_1}{r_2})^j + \overline{C}_i r_2}, \tag{3.3}$$

with $\overline{C}_i$ as in (2.8)*. Then the solution to this equation is given by*

$$\begin{cases} x_{mk} = x_0 \prod_{j=0}^{m-1} \prod_{i=0}^{k-1} \beta_i(j) \\ x_{mk+1} = \beta_0(m) \cdot x_{mk} \\ x_{mk+2} = \beta_0(m)\beta_1(m) \cdot x_{mk} \\ \quad \vdots \\ x_{mk+(k-1)} = \beta_0(m) \cdots \beta_{k-2}(m) \cdot x_{mk}. \end{cases}$$

for $m = 0, 1, 2, 3, \ldots$

Proof. From (2.5) and (2.7), we obtain

$$y_{mk+i} = \frac{(1 + \frac{a}{b}r_1)(\frac{r_1}{r_2})^m + \overline{C}_i(1 + \frac{a}{b}r_2)}{r_1(\frac{r_1}{r_2})^m + \overline{C}_i r_2}.$$

Since we defined $y_n = \frac{x_{n+1}}{x_n}$ in (2.2), then

$$x_{mk+i+1} = x_{mk+i} \cdot \left(\frac{(1+\frac{a}{b}r_1)(\frac{r_1}{r_2})^m + \overline{C}_i(1+\frac{a}{b}r_2)}{r_1(\frac{r_1}{r_2})^m + \overline{C}_i r_2} \right).$$

By applying this equality recursively for all non-negative integers m and k, and for $i = 0, 1, 2, 3, \dots, k-1$, we immediately obtain the Theorem's result. ■

4. Asymptotic behavior of the solution to equation (3.1)

For the analysis of the global behavior of (3.1), let us consider the following additional conditions:

$$(\mathrm{H}_1): \begin{cases} b + ar_1 \neq br_1\left(\frac{x_{-k+i+1}}{x_{-k+i}}\right) \text{ for every } i = 0, 1, \dots, k-1, \text{ where } r_1 \\ \text{is defined as in (3.2), and } b \neq ac. \end{cases}$$

$$(\mathrm{H}_2): \begin{cases} \overline{C}_i \neq \frac{(1+\frac{a}{b}r_1)}{(1+\frac{a}{b}r_2)}\left(\frac{r_1}{r_2}\right)^j \text{ for all } i \text{ and for all } j \geq 0. \end{cases}$$

We can see that r_2, as given in (3.2) with $b \neq ac$, satisfies

$$\begin{aligned} \frac{1}{r_2} + \frac{a}{b} &= \frac{2(b-ac)}{b((a+c)+\sqrt{(a-c)^2+4b})} + \frac{a}{b} \\ &= \frac{\sqrt{(a-c)^2+4b}-(a+c)}{2b} + \frac{a}{b} = \frac{\sqrt{(a-c)^2+4b}+(a-c)}{2b}. \end{aligned}$$

We also see that $\frac{1}{r_2} + \frac{a}{b} > 0$. Moreover,

$$\begin{aligned} &\frac{1}{r_2} + \frac{a}{b} < 1 \Leftrightarrow \sqrt{(a-c)^2+4b} < 2b-(a-c) \Leftrightarrow 2b-(a-c) > 0 \quad \text{and} \\ &(2b-(a-c))^2 > (a-c)^2+4b \Leftrightarrow 2b-(a-c) > 0 \quad \text{and} \quad b-(a-c) > 1 \\ &\Leftrightarrow b-(a-c) > 0. \end{aligned}$$

From this, and in the same manner for the remaining cases, we have

a) $\frac{1}{r_2} + \frac{a}{b} < 1 \iff b-(a-c) > 1.$

b) $\frac{1}{r_2} + \frac{a}{b} > 1 \iff b-(a-c) < 1.$

c) $\frac{1}{r_2} + \frac{a}{b} = 1 \iff b-(a-c) = 1.$

Theorem 4.1. *Let $\{x_n\}_{n=-k}^{\infty}$ be the solution to* (3.1) *such that the initial conditions $x_{-k}, \dots, x_0$ satisfy (H) and (H_1). Then,*

1. *If $b-(a-c) > 1$, then $\{x_n\}_{n=-k}^{\infty}$ converges to 0.*

2. *If $b-(a-c) < 1$ and the initial conditions satisfy (H_2) as well, then $\{x_n\}_{n=-k}^{\infty}$ is unbounded.*

3. *If $b-(a-c) = 1$, then $\{x_n\}_{n=-k}^{\infty}$ converges to a finite limit.*

Proof. From conditions, we have $\overline{C}_i \neq 0$ for all i. On the other hand, since $|r_1/r_2| < 1$, it follows for all i that $\beta_i(j) \to \frac{1}{r_2} + \frac{a}{b}$ if $j \to \infty$, where $\beta_i(j)$ is as defined in (3.3).

1. If $b-(a-c)>1$, then $\frac{1}{r_2}+\frac{a}{b}<1$. Hence, there exist $0<\varepsilon<1$ and $j_0\in\mathbb{N}$ such that $|\beta_i(j)|<\varepsilon$ for all $j\geq j_0$ and for all i. Then, for large enough values of m, we have

$$\begin{aligned} |x_{mk}| &= |x_0|\left|\prod_{j=0}^{j_0-1}\prod_{i=0}^{k-1}\beta_i(j)\right|\left|\prod_{j=j_0}^{m-1}\prod_{i=0}^{k-1}\beta_i(j)\right| \\ &< |x_0|\left|\prod_{j=0}^{j_0-1}\prod_{i=0}^{k-1}\beta_i(j)\right|\cdot\varepsilon^{k(m-j_0)}. \end{aligned}$$

We conclude that as m tends to infinity, then x_{km} converges to 0. Moreover, for $i\in\{1,2,\ldots,k-1\}$, we have

$$x_{mk+i} = x_{mk}\cdot\left|\prod_{l=0}^{i-1}\beta_l(m)\right|.$$

Therefore, $\{x_n\}_{n=-k}^{\infty}$ tends to 0.

2. If $b-(a-c)<1$, then $\frac{1}{r_2}+\frac{a}{b}>1$. Hence, there exist $1<\varepsilon_1<\frac{1}{r_2}+\frac{a}{b}$ and $j_1\in\mathbb{N}$ such that $\beta_i(j)>\varepsilon_1>1$ for all $j\geq j_1$ and for all i. Moreover, by condition (H_2), we have $\beta_i(j)\neq 0$ for all i and for all j. Then, for large enough values of m, we have

$$\begin{aligned} |x_{mk}| &= |x_0|\left|\prod_{j=0}^{j_1-1}\prod_{i=0}^{k-1}\beta_i(j)\right|\left|\prod_{j=j_1}^{m-1}\prod_{i=0}^{k-1}\beta_i(j)\right| \\ &> |x_0|\left|\prod_{j=0}^{j_1-1}\prod_{i=0}^{k-1}\beta_i(j)\right|\cdot\varepsilon_1^{k(m-j_1)}. \end{aligned}$$

We conclude that $|x_{km}|\to\infty$ when $m\to\infty$. Moreover, for $i\in\{1,2,\ldots,k-1\}$, we have

$$x_{mk+i} = x_{mk}\cdot\prod_{l=0}^{i-1}\beta_l(m).$$

Therefore, the solution set $\{x_n\}_{n=-k}^{\infty}$ is unbounded.

3. If $b-(a-c)=1$, then $\frac{1}{r_2}+\frac{a}{b}=1$. Hence, there exists $j_2\in\mathbb{N}$ such that $\beta_i(j)>0$ for $j\geq j_2$ and for all i. Then, we have

$$\begin{aligned} x_{km} &= x_0\left(\prod_{j=0}^{j_2-1}\prod_{i=0}^{k-1}\beta_i(j)\right)\left(\prod_{j=j_2}^{m-1}\prod_{i=0}^{k-1}\beta_i(j)\right) \\ &= x_0\left(\prod_{j=0}^{j_2-1}\prod_{i=0}^{k-1}\beta_i(j)\right)\exp\left(\sum_{j=j_2}^{m-1}\sum_{i=0}^{k-1}\ln(\beta_i(j))\right). \end{aligned}$$

Let us define

$$
\begin{aligned}
a_j &:= \sum_{i=0}^{k-1} \ln(\beta_i(j)) = \sum_{i=0}^{k-1} \ln\left(\frac{1+\frac{1}{\overline{C}_i r_2}\left(1+\frac{a}{b}r_1\right)\left(\frac{r_1}{r_2}\right)^j}{1+\frac{r_1}{\overline{C}_i r_2}\left(\frac{r_1}{r_2}\right)^j} \right) \\
&= \sum_{i=0}^{k-1}\left(\ln\left(1+\frac{1}{\overline{C}_i r_2}(1+\frac{a}{b}r_1)(\frac{r_1}{r_2})^j\right) - \ln\left(1+\frac{r_1}{\overline{C}_i r_2}(\frac{r_1}{r_2})^j\right)\right) \\
&= \sum_{i=0}^{k-1}\left(\left(\frac{1}{\overline{C}_i r_2}\left(1+\frac{a}{b}r_1\right)\left(\frac{r_1}{r_2}\right)^j + \mathcal{O}\left((r_1/r_2)^{2j}\right)\right)\right. \\
&\left.- \left(\frac{r_1}{\overline{C}_i r_2}\left(\frac{r_1}{r_2}\right)^j + \mathcal{O}\left((r_1/r_2)^{2j}\right)\right)\right) \\
&= \sum_{i=0}^{k-1}\left(\frac{1}{\overline{C}_i r_2}\left(1+\frac{a-b}{b}r_1\right)\left(\frac{r_1}{r_2}\right)^j + \mathcal{O}((r_1/r_2)^{2j})\right) \\
&= \frac{1}{r_2}\left(1+\frac{a-b}{b}r_1\right)\left(\sum_{i=0}^{k-1}\frac{1}{\overline{C}_i}\right)\cdot\left(\frac{r_1}{r_2}\right)^j + \mathcal{O}\left((r_1/r_2)^{2j}\right).
\end{aligned}
$$

Then, we have

$$\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| = \left|\frac{r_1}{r_2}\right| < 1.$$

By D'Alembert's ratio test, the series $\sum_{j=j_2}^{\infty}\sum_{i=0}^{k-1}\ln(\beta_i(j))$ converges. Hence, there exists $v\in\mathbb{R}$ such that

$$\lim_{m\to\infty} x_{km} = v.$$

In the same way, for $i\in\{1,\ldots,k-1\}$, we have

$$x_{mk+i} = x_{mk}\cdot\prod_{l=0}^{i-1}\beta_l(m) \to v \quad \text{when} \quad m\to\infty.$$

Therefore, the solution set $\{x_n\}_{n=-k}^{\infty}$ converges to a finite limit.

■

5. Numerical Results

Numerical simulations performed with MATLAB for the three cases stated in Theorem 4.1 are shown in the following examples.

Example 1. Consider the equation

$$x_{n+1} = \frac{x_n(x_{n-4}+7.3x_{n-3})}{3.5x_{n-3}+5.8x_{n-4}}.$$

In this case we have $a=7.3$, $b=3.5$, $c=5.8$ and $k=4$. Also, we can see that $b-a+c>1$. Table 1 shows convergence to zero.

Table 1: Numerical results for Example 1.

n	x_n	n	x_n
-4	2.1	10	-0.083543908285124
-3	1	20	0.012701017754877
-2	8.5	50	0.003594428250519
-1	-3.3	100	$2.862071648505816 \times 10^{-8}$
0	-1.7	200	$1.691446180919758 \times 10^{-18}$
1	-1.019132653061225	500	$3.491237927069944 \times 10^{-49}$
2	-1.807491245443325	999	$3.185169006739856 \times 10^{-100}$

Example 2 Consider the equation

$$x_{n+1} = \frac{x_n(x_{n-3} + 0.8x_{n-2})}{0.2x_{n-2} + 0.1x_{n-3}}.$$

In this case, we have $a = 0.8, b = 0.2, c = 0.1, k = 3$. Also, we can see that $b - a + c < 1$. Table 2 shows the solution set is unbounded.

Table 2: Numerical results for Example 2.

n	x_n	n	x_n
-3	2.8	5	$3.976943951329059 \times 10^{3}$
-2	7.5	10	$7.870828852071307 \times 10^{6}$
-1	1.3	20	$3.259245595490367 \times 10^{13}$
0	0.7	50	$2.322461318837964 \times 10^{33}$
1	3.460674157303371	100	$2.844463544208173 \times 10^{66}$
2	29.261541884525528	150	$3.483792297723871 \times 10^{99}$
3	$2.015795107600647 \times 10^{2}$	200	$4.266818183833936 \times 10^{132}$

Example 3 Consider the equation

$$x_{n+1} = \frac{x_n(x_{n-5} + 1.5x_{n-4})}{1.7x_{n-4} + 0.8x_{n-5}}.$$

In this case we have $a = 1.5$, $b = 1.7$, $c = 0.8$, $k = 5$. Also, we can see that $b - a + c = 1$. Table 3 shows convergence to a finite limit approximately equal to 2.804367096028192.

Table 3: Numerical results for Example 3.

n	x_n	n	x_n
-5	3.1	2	2.824563238832514
-4	2.1	20	2.804362901181129
-3	1.8	50	2.804367096027094
-2	6.5	100	2.804367096028192
-1	3.3	200	2.804367096028192
0	2.7	500	2.804367096028192
1	2.789256198347107	999	2.804367096028192

6. Conclusion

In Theorem 3.1 we found an explicit solution for equation (1.1) when $\overline{a} \geq 0$, $a > 0$, $b > 0$, $c > 0$, and $k \geq 1$ is an integer. The idea behind the construction of such a solution was to transform the given equation into a Riccati difference equation, which can be easily transformed into a linear difference equation with constant coefficients.

Similarly, in Theorem 4.1 we obtained results concerning the asymptotic behaviour of the solutions to (1.1). We determined that solutions can be convergent or divergent, depending on whether the value of $b - a + c$ is greater than, less than or equal to 1, when $\overline{a} = 1$. We also performed some numerical experiments in order to verify such behaviours for different values of a, b, c and k.

The author considers that similar techniques can be used to obtain explicit solutions, or at least results about the global behavious of such solutions, for the case when $\overline{a}$, a, b and/or c are negatives, or when these coefficients are linear on n. The author conjectures that the first case could give rise to periodical solutions, while the second case can be dealt with by converting the resulting Riccati equation into a Cauchy-Euler equation.

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Common fixed point theorem for set of quasi triangular α-orbital admissible mappings in complete metric space with application

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Abstract. The purpose of this paper is to construct a common fixed point theorem for pair of quasi triangular α-orbital admissible with an interpolative (φ, ψ)- Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction mappings with reference to simulation function in complete metric space. We adopt an example to validate our main result. Our result extends the result of M. S. Khan et al. [15]. As an application, we provide the existence of a solution for a nonlinear Fredholm integral equations.

AMS Subject Classifications: 47H10, 54H25.

Keywords: Interpolative (φ, ψ)-type $\mathcal{Z}$-contraction, altering distance function, comparison function, simulation function, quasi triangular α-orbital admissible mappings.

Contents

1. Introduction

The Banach contraction principle is pivotal tools in fixed point theory. Many inventors expanded and generalized the Banach contraction principle to many orientations [3, 5, 24, 27, 28]. Samet et al. [25] found the conception of $\alpha - \psi$ contraction type mapping and take advantage of their new concept to established and found several fixed point theorems. Several inventors used the concept of α-admissible mapping to established new results in many spaces [10, 21, 22, 26, 30]. In 2014, Popescu [20] found the two new concept α-orbital admissible and triangular α-orbital admissible and gave the result each α-admissible mapping is an α-orbital admissible mapping and each triangular α-admissible mapping is an triangular α-orbital admissible mapping. Many inventors gave the fixed point and common fixed point result for α-orbital admissible mapping [1, 7, 9, 18, 19]. In 2015, Khojasteh et al.[17] found the notion of simulation function. In the same year, Argoubi et al. [6] clarified the conception of simulation function. Many inventors found the fixed point and common fixed point result for simulation function in discrete spaces [2, 4, 11, 12, 14, 23, 29].

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2. Preliminaries

We recall some useful definitions that will be needed in the sequel.

Definition 2.1. *[25] Let $Q : Y \to Y$ be a mapping and $\alpha : Y \times Y \to [0, \infty)$ be a function. Then Q is α-admissible if $\alpha(u, v) \geq 1$ implies $\alpha(Qu, Qv) \geq 1$.*

Definition 2.2. *[13] Let $Q : Y \to Y$ be a function and $\alpha : Y \times Y \to [0, \infty)$ be a function. Then Q is said to be triangular α-admissible if Q fulfills the following conditions:*

1. *Q is α-admissible,*
2. *if $\alpha(u, w) \geq 1$ and $\alpha(w, v) \geq 1$ implies $\alpha(u, v) \geq 1$.*

Qawagneh et al. [22] introduced the notion of triangular α-admissible for set of self mappings on Y.

Definition 2.3. *[22] Let $H, Q : Y \to Y$ be two mappings and $\alpha : Y \times Y \to [0, \infty)$ be a function such that the following conditions hold:*

1. *if $\alpha(u, v) \geq 1$ then $\alpha(Hu, Qv) \geq 1$ and $\alpha(QHu, HQv) \geq 1$;*
2. *if $\alpha(u, w) \geq 1$ and $\alpha(w, v) \geq 1$ implies $\alpha(u, v) \geq 1$.*

Then we say that the pair (H, Q) is triangular α-admissible.

Definition 2.4. *[20] Let $Q : Y \to Y$ be a mapping and $\alpha : Y \times Y \to [0, \infty)$ be a function. Then Q is said to be α-orbital admissible if $\alpha(u, Qu) \geq 1$ implies $\alpha(Qu, Q^2u) \geq 1$.*

Definition 2.5. *[20] Let $Q : Y \to Y$ be a mapping and $\alpha : Y \times Y \to [0, \infty)$ be a function. Then Q is said to be triangular α-orbital admissible if Q satisfies the following conditions:*

1. *if Q is α-orbital admissible,*
2. *if $\alpha(u, v) \geq 1$ and $\alpha(v, Qv) \geq 1$ implies $\alpha(u, Qv) \geq 1$.*

Definition 2.6. *[19] Let $H, Q : Y \to Y$ be two mappings and $\alpha_s : Y \times Y \to [0, \infty)$ be a function such that the following condition hold:*

1. *if $\alpha_s(u, Qu) \geq s^2$ and $\alpha_s(u, Hu) \geq s^2$ then $\alpha_s(Qu, HQu) \geq s^2$ and $\alpha_s(Hu, QHu) \geq s^2$.*

Then the set (H, Q) is α_s-orbital admissible.

Definition 2.7. *[19] Let $H, Q : Y \to Y$ be two mappings and $\alpha_s : Y \times Y \to [0, \infty)$ be a function such that the following conditions hold:*

1. *the self mappings H, Q are α_s-orbital admissible,*
2. *if $\alpha_s(u, v) \geq s^2$, $\alpha_s(v, Hv) \geq s^2$ and $\alpha_s(v, Qv) \geq s^2$ implies $\alpha_s(u, Hv) \geq s^2$ and $\alpha_s(u, Qv) \geq s^2$.*

Then the set (H, Q) is triangular α_s-orbital admissible.

M. S. Khan et al. [15] introduced the concept of quasi triangular α-orbital admissible mappings as follows:

Definition 2.8. *[15] Let $Q : Y \to Y$ be a mapping and $\alpha : Y \times Y \to [0, \infty)$ be a function. Then Q is said to be quasi triangular α-orbital admissible if Q satisfies the following conditions:*

1. *if Q is α-orbital admissible,*
2. *if $\alpha(u, v) \geq 1$ implies $\alpha(u, Qv) \geq 1$.*

Definition 2.9. *[17] A mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called a simulation function, if it fulfils the following conditions:*

1. $\zeta(0,0) = 0$;
2. $\zeta(v,u) < u - v$ *for all* $u, v > 0$;
3. *if* $\{v_n\}$, $\{u_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to+\infty} v_n = \lim_{n\to+\infty} u_n > 0$, *then* $\lim_{n\to+\infty} sup\, \zeta(v_n, u_n) < 0$.

The set of all simulation functions is denoted by $\mathcal{Z}$.

Definition 2.10. *[17] Let (Y, d) be a metric space and $Q : Y \to Y$ be mapping. if there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta(d(Qu, Qv), d(u, v)) \geq 0.$$

for all $u, v \in Y$. Then Q is called $\mathcal{Z}$-contraction with respect to ζ.

Definition 2.11. *[16] A continuous function $\varphi : [0,\infty) \to [0,\infty)$ is called an altering distance if it is non-decreasing and $\varphi(l) = 0$ if and only if $l = 0$.*

Definition 2.12. *[8] A function $\psi : [0,\infty) \to [0,\infty)$ is called comparison function if it is monotonically increasing and $\psi^n(l) \to 0$ as $n \to \infty$ for all $l > 0$.*

M. S. Khan et al.[15] gave (φ, ψ)-type $\mathcal{Z}$-contraction with respect to simulation function ζ using an interpolative (φ, ψ) approach in the setting of metric spaces as follows:

Definition 2.13. *[15] A mapping $Q : Y \to Y$ is called an interpolative (φ, ψ)-Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction with respect to ζ if there exists $\alpha : Y \times Y \to \mathbb{R}, \zeta \in \mathcal{Z}, \varphi \in \Phi, \psi \in \Psi, \theta_1, \theta_2 \in (0,1)$ such that $\varphi(t) > \psi(t)$, for $t > 0, \psi > 0$ and $\theta_1 + \theta_2 < 1$ fulfilling the inequality*

$$\zeta(\alpha(u,v)\varphi(d(Qu, Qv)), \psi(B(u,v))) \geq 0 \;\; for\; all\; u,\; v \in Y,$$

where

$$B(u,v) = [d(u,v)]^{\theta_1}.[\frac{1}{2}(d(u,Qu) + d(v,Qv))]^{\theta_2}.[\frac{1}{2}(d(u,Qv) + d(v,Qu))]^{1-\theta_1-\theta_2}$$

In this paper, we construct a common fixed point theorem for set of quasi triangular α-orbital admissible mappings which form an interpolative (φ, ψ)-Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction with reference to simulation function in complete metric space.

3. Main Result

In this section, we introduced the conception of quasi triangular α-orbital admissible mapping for set of self mappings H and Q on Y and discuss (φ, ψ)-type $\mathcal{Z}$-contraction with reference to simulation function.

Definition 3.1. *Let $H, Q : Y \to Y$ be two mappings and $\alpha : Y \times Y \to [0,\infty)$ be a function such that the following conditions hold.*

1. *if* $\alpha(u, Qu) \geq 1$ *and* $\alpha(u, Hu) \geq 1$ *then* $\alpha(Qu, HQu) \geq 1$ *and* $\alpha(Hu, QHu) \geq 1$;
2. *if* $\alpha(u, v) \geq 1$ *implies* $\alpha(u, Qv) \geq 1$ *and* $\alpha(u, Hv) \geq 1$.

Then the pair (H, Q) is called quasi triangular α-orbital admissible.

In the following example shows that the mapping (H,Q) is quasi triangular α-orbital admissible but it is not a triangular α-admissible.

Example 3.2. *Let $Y=\{0,1,2\}$ with usual metric $d(u,v)=|u-v|$. Let $H:Y\to Y$, $Q:Y\to Y$ and $\alpha:Y\times Y\to\mathbb{R}$ be mappings defined by*

$$HY=\begin{pmatrix}0&1&2\\1&0&0\end{pmatrix},\ QY=\begin{pmatrix}0&1&2\\1&0&2\end{pmatrix},\ \alpha(u,v)=\begin{cases}1, \text{ if } (u,v)\in A,\\ 0, \text{ otherwise}\end{cases}$$

where, $A=\{(0,0),(0,1),(1,0),(1,1),(1,2)\}$. Since $(0,1),(1,0)\in A$, then we have $\alpha(0,Q0)=\alpha(Q0,HQ0)=\alpha(1,0)=1$, $\alpha(0,H0)=\alpha(H0,QH0)=\alpha(1,0)=1$ and $\alpha(1,Q1)=\alpha(Q1,HQ1)=\alpha(0,1)=1$, $\alpha(1,H1)=\alpha(H1,QH1)=\alpha(0,1)=1$. Then (H,Q) is α-orbital admissible mappings. Further, we have

$\alpha(0,1)=\alpha(0,Q1)=\alpha(0,0)=1$ *and* $\alpha(0,1)=\alpha(0,H1)=\alpha(0,0)=1$,
$\alpha(1,0)=\alpha(1,Q0)=\alpha(1,1)=1$ *and* $\alpha(1,0)=\alpha(1,H0)=\alpha(1,1)=1$
$\alpha(1,2)=\alpha(1,Q2)=\alpha(1,2)=1$ *and* $\alpha(1,2)=\alpha(1,H2)=\alpha(1,0)=1$.

Hence, (H,Q) is quasi triangular α-orbital admissible mappings. Since $\alpha(u,v)=\alpha(1,2)=1$, $\alpha(v,Qv)=\alpha(2,Q2)=\alpha(2,2)=0$ and $\alpha(v,Hv)=\alpha(2,H2)=\alpha(2,0)=0$ but $\alpha(1,2)=\alpha(1,Q2)=\alpha(1,2)=1$ and $\alpha(1,2)=\alpha(1,H2)=\alpha(1,0)=1$. This shows that the condition $\alpha(v,Qv)$ and $\alpha(v,Hv)$ for triangular α-orbital admissible are not necessity for quasi triangular α-orbital admissible. On the other hand, we have $\alpha(1,2)=1,\alpha(H1,Q2)=\alpha(0,2)=0$ and $\alpha(QH1,HQ2)=\alpha(1,0)=1$ as $(0,2)\nexists Y$, so (H,Q) is not α-admissible mapping. Further, we have $\alpha(0,1)=\alpha(1,2)=1$, but $\alpha(0,2)=0$, so (H,Q) is not triangular α-admissible mapping.

Lemma 3.3. *Let $H,Q:Y\to Y$ be two mappings and $\alpha:Y\times Y\to[0,\infty)$ such that the set (H,Q) is quasi triangular α-orbital admissible. Assume that there exists $u_0\in Y$ in this manner $\alpha(u_0,Hu_0)\geq 1$. Define a sequence $\{u_n\}$ in Y by $Hu_{2n}=u_{2n+1}$ and $Qu_{2n+1}=u_{2n+2}$. Then $\alpha(u_n,u_m)\geq 1$ for all $m,n\in\mathbb{N}\cup\{0\}$ with $n<m$.*

Proof. Since $\alpha(u_0,Hu_0)=\alpha(u_0,u_1)\geq 1$ and H,Q are α-orbital admissible self mappings,

$$\begin{aligned}\alpha(u_0,Hu_0)&\geq 1 \quad implies\\ \alpha(Hu_0,QHu_0)&=\alpha(u_1,Qu_1)=\alpha(u_1,u_2)\geq 1\\ and\ \ \alpha(u_1,Qu_1)&\geq 1 \quad implies\\ \alpha(Qu_1,HQu_1)&=\alpha(u_2,Hu_2)=\alpha(u_2,u_3)\geq 1\\ also\ \ \alpha(u_2,Hu_2)&\geq 1 \quad implies\\ \alpha(Hu_2,QHu_2)&=\alpha(u_3,Qu_3)=\alpha(u_3,u_4)\geq 1\end{aligned}$$

Applying the above argument repeatedly, we obtain $\alpha(u_n,u_{n+1})\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$. Since (H,Q) is quasi triangular α-orbital admissible mapping and $\alpha(u_n,u_{n+1})\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$, then we get $\alpha(u_n,Qu_{n+1})=\alpha(u_n,u_{n+2})\geq 1$ and $\alpha(u_n,Hu_{n+1})=\alpha(u_n,u_{n+2})\geq 1$. By continuing the process, we get that $\alpha(u_n,u_m)\geq 1$ for all $m,n\in\mathbb{N}\cup\{0\}$ with $n<m$.

Definition 3.4. *The mappings $H,Q:Y\to Y$ are called an interpolative (φ,ψ)-Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction with respect to ζ if there exists $\alpha:Y\times Y\to\mathbb{R},\zeta\in\mathcal{Z},\varphi\in\Phi,\psi\in\Psi,\theta_1,\theta_2,\theta_3\in(0,1)$ in this manner $\varphi(t)>\psi(t)$, for $t>0,\psi>0$ and $\theta_1+\theta_2+\theta_3<1$ fulfilling the inequality*

$$\zeta(\alpha(u,v)\varphi(d(Hu,Qv)),\psi(B(u,v)))\geq 0 \ \ for\ all\ u,v\in Y, \tag{3.1}$$

where

$$B(u,v)=[d(u,v)]^{\theta_1}.[\frac{1}{2}(d(u,Hu)+d(v,Qv))]^{\theta_2}.[\frac{1}{2}(d(u,Qv)+d(v,Qu))]^{\theta_3}.$$
$$[\frac{1}{2}(d(u,Hv)+d(v,Hu))]^{1-\theta_1-\theta_2-\theta_3}$$

Now, we state and prove our main results as follows:

Theorem 3.5. *Let H and Q be self mappings on a metric space (Y,d) which is complete. Suppose that (H,Q) is a quasi triangular α-orbital admissible and forms an interpolative (φ,ψ)-Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction with respect to ζ. If there exists $u_0\in Y$ such that $\alpha(u_0,Hu_0)\geq 1$ and H and Q are continuous, then the mappings H and Q have a unique common fixed point.*

Proof. Let $u_0\in Y$ be such that $\alpha(u_0,Hu_0)\geq 1$. Define a sequence $\{u_n\}$ in Y such that $u_{2n+1}=Hu_{2n}$ and $u_{2n+2}=Qu_{2n+1}$ for all $n\in\mathbb{N}$. If $u_{n_0}=u_{n_0+1}$ for some $n_0\in\mathbb{N}$, then it is very easy to show that H and Q have a common fixed point. Hereof, uniquitously the proof we shall assume that $u_n\neq u_{n+1}$ and hence we have $d(u_n,u_{n+1})>0$ for all $n\in\mathbb{N}$. Now, since the pair (H,Q) is α-orbital admissible, then

$$\begin{aligned}\alpha(u_0,Hu_0)&\geq 1 \quad implies\\ \alpha(Hu_0,QHu_0)&=\alpha(u_1,Qu_1)=\alpha(u_1,u_2)\geq 1\\ and\ \ \alpha(u_1,Qu_1)&\geq 1 \quad implies\\ \alpha(Qu_1,HQu_1)&=\alpha(u_2,Hu_2)=\alpha(u_2,u_3)\geq 1\\ also\ \ \alpha(u_2,Hu_2)&\geq 1 \quad implies\\ \alpha(Hu_2,QHu_2)&=\alpha(u_3,Qu_3)=\alpha(u_3,u_4)\geq 1\end{aligned}$$

Applying the above argument repeatedly, we get $\alpha(u_n,u_{n+1})\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$. By the definition of quasi triangular α-admissibility, we can find that for any $n,m\in\mathbb{N}$ with $m>n$, we have $\alpha(u_n,u_m)\geq 1$.

Suppose $u_{2n}\neq u_{2n+1}$ for all $n\in\mathbb{N}$, by Lemma 3.3, we have $\alpha(u_{2n},u_{2n+1})\geq 1$, for all $n\in\mathbb{N}$. From (3.1), we obtain

$$\begin{aligned}0&\leq\zeta\Big(\alpha(u_{2n},u_{2n+1})\varphi(d(Hu_{2n},Qu_{2n+1})),\psi(B(u_{2n},u_{2n+1}))\Big)\\ &=\zeta\Big(\alpha(u_{2n},u_{2n+1})\varphi(d(u_{2n+1},u_{2n+2})),\psi(B(u_{2n},u_{2n+1}))\Big)\\ &<\psi(B(u_{2n},u_{2n+1}))-\alpha(u_{2n},u_{2n+1})\varphi(d(u_{2n+1},u_{2n+2}))\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}B(u_{2n},u_{2n+1})&=[d(u_{2n},u_{2n+1})]^{\theta_1}.[\frac{1}{2}(d(u_{2n},Hu_{2n})+d(u_{2n+1},Qu_{2n+1}))]^{\theta_2}.[\frac{1}{2}(d(u_{2n},Qu_{2n+1})\\ &\quad+d(u_{2n+1},Qu_{2n}))]^{\theta_3}.[\frac{1}{2}(d(u_{2n},Hu_{2n+1})+d(u_{2n+1},Hu_{2n}))]^{1-\theta_1-\theta_2-\theta_3}\\ &=[d(u_{2n},u_{2n+1})]^{\theta_1}.[\frac{1}{2}(d(u_{2n},u_{2n+1})+d(u_{2n+1},u_{2n+2}))]^{\theta_2}\\ &\quad.[\frac{1}{2}(d(u_{2n},u_{2n+2})]^{1-\theta_1-\theta_2}\end{aligned}\tag{3.3}$$

Consequently, we arrive

$$\begin{aligned}\varphi(d(u_{2n+1},u_{2n+2})) &\leq \alpha(u_{2n},u_{2n+1})\varphi(d(u_{2n+1},u_{2n+2}))\\ &< \psi(B(u_{2n},u_{2n+1}))\\ &= \psi\big([d(u_{2n},u_{2n+1})]^{\theta_1}.[\frac{1}{2}(d(u_{2n},u_{2n+1})+d(u_{2n+1},u_{2n+2}))]^{\theta_2}.\\ &\quad [\frac{1}{2}(d(u_{2n},u_{2n+2})]^{1-\theta_1-\theta_2}\big)\\ &\leq \psi\big([d(u_{2n},u_{2n+1})]^{\theta_1}.[\frac{1}{2}(d(u_{2n},u_{2n+1})+d(u_{2n+1},u_{2n+2}))]^{1-\theta_1}\big). \qquad (3.4)\end{aligned}$$

Suppose $d(u_{2n},u_{2n+1}) < d(u_{2n+1},u_{2n+2})$, for $n \geq 1$, then from (3.4), we obtain

$$\varphi(d(u_{2n+1},u_{2n+2})) \leq \psi(d(u_{2n+1},u_{2n+2})) < \varphi(d(u_{2n+1},u_{2n+2})).$$

This is a contradiction. Accordingly, we obtain

$$d(u_{2n+1},u_{2n+2}) \leq d(u_{2n},u_{2n+1}),\ for\ all\ n \geq 1.$$

Identically, we can show that $d(u_{2n},u_{2n+1}) \leq d(u_{2n-1},u_{2n})$. So, we conclude that $d(u_n,u_{n+1}) \leq d(u_{n-1},u_n)$. Hence $d(u_n,u_{n+1})$ is a monotonic decreasing sequence of positive real numbers. So, there exists $l \geq 0$ such that $\lim_{n\to+\infty} d(u_n,u_{n+1}) = l$. Now, we show that $l = 0$. We claim that $l > 0$. Now, we have

$$\begin{aligned}0 &\leq \zeta\Big(\alpha(u_{n-1},u_n)\varphi(d(u_n,u_{n+1})), \psi(B(u_{n-1},u_n))\Big)\\ &< \psi(B(u_{n-1},u_n)) - \alpha(u_{n-1},u_n)\varphi(d(u_n,u_{n+1})). \qquad (3.5)\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\varphi(d(u_n,u_{n+1}) &\leq \alpha(u_{n-1},u_n)\varphi(d(u_n,u_{n+1})) \leq \psi(B(u_{n-1},u_n))\\ &\leq \varphi(B(u_{n-1},u_n))\\ &\leq \varphi(d(u_{n-1},u_n)) \qquad (3.6)\end{aligned}$$

Letting limit as $n \to +\infty$ in (3.6), we get

$$lim_{n\to+\infty}\alpha(u_{n-1},u_n)\varphi(d(u_n,u_{n+1})) = lim_{n\to+\infty}\psi(B(u_{n-1},u_n)) = \varphi(l). \qquad (3.7)$$

Setting $s_n = \alpha(u_{n-1},u_n)\varphi(d(u_n,u_{n+1}))$, $t_n = \psi(B(u_{n-1},u_n))$ in (3.5), then by definition of simulation function

$$0 \leq lim_{n\to+\infty} sup \zeta(\alpha(u_{n-1},u_n)\varphi(d(u_n,u_{n+1})), \psi(B(u_{n-1},u_n))) < 0.$$

Which is a contradiction and thus we have $\lim_{n\to+\infty} d(u_n,u_{n+1}) = 0$. Now, we show that $\{u_n\}$ is a Cauchy sequence. Suppose not, there exists $\epsilon > 0$ for which we can find two sequences m_k and n_k, for all $k \geq 1$ with $u_{m_k} > u_{n_k} \geq k$ such that $d(u_{n_k},u_{m_k}) \geq \epsilon$. Further, we assume that m_k is the smallest number greater than n_k, then $d(u_{n_k},u_{m_{k-1}}) < \epsilon$.

By triangular inequality, we get

$$\epsilon \leq d(u_{n_k},u_{m_k}) \leq d(u_{n_k},u_{m_{k-1}}) + d(u_{m_{k-1}},u_{m_k}) < \epsilon + d(u_{m_{k-1}},u_{m_k}).$$

Taking limit as $k \to +\infty$, we obtain

$$\lim_{k\to+\infty} d(u_{n_k},u_{m_k}) = \epsilon. \qquad (3.8)$$

Again by triangular inequality, we obtain

$$d(u_{n_k}, u_{m_k}) \le d(u_{n_k}, u_{n_{k+1}}) + d(u_{n_{k+1}}, u_{m_{k+1}}) + d(u_{m_{k+1}}, u_{m_k}).$$

Also we obtain

$$d(u_{n_{k+1}}, u_{m_{k+1}}) \le d(u_{n_{k+1}}, u_{n_k}) + d(u_{n_k}, u_{m_k}) + d(u_{m_k}, u_{m_{k+1}}).$$

By using the above two inequalities and taking limit as $k \to +\infty$ with (3.8), we get

$$\lim_{k\to+\infty} d(u_{n_{k+1}}, u_{m_{k+1}}) = \epsilon. \tag{3.9}$$

Furthermore, we obtain

$$d(u_{n_k}, u_{m_k}) \le d(u_{n_k}, u_{n_{k+1}}) + d(u_{n_{k+1}}, u_{m_k}) \le d(u_{n_k}, u_{m_k}) + 2d(u_{m_k}, u_{m_{k+1}}).$$

Taking limit as $k \to +\infty$, we obtain

$$\lim_{k\to+\infty} d(u_{n_{k+1}}, u_{m_k}) = \epsilon. \tag{3.10}$$

Similarly, we get

$$d(u_{n_k}, u_{m_k}) \le d(u_{n_k}, u_{m_{k+1}}) + d(u_{m_{k+1}}, u_{m_k}) \le d(u_{n_k}, u_{m_k}) + 2d(u_{m_k}, u_{m_{k+1}}).$$

Taking limit as $k \to +\infty$, we get

$$\lim_{k\to+\infty} d(u_{n_k}, u_{m_{k+1}}) = \epsilon. \tag{3.11}$$

Since (H, Q) is quasi triangular α-orbital admissible, by lemma 3.3, we get $B(u_{n_k}, u_{m_k}) \ge 1$, for all numbers m_k, n_k such that $m_k > n_k$, where $k \ge 1$. From (3.1), we get

$$\begin{aligned} 0 &\le \zeta\Big(\alpha(u_{n_k}, u_{m_k})\varphi(d(Hu_{n_k}, Qu_{m_k}), \psi(B(u_{n_k}, u_{m_k}))\Big) \\ &= \zeta\Big(\alpha(u_{n_k}, u_{m_k})\varphi(d(u_{n_{k+1}}, u_{m_{k+1}}), \psi(B(u_{n_k}, u_{m_k}))\Big) \\ &< \psi(B(u_{n_k}, u_{m_k})) - \alpha(u_{n_k}, u_{m_k})\varphi(d(u_{n_{k+1}}, u_{m_{k+1}}). \end{aligned}$$

Consequently,

$$\begin{aligned} \varphi(d(u_{n_{k+1}}, u_{m_{k+1}}) &\le \alpha(u_{n_k}, u_{m_k})\varphi(d(u_{n_{k+1}}, u_{m_{k+1}}) \\ &\le \psi(B(u_{n_k}, u_{m_k})) < \varphi(B(u_{n_k}, u_{m_k})), \end{aligned}$$

where

$$\begin{aligned} B(u_{n_k}, u_{m_k}) = [d(u_{n_k}, u_{m_k})]^{\theta_1}.[\frac{1}{2}(d(u_{n_k}, Hu_{n_k}) + d(u_{m_k}, Qu_{m_k}))]^{\theta_2}.[\frac{1}{2}(d(u_{n_k}, Qu_{m_k}) \\ + d(u_{m_k}, Qu_{n_k}))]^{\theta_3}.[\frac{1}{2}(d(u_{n_k}, Hu_{m_k}) + d(u_{m_k}, Hu_{n_k}))]^{1-\theta_1-\theta_2-\theta_3} \end{aligned}$$

Taking limit as $k \to +\infty$ together with (3.8), (3.9), (3.10) and (3.11), we get

$$0 \le \varphi(\epsilon) < \varphi(0) = 0 \Rightarrow \varphi(\epsilon) = 0 \; if \; and \; only \; if \; \epsilon = 0.$$

Which is a contradiction and hence $\{u_n\}$ is a Cauchy sequence in Y. Since Y is complete, there exists $w \in Y$ such that $\lim_{n\to\infty} u_n = w$. Since H and Q are continuous, we find that $Hw = \lim_{n\to\infty} Hu_n = \lim_{n\to\infty} u_{n+1} = w$ and $Qw = \lim_{n\to\infty} Qu_n = \lim_{n\to\infty} u_{n+1} = w$. Therefore w is the common fixed point of H and Q.

To demonstrate the uniqueness of the common fixed point, we suppose that w^* is another common fixed point of H and Q and $\alpha(w, w^*) \geq 1$. Assume that $w \neq w^*$. From (3.1), we get

$$\begin{aligned}\zeta(\alpha(w,w^*)\varphi(d(Hw,Qw^*)), \psi(B(w,w^*))) &\geq 0\\ \zeta(\alpha(w,w^*)\varphi(d(w,w^*)), \psi(B(w,w^*))) &\geq 0\\ \psi(B(w,w^*)) - \alpha(w,w^*)\varphi(d(w,w^*)) &\geq 0\\ -\alpha(w,w^*)\varphi(d(w,w^*)) &\geq 0.\end{aligned}$$

Which is contradiction and therefore the mappings H and Q have a unique common fixed point.

Remark 3.6. *For $H = Q$ in Theorem 3.5, we get the following result of M. S. Khan et al.[15]*

Corollary 3.7. *Let Q be a self mapping on a metric space (Y, d) which is complete. Suppose that Q is quasi triangular α-orbital admissible and forms an interpolative (φ, ψ)-Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction with respect to ζ. If there exists $u_0 \in Y$ such that $\alpha(u_0, Qu_0) \geq 1$ and Q is continuous, then Q has a unique fixed point.*

Remark 3.8. *Setting $\zeta(u, v) = \psi(v) - u$ for all $u, v > 0$ in Theorem 3.5, we get the following result.*

Corollary 3.9. *Let $H, Q : Y \to Y$ be self mappings on a metric space (Y, d) which is complete. If there exists $\alpha : Y \times Y \to \mathbb{R}, \varphi \in \Phi, \psi \in \Psi, \theta_1, \theta_2, \theta_3 \in (0, 1)$ such that $\varphi(t) > \psi(t)$, for $t > 0, \psi > 0$ and $\theta_1 + \theta_2 + \theta_3 < 1$ satisfying the inequality*

$$\alpha(u,v)\varphi(d(Hu,Qv)) \leq \psi(B(u,v)) \ \ for\ all\ u,v \in Y.$$

If there exists $u_0 \in Y$ such that $\alpha(u_0, Hu_0) \geq 1$ and H and Q are continuous. Then the mappings H and Q have a unique common fixed point.

Remark 3.10. *By letting $\alpha(u, v) = 1$ for all $u, v \in Y$ and $\varphi = I_Y$ in Corollary 3.9, we find the following result.*

Corollary 3.11. *Let $H, Q : Y \to Y$ be self mappings on a metric space (Y, d) which is complete. If there exists $\psi \in \Psi, \theta_1, \theta_2, \theta_3 \in (0, 1)$ such that $\theta_1 + \theta_2 + \theta_3 < 1$ satisfying the inequality*

$$d(Hu,Qv) \leq \psi(B(u,v)) \ \ for\ all\ u,v \in Y.$$

Then the mappings H and Q have a unique common fixed point.

Now, we illustrate an example to validate our main Theorem 3.5.

Example 3.12. *Let $Y = (-1, 1]$ and $d : Y \times Y \to \mathbb{R}$ defined by $d(u, v) = |u - v|$. Define the mappings $H, Q : Y \to Y$ by*

$$HY = \begin{cases} \frac{u}{3},\ \text{if}\ u \in (-1,0)\\ \frac{u}{9},\ \text{if}\ u \in [0,1]\end{cases},\ QY = \begin{cases} \frac{u}{2},\ \text{if}\ u \in (-1,0)\\ \frac{u}{3},\ \text{if}\ u \in [0,1].\end{cases}$$

Also, we define the function $\alpha : Y \times Y \to [0, \infty)$ by

$$\alpha(u,v) = \begin{cases} 1,\ \text{if}\ u,v \in [0,1]\\ 0,\ \text{otherwise.}\end{cases}$$

Taking $\zeta(u, v) = \psi(v) - u$, for all $u, v > 0$ in Theorem 3.5, we get

$$\alpha(u,v)\varphi(d(Hu,Qv)) \leq \psi(B(u,v)),$$

for all $u, v \in Y$. Let $\varphi(t) = t$, $\psi(t) = kt$, where $k = \frac{1}{\sqrt{3}}, \theta_1 = \frac{1}{2}, \theta_2 = \frac{1}{4}, \theta_3 = \frac{1}{6}$, then $\varphi(t) \geq \psi(t)$. Since $0 \leq u, v \leq 1$, then we get

$$
\begin{aligned}
&0 \leq |u-v| \leq 1 \Rightarrow 0 \leq |u-v|^{\frac{1}{2}} \leq 1,\\
&0 \leq \frac{1}{2}[|u-Hu| + |v-Qv|] = [(\frac{1}{9})(4u+3v)]^{\frac{1}{4}} \leq (\frac{7}{9})^{\frac{1}{4}},\\
&0 \leq \frac{1}{2}[|u-Qv| + |v-Qu|] = [\frac{1}{6}(|3u-v| + |3v-u|)]^{\frac{1}{6}} \leq (\frac{2}{3})^{\frac{1}{6}} \ \ and\\
and \ \ &0 \leq \frac{1}{2}[|u-Hv| + |v-Hu] = [\frac{1}{18}(|9u-v| + |9v-u|)]^{\frac{1}{12}} \leq (\frac{8}{9})^{\frac{1}{12}}.
\end{aligned}
$$

By simple calculation for all $u, v \in Y$, we obtain

$$
\begin{aligned}
\alpha(u,v)\varphi(d(Hu,Qv)) &= \alpha(u,v)|Hu-Qv| = \frac{3}{9}|u-3v| = \frac{1}{3}|u-3v|\\
&\leq \frac{1}{\sqrt{3}}|u-v|^{\frac{1}{2}}.[(\frac{1}{9})(4u+3v)]^{\frac{1}{4}}.[\frac{1}{6}(|3u-v| + |3v-u|)]^{\frac{1}{6}}.\\
&\ [\frac{1}{18}(|9u-v| + |9v-u|)]^{\frac{1}{12}}\\
&= \psi(B(u,v)).
\end{aligned}
$$

Therefore the set (H, Q) is an interpolative (φ, ψ)-Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction with reference to ζ. If $\{u_n\}$ is a sequence in Y such that $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\{u_n\} \subseteq [0,1]$ for all $n \in \mathbb{N}$. Since $([0,1], d)$ is a complete metric space, then the sequence $\{u_n\}$ converges to u in $[0,1] \subseteq Y$. If $\alpha(u,v) \geq 1$, then $u, v \in [0,1]$. So, $Hu, Qv, QHu, HQv \in [0,1]$. Therefore, $\alpha(u, Qu) = 1$ and $\alpha(u, Hu) = 1$ then $\alpha(Qu, HQu) = 1$ and $\alpha(Hu, QHu) = 1$. Also if $\alpha(u,v) = 1$ implies $\alpha(u, Qv) = 1$ and $\alpha(u, Hv) = 1$. This implies that the pair (H, Q) is a quasi triangular α-orbital admissible in Y.
Let $\{u_n\} \subseteq [0,1]$ for all $n \in \mathbb{N}$. This implies that

$$\lim_{n\to\infty} Hu_n = \lim_{n\to\infty} \frac{1}{9}u_n = \frac{1}{9}u = Hu,$$

and

$$\lim_{n\to\infty} Qu_n = \lim_{n\to\infty} \frac{1}{3}u_n = \frac{1}{3}u = Qu,$$

This implies that the mappings H and Q are continuous. Thus, all supposition of Theorem 3.5 are fulfilled. Hence H and Q have a unique common fixed point $u = 0$.

In the following theorem, we put back the continuity of H and Q with the notion of α-regularity.

Theorem 3.13. *Let H and Q be self mappings on a metric space (Y, d) which is complete. Suppose that (H, Q) is a quasi triangular α-orbital admissible and forms an interpolative (φ, ψ)-Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction with respect to ζ. If there exists $u_0 \in Y$ such that $\alpha(u_0, Hu_0) \geq 1$ and $\{u_n\}$ in Y is α-regular, then the mappings H and Q have a unique common fixed point.*

Proof. Let $u_0 \in Y$ be such that $\alpha(u_0, Hu_0) \geq 1$. Define a sequence $\{u_n\}$ in Y such that $u_{2n+1} = Hu_{2n}$ and $u_{2n+2} = Qu_{2n+1}$ for all $n \in \mathbb{N}$. Since the pair (H, Q) is α-orbital admissible, we find that $\alpha(u_n, u_{n+1}) \geq 1$, for all $n \in \mathbb{N}$. We suppose that $u_n \neq u_{n+1}$ and hence we have $d(u_n, u_{n+1}) > 0$ for all $n \in \mathbb{N}$. By repeating the process as in the proof of Theorem 3.5, we derived that $\{u_n\}$ converges to w. Since $\{u_n\}$ in Y is α-regular, then there exists a subsequence u_{n_k} of $\{u_n\}$ such that $\alpha(u_{n_k}, w) \geq 1$, for each $k \in \mathbb{N} \cup \{0\}$. From (3.1), we get

$$
\begin{aligned}
\zeta(\alpha(u_{2n_k}, w)\varphi(d(Hu_{2n_k}, Qw)), \psi(B(u_{2n_k}, w)) &\geq 0\\
\zeta(\alpha(u_{2n_k}, w)\varphi(d(u_{2n_{k+1}}, Qw)), \psi(B(u_{2n_k}, w)) &\geq 0\\
\psi(B(u_{2n_k}, w)) - \alpha(u_{2n_k}, w)\varphi(d(u_{2n_{k+1}}, Qw) &\geq 0
\end{aligned}
$$

Consequently, we arrive

$$\varphi(d(u_{2n_{k+1}},Qw) \leq \alpha(u_{2n_k},w)\varphi(d(u_{2n_{k+1}},Qw)) < \psi(B(u_{2n_k},w)) < \varphi(B(u_{2n_k},w))$$

where

$$\begin{aligned}B(u_{2n_k},w) &= [d(u_{2n_k},w)]^{\theta_1}.[\frac{1}{2}(d(u_{2n_k},Hu_{2n_k}) + d(w,Qw))]^{\theta_2}.[\frac{1}{2}(d(u_{2n_k},Qw)\\ &\quad + d(w,Qu_{2n_k}))]^{\theta_3}.[\frac{1}{2}(d(u_{2n_k},Hw) + d(w,Hu_{2n_k}))]^{1-\theta_1-\theta_2-\theta_3}\\ &= [d(u_{2n_k},w)]^{\theta_1}.[\frac{1}{2}(d(u_{2n_k},u_{2n_{k+1}}) + d(w,Qw))]^{\theta_2}.[\frac{1}{2}(d(u_{2n_k},Qw)\\ &\quad + d(w,u_{2n_{k+1}}))]^{\theta_3}.[\frac{1}{2}(d(u_{2n_k},Hw) + d(w,u_{2n_{k+1}}))]^{1-\theta_1-\theta_2-\theta_3}\end{aligned}$$

Taking $k \to +\infty$, we get $\varphi(d(w,Qw)) = 0$ which implies $d(w,Qw) = 0$. This shows that w is a fixed point of Q. Similarly, we can show that $(Hw,w) = 0$. Hence the mappings H and Q have a common fixed point.

To demonstrate the uniqueness of the common fixed point, we suppose that w^* is another common fixed point of H and Q and $\alpha(w,w^*) \geq 1$. Assume that $w \neq w^*$. From (3.1), we get

$$\begin{aligned}\zeta(\alpha(w,w^*)\varphi(d(Hw,Qw^*)),\psi(B(w,w^*))) &\geq 0\\ \zeta(\alpha(w,w^*)\varphi(d(w,w^*)),\psi(B(w,w^*))) &\geq 0\\ \psi(B(w,w^*)) - \alpha(w,w^*)\varphi(d(w,w^*)) &\geq 0\\ -\alpha(w,w^*)\varphi(d(w,w^*)) &\geq 0.\end{aligned}$$

which is contradiction and hence the mappings H and Q have a unique common fixed point.

Remark 3.14. *For $H = Q$ in Theorem 3.13, we get Theorem 2.2 of M. S. Khan et al. [15]*

Corollary 3.15. *Let Q be a self mapping on a metric space (Y,d) which is complete. Suppose that Q is quasi triangular α-orbital admissible and forms an interpolative (φ,ψ)-Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction with respect to ζ. If there exists $u_0 \in Y$ such that $\alpha(u_0,Qu_0) \geq 1$ and $\{u_n\}$ in Y is α-regular, then Q has a unique fixed point in Y.*

Remark 3.16. *Setting $\zeta(u,v) = \psi(v) - u$ for all $u,v > 0$ in Theorem 3.13, we get the following result.*

Corollary 3.17. *Let $H,Q : Y \to Y$ be self mappings on a metric space (Y,d) which is complete. If there exists $\alpha : Y \times Y \to \mathbb{R}, \varphi \in \Phi, \psi \in \Psi, \theta_1,\theta_2,\theta_3 \in (0,1)$ such that $\varphi(t) > \psi(t)$, for $t > 0, \psi > 0$ and $\theta_1+\theta_2+\theta_3 < 1$ satisfying the inequality*

$$\alpha(u,v)\varphi(d(Hu,Qv)) \leq \psi(B(u,v)) \;\; for\ all\ u,v \in Y.$$

If there exists $u_0 \in Y$ such that $\alpha(u_0,Hu_0) \geq 1$ and $\{u_n\}$ in Y is α-regular. Then the mappings H and Q have a unique common fixed point.

Remark 3.18. *By letting $\alpha(u,v) = 1$ for all $u,v \in Y$ and $\varphi = I_Y$ in Corollary 3.17, we get the following result.*

Corollary 3.19. *Let $H,Q : Y \to Y$ be two self mappings on a complete metric space. If there exists $\psi \in \Psi, \theta_1,\theta_2,\theta_3 \in (0,1)$ such that $\theta_1+\theta_2+\theta_3 < 1$, for $t > 0, \psi > 0$ satisfying the inequality*

$$d(Hu,Qv) \leq \psi(B(u,v)) \;\; for\ all\ u,v \in Y.$$

Then the mappings H and Q have a unique common fixed point.

4. Application

We apply our outcome to find an existence theorem for Fredholm integral equations. Let $Y = \mathrm{C}[a,b]$ be a set of all real continuous functions on $[a,b]$ equipped with metric $d(e,j) = max_{t\in[a,b]}|e(t) - j(t)|$ for all $e, j \in \mathrm{C}[a,b]$. Then (Y,d) is a complete metric space.
Now, we consider Fredholm integral equations

$$u(t) = h(t) + \int_a^b K(t,s,u(s))ds \tag{4.1}$$

$$v(t) = h(t) + \int_a^b K(t,s,v(s))ds \tag{4.2}$$

where $t, s \in [a,b]$. Assume that $K : [a,b] \times [a,b] \times Y \to \mathbb{R}$ and $h : [a,b] \to \mathbb{R}$ continuous.

Theorem 4.1. *Let (Y,d) be a metric space equipped with metric $d(e,j) = max_{t\in[a,b]}|e(t)-j(t)|$ for all $e, j \in Y$ and $H, Q : Y \to Y$ are operator on Y defined by*

$$Hu(t) = h(t) + \int_a^b K(t,s,u(s))ds \tag{4.3}$$

$$Qv(t) = h(t) + \int_a^b K(t,s,v(s))ds \tag{4.4}$$

where $t, s \in [a,b]$. Assume that $K : [a,b] \times [a,b] \times Y \to \mathbb{R}$ and $h : [a,b] \to \mathbb{R}$ is continuous. Further, assume that the following conditions hold:
(i) If there exists a continuous function $q : [a,b] \times [a,b] \to [0,\infty)$, $\theta_1, \theta_2, \theta_3 \in (0,1)$ with $\theta_1 + \theta_2 + \theta_3 < 1$ that for all $u, v \in Y, s, t \in [a,b]$ fulfilling the following inequality

$$|K(t,s,u(s)) - K(t,s,v(s))| \leq q(t,s)M(u(s),v(s)) \tag{4.5}$$

$$\begin{aligned} where \quad M(u(s),v(s)) = {} & [|u(s) - v(s)|]^{\theta_1}.[\frac{1}{2}(|u(s) - Hu(s)| + |v(s) - Qv(s)|)]^{\theta_2}. \\ & [\frac{1}{2}(|u(s) - Qv(s)| + |v(s) - Qu(s)|)]^{\theta_3}[\frac{1}{2}(|u(s) - Hv(s)| \\ & + |v(s) - Hu(s)|)]^{1-\theta_1-\theta_2-\theta_3} \end{aligned}$$

(ii) If there exists $k \in [0,1)$ and $\alpha : Y \times Y \to (0,\infty)$ such that for each $u \in Y$, we have

$$max_{t\in[a,b]} \int_a^b q(t,s)ds \leq \frac{k}{\alpha(u,v)}.$$

(iii) If there exists $u_0 \in Y$ such that $\alpha(u_0, Hu_0) \geq 1$.

Then the integral equations have a unique common solution in Y.

Proof. From $(4.3), (4.4)$ and (4.5), we obtain

$$\begin{aligned}
|Hu(t) - Qv(t)| &= |\int_a^b K(t,s,u(s))ds - \int_a^b K(t,s,v(s))ds| \\
&= \int_a^b |K(t,s,u(s)) - K(t,s,v(s))|ds \\
&\leq \int_a^b q(t,s)M(u(s),v(s))ds \\
&\leq \int_a^b q(t,s)([|u(s)-v(s)|]^{\theta_1}.[\frac{1}{2}(|u(s)-Hu(s)|+|v(s)-Qv(s)|)]^{\theta_2}. \\
&\quad [\frac{1}{2}(|u(s)-Qv(s)|+|v(s)-Qu(s)|)]^{\theta_3}.[\frac{1}{2}(|u(s)-Hv(s)| \\
&\quad +|v(s)-Hu(s)|)]^{1-\theta_1-\theta_2-\theta_3})ds.
\end{aligned}$$

Taking maximum on both sides for all $t \in [a,b]$, we get

$$\begin{aligned}
d(Hu,Qv) &= max_{t\in[a,b]}|Hu(t)-Qv(t)| \\
&\leq max_{t\in[a,b]}\int_a^b q(t,s)([|u(s)-v(s)|]^{\theta_1}.[\frac{1}{2}(|u(s)-Hu(s)|+|v(s)-Qv(s)|)]^{\theta_2}. \\
&\quad [\frac{1}{2}(|u(s)-Qv(s)|+|v(s)-Qu(s)|)]^{\theta_3}. \\
&\quad [\frac{1}{2}(|u(s)-Hv(s)|+|v(s)-Hu(s)|)]^{1-\theta_1-\theta_2-\theta_3})ds \\
&\leq (max_{t\in[a,b]}([|u(s)-v(s)|]^{\theta_1}.[\frac{1}{2}(|u(s)-Hu(s)|+|v(s)-Qv(s)|)]^{\theta_2}. \\
&\quad [\frac{1}{2}(|u(s)-Qv(s)|+|v(s)-Qu(s)|)]^{\theta_3}.[\frac{1}{2}(|u(s)-Hv(s)| \\
&\quad +|v(s)-Hu(s)|)]^{1-\theta_1-\theta_2-\theta_3}))\int_a^b q(t,s)ds \\
&\leq [d(u,v)]^{\theta_1}.[\frac{1}{2}(d(u,Hu)+d(v,Qv))]^{\theta_2}.[\frac{1}{2}(d(u,Qv)+d(v,Qu))]^{\theta_3}. \\
&\quad [\frac{1}{2}(d(u,Hv)+d(v,Hu))]^{1-\theta_1-\theta_2-\theta_3} max_{t\in[a,b]}\int_a^b q(t,s)ds \\
&\leq \frac{k}{\alpha(u,v)}B(u,v) \\
or \;\; \alpha(u,v)d(Hu,Qv) &\leq kB(u,v).
\end{aligned}$$

Since $Y = \mathrm{C}[a,b]$ is complete metric space. Hence, all the suppositions of Theorem 3.5 are satisfied by setting $\zeta(v,u) = \psi(u) - v$ with $\psi(l) = kl$ and $\varphi(l) = l$ for all $l > 0$, where $k \in [0,1)$ and hence the integral equations have a unique common solution.

5. Conclusion

From our investigations, we conclude that the existence and uniqueness of common fixed point theorem for pair of quasi triangular α-orbital admissible with an interpolative (φ,ψ)- Banach-Kannan-Chatterjea type $\mathcal{Z}$-contraction mappings with reference to simulation function in complete metric space. As an application, we find the existence and uniqueness of common solution for nonlinear Fredholm integral equations. An example is given in support of our main result. Our result provides new path for the researchers in the concerned field.

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Statistical extension some types of symmetrically continuity

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Abstract. In this paper, the notions of symmetric continuity, weak continuity, and weak symmetric continuity were introduced in [P. Pongsriiam and T.Thongsiri, Weakly symmetrically continuous function, Chamchuri Journal of Mathematics, vol 8(2016),49-65] are generalized by using natural density defined on $\mathbb{N}$. Among the others, some basic properties of a generalized form of symmetrically continuity is investigated with several useful examples.

AMS Subject Classifications: 26A15, 40A35, 54C08

Keywords: Continuity, symmetric continuity, statistical strong weak symmetric continuity, statistical weak continuity, statistical weak symmetric continuity.

Contents

1. Introduction

The conception of continuity is one of the essential notions of mathematical analysis. Let X be a nonempty subset of $\mathbb{R}$ and $\phi : X \to \mathbb{R}$ be a function. Continuity of the function ϕ at a point $\xi_0 \in X$ can be checked in two ways:

(I) For all $\epsilon > 0$, there is a $\delta > 0$ such that

$$| \phi(\xi) - \phi(\xi_0) | < \epsilon$$

holds for all ξ which is satisfying $| \xi - \xi_0 | < \delta$.

(II) If $\phi(\xi_n)$ tends to $\phi(\xi_0)$ when $n \to \infty$ holds for all sequence (ξ_n) tends to ξ_0 when $(n \to \infty)$.

The statement given in (I) is known as the Cauchy definition of continuity and (II) as the Heine definition of continuity. It is well known that definitions (I) and (II) are equivalent on the space, which has a countable basis.

It is more important to classify the discontinuity at that point rather than investigate the continuity of the function. There are three discontinuity types at a point: removable discontinuity, jump discontinuity, and infinite discontinuity. In 1958, Pervin and Levine [20] showed that a function with a removable discontinuity is

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continuous under certain conditions. In addition, in 1960, Halfer [12] proved, with minor modification, given by Pervin and Levine [20] that the continuity and the removable discontinuity are equivalent under certain conditions.

In recent years, a characterization of symmetrical continuous functions at points of removable discontinuity has been intensively studied. The symmetric continuity of functions emerged as an application of trigonometric series theory. Mazurkiewicz [15] first gave symmetric continuity of functions [15] in 1919. Afterward, many studies have been done in this direction [2, 4, 11, 13, 19, 21, 24, 25, 30]. Afterwards, many studies have been done on this direction [2, 4, 11, 13, 19, 21, 24, 25, 30].

Let X be a nonempty subset of $\mathbb{R}$. A function $\phi : X \to \mathbb{R}$ is called at a point $\xi_0 \in X$

(I) symmetrically continuous if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\phi(\xi_0 + \lambda) - \phi(\xi_0 - \lambda)| < \epsilon$$

holds, for every $|\lambda| < \delta$. This can be also checked as $\lim_{\lambda \to 0} \phi(\xi_0 + \lambda) - \phi(\xi_0 - \lambda) = 0$.

(II) weakly continuous if there are sequence $\xi_n \nearrow \xi_0$ and sequence $\eta_n \searrow \xi_0$ so that

$$\lim_{n\to\infty} \phi(\xi_n) = \lim_{n\to\infty} \phi(\eta_n) = \phi(\xi_0)$$

(III) weakly symmetrically continuous if there is a sequence $(\lambda_n) \subset \mathbb{R}^+$ with $(\lambda_n) \to 0$, $n \to \infty$ such that

$$\lim_{n\to\infty} (\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)) = 0.$$

In addition to symmetric continuity of functions, there are many studies on weak continuity [18, 22] and weak symmetric continuity of functions [23, 29]. To ensure coordination between published studies, we will stick to the notations used in the study [23]; SC for the set of symmetrically continuous functions, WC for the set of weakly continuous functions and WSC for the set of weakly symmetrically continuous functions.

With the help of the definition of natural density given below, these spaces will be expanded and larger spaces will be obtained. The smallness of a subset of natural numbers depends on its natural density. Natural density of a subset A of natural numbers is determined by (if limit exists)

$$\delta(A) := \lim_{n\to\infty} \frac{1}{n} |\{k \in A : k \leq n\}|$$

where $|\{k \in A : k \leq n\}|$ denotes the number of elements of A.

Considering the definition of natural density, it can be say that a number sequence (ξ_k) is statistical convergent $\xi \in \mathbb{R}$ if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n\colon |\ \xi_k - \xi\ | \geq \epsilon\}| = 0.$$

It is denoted by the symbol $st - \lim \xi_k = \xi$.

Statistical convergence was first defined by Fast [8] and Steinhaus [28] in 1951. Later, in 1959, Schoenberg [27] statistical convergence was reintroduced. In [9], Fridy gave specific results on statistical convergence. Last ten decades, in literature there are several studies in different directions on statistical convergence [1, 3, 5, 7, 10, 14, 16, 17, 26].

The aim of this paper by using natural density to give the statistical version of continuous function, weakly continuous function, weakly symmetrically continuous function, and strong weakly symmetrically continuous function. Then, investigate the relationship between these new type continuities regarding inclusion with some counterexamples.

Throughout this paper, we will consider X as a nonempty subset of $\mathbb{R}$.

Definition 1.1. *[6] The function $\phi : X \to \mathbb{R}$ is called to be statistical continuous at a point ξ_0 if for all sequence (ξ_n) in $\mathbb{R}$ such that $\lim_{n\to\infty} \xi_n = \xi_0$ implies that $\forall\epsilon > 0$,*

$$\delta(\{n: |\phi(\xi_n) - \phi(\xi_0)| \geq \epsilon\}) = 0$$

holds.

Let

$$L_{\xi_0}(X) := \{(\xi_n) \subset X : (\xi_n) \text{ strictly increasing and } \lim_{n\to\infty} \xi_n = \xi_0\}$$
$$U_{\xi_0}(X) := \{(\eta_n) \subset X : (\eta_n) \text{ strictly decreasing and } \lim_{n\to\infty} \eta_n = \xi_0\}.$$

Definition 1.2. *The function $\phi : X \to \mathbb{R}$ is called to be statistical weakly continuous at a point ξ_0 if the undermentioned statements hold:*

1. *if $L_{\xi_0}(X) \neq \emptyset$, then there exists $(\xi_n) \in L_{\xi_0}(X)$ such that $\forall\epsilon > 0$,*

$$\delta(\{n: |\phi(\xi_n) - \phi(\xi_0)| \geq \epsilon\}) = 0,$$

holds,

2. *if $U_{\xi_0}(X) \neq \emptyset$, then there exists $(\eta_n) \in U_{\xi_0}(X)$ such that $\forall\epsilon > 0$,*

$$\delta(\{n: |\phi(\eta_n) - \phi(\xi_0)| \geq \epsilon\}) = 0.$$

holds.

Let

$$S_{\xi_0}(X) := \{(\lambda_n) \subset \mathbb{R}^+ : \lim_{n\to\infty} \lambda_n = 0 \text{ and } \xi_0 + \lambda_n, \xi_0 - \lambda_n \in X\}.$$

Definition 1.3. *The function $\phi : X \to \mathbb{R}$ is said to be statistical weakly symmetrically continuous at ξ_0 if $S_{\xi_0}(X) \neq \emptyset$, then there exists a sequence $(\lambda_n) \in S_{\xi_0}(X)$ such that $\forall\epsilon > 0$,*

$$\delta(\{n: |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

holds.

Definition 1.4. *The function $\phi : X \to \mathbb{R}$ is said to be statistical strong weakly symmetrically continuous at the point ξ_0 if for all real valued sequence (λ_n) with $\xi_0 + \lambda_n, \xi_0 - \lambda_n \in X$ and $\lim_{n\to\infty} \lambda_n = 0$ such that $\forall\epsilon > 0$,*

$$\delta(\{n: |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0$$

holds.

Symbolically $\mathscr{C}^{st}$, $\mathscr{W}\mathscr{C}^{st}$, $\mathscr{W}\mathscr{S}\mathscr{C}^{st}$ and $\mathscr{S}\mathscr{W}\mathscr{S}\mathscr{C}^{st}$ will be used for the set of statistical continuous functions, statistical weakly continuous functions, statistical weakly symmetrically continuous functions and statistical strong weakly symmetrically continuous functions, respectively.

Lemma 1.5. *Let $\phi : X \to \mathbb{R}$ be a function and $\xi_0 \in X$. The undermentioned statements are true:*

(i) $\phi \in \mathscr{W}\mathscr{S}\mathscr{C}^{st}$ if and only if there exists such a set

$$T = \{t_1 < t_2 < ... < t_n < ...\}$$

that $\delta(T) = 1$ and $\lim_{n\to\infty}(\phi(\xi_0 + \lambda_{t_n}) - \phi(\xi_0 - \lambda_{t_n})) = 0$.

(ii) $\phi \in \mathscr{SWSC}^{st}$ *if and only if there exists such a set*

$$T = \{t_1 < t_2 < ... < t_n < ...\}$$

that $\delta(T) = 1$ *and* $\lim_{n\to\infty}(\phi(\xi_0 + \lambda_{t_n}) - \phi(\xi_0 - \lambda_{t_n})) = 0$.

(iii) $\phi \in \mathscr{WC}^{st}$ *if and only if there exists such a set*

$$T = \{t_1 < t_2 < ... < t_n < ...\}$$

that $\delta(T) = 1$ *and* $\lim_{n\to\infty}\phi(\xi_{t_n}) = \lim_{n\to\infty}\phi(\eta_{t_n}) = \phi(\xi_0)$.

Proof. We are going to bestow upon only the proof of (i). Statements (ii) and (iii) can be proved by following the same steps given in (i).

Assume that $S_{\xi_0}(X) \neq \emptyset$ and $\exists(\lambda_t) \in S_{\xi_0}(X)$ such that $\forall\epsilon > 0$,

$$\delta(\{t \in \mathbb{N} : |\phi(\xi_0 + \lambda_t) - \phi(\xi_0 - \lambda_t)| \geq \epsilon\}) = 0$$

holds. Put a set for $j = 1, 2, ...,$

$$T_j := \{n \in \mathbb{N} : |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| < \frac{1}{j}\}.$$

It is clear that

$$T_1 \supset T_2 \supset ... \supset T_j \supset T_{j+1} \supset ..., \tag{1.1}$$

satisfies and for all $j \in \mathbb{N}$

$$\delta(T_j) = 1. \tag{1.2}$$

Let an arbitrary element $s_1 \in T_1$. Considering (1.2) there exists $s_2 \in T_2$ satisfying $s_2 > s_1$ and for all $n \geq s_2$ we have $\frac{T_2(n)}{n} > \frac{1}{2}$. Further, according to (1.2) there exists $s_3 \in T_3$ with $s_3 > s_2$, such that for all $n \geq s_3$, we have

$$\frac{T_3(n)}{n} > \frac{2}{3}.$$

Thus, we obtain a sequence of positive integers

$$s_1 < s_2 < ... < s_j < s_{j+1} < ...$$

that $s_j \in T_j$ $(j = 1, 2, ...)$ and for all $n \geq s_j$

$$\frac{T_j(n)}{n} > \frac{j-1}{j} \tag{1.3}$$

holds.

Let us consider the set T as follows: Each natural number of the interval $(1, s_1)$ belongs to T further, any natural number of the interval (s_j, s_{j+1}) belongs to T if and only if it belongs to T_j $(j = 1, 2, ...)$. According to the equations (1.1) and (1.3) for each n, $s_j \leq n < s_{j+1}$ we get

$$\frac{T(n)}{n} \geq \frac{T_j(n)}{n} > \frac{j-1}{j}$$

From this calculation it is apparent that $\delta(T) = 1$. Let $\epsilon > 0$. There exists a natural number j such that $\frac{1}{j} < \epsilon$. Let $n \geq s_j$, $n \in T$. Then, there exists such a number $l \geq j$ that $s_l \leq n < s_{l+1}$. From the definition of T, we have $n \in T_l$.

Hence,

$$|\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| < \frac{1}{l} \leq \frac{1}{j} < \epsilon$$

Therefore,

$$|\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| < \epsilon$$

for each $n \in T$ with $n \geq s_j$, i.e.,

$$\lim_{t \to \infty} (\phi(\xi_0 + \lambda_t) - \phi(\xi_0 - \lambda_t)) = 0.$$

For to prove converse implication, assume that there exists a set $T = \{t_1 < t_2 < ... < t_n < ...\} \subset \mathbb{N}$ with $\delta(T) = 1$ such that

$$\lim_{n \to \infty} (\phi(\xi_0 + \lambda_{t_n}) - \phi(\xi_0 - \lambda_{t_n})) = 0$$

is satisfied. So, for any $\epsilon > 0$, it can choose a number $n_0 \in \mathbb{N}$ that for each $n > n_0$ we have

$$|\phi(\xi_0 + \lambda_{t_n}) - \phi(\xi_0 - \lambda_{t_n})| < \epsilon. \tag{1.4}$$

Put $A_\epsilon = \{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}$. Then, from (1.4) we get

$$A_\epsilon \subset \mathbb{N} - \{t_{n_0+1}, t_{n_0+2}, ...\}.$$

Therefore $\delta(A_\epsilon) = 0$ and this completed the proof. ■

Theorem 1.6. *Let $\phi : X \to \mathbb{R}$ be a function. If $\phi \in \mathscr{C}^{st}$ then $\phi \in \mathscr{W}\mathscr{S}\mathscr{C}^{st}$.*

Proof. Let ϕ be statistical continuous at ξ_0. Then, for every sequence (ξ_n) in $\mathbb{R}$ for which $\xi_n \to \xi_0$ $(n \to \infty)$ implies that $\forall \epsilon > 0$,

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_n) - \phi(\xi_0)| \geq \epsilon\}) = 0. \tag{1.5}$$

Since (1.5) is provided for every sequence (ξ_n) in $\mathbb{R}$ which is convergent to ξ_0 then, we can choose $(\xi_n) = (\xi_0 + \lambda_n)$ such that $(\lambda_n) \in \mathbb{R}^+$ and $\lambda_n \to 0$. Therefore,

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\}) = 0. \tag{1.6}$$

Similarly, we can choose $(\xi_n) = (\xi_0 - \lambda_n)$ such that $(\lambda_n) \in \mathbb{R}^+$ where $\lambda_n \to 0$ and equation (1.5) implies that

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 - \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\}) = 0. \tag{1.7}$$

So, $S_{\xi_0}(X) \neq \emptyset$ and from (1.6) and (1.7) we have

$$\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\} \subseteq$$
$$\subseteq \{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\} \cup \{n \in \mathbb{N}\colon |\phi(\xi_0 - \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\}$$

and related inequality

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) \leq$$
$$\leq \delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\}) + \delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 - \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\})$$

holds. This implies that

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Hence, ϕ is statistical weakly symmetrically continuous at ξ_0. ■

Theorem 1.7. *Let $\phi : X \to \mathbb{R}$ be a function. If $\phi \in \mathscr{C}^{st}$, then $\phi \in \mathscr{W}\mathscr{C}^{st}$.*

Proof. If $\phi \in \mathscr{C}^{st}$ then, for every real valued sequence (ξ_n) in X for which $\xi_n \to \xi_0$ $(n \to \infty)$ implies that $\forall\epsilon > 0$,

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_n) - \phi(\xi_0)| \geq \epsilon\}) = 0.$$

If $L_{\xi_0}(X)$ and $U_{\xi_0}(X)$ are not empty, then there are $(\xi_n) \in L_{\xi_0}(X)$ and $(\eta_n) \in U_{\xi_0}(X)$ such that $\xi_n \to \xi_0$ and $\eta_n \to \xi_0$ holds. Since ϕ statistical continuous, then

$$\delta(\{n\colon |\phi(\xi_n) - \phi(\xi_0)| \geq \epsilon\}) = 0$$

and

$$\delta(\{n\colon |\phi(\eta_n) - \phi(\xi_0)| \geq \epsilon\}) = 0$$

are satisfied. This prove our assertion. ■

Theorem 1.8. *Let $\phi : X \to \mathbb{R}$ be a function. If $\phi \in \mathscr{C}^{st}$, then $\phi \in \mathscr{S}\mathscr{W}\mathscr{S}\mathscr{C}^{st}$.*

Proof. Let ϕ be a statistical continuous function at ξ_0. Then, for every sequence (ξ_n) in $\mathbb{R}$ for which $\xi_n \to \xi_0$ $(n \to \infty)$ implies that $\forall\epsilon > 0$,

$$\delta(\{n\colon |\phi(\xi_n) - \phi(\xi_0)| \geq \epsilon\}) = 0. \tag{1.8}$$

If we choose $(\xi_n) = (\xi_0 + \lambda_n)$ for $\lambda_n \to 0$ when $n \to \infty$, then, (1.8) implies that $\forall\epsilon > 0$,

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\}) = 0. \tag{1.9}$$

Similarly, if we choose $(\xi_n) = (\xi_0 - \lambda_n)$ for $\lambda_n \to 0$ when $n \to \infty$, $\forall\epsilon > 0$, from (1.8) we have

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 - \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\}) = 0. \tag{1.10}$$

Therefore, $\forall\epsilon > 0$ we have

$$\begin{aligned} &\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\} \subseteq \\ &\subseteq \{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\} \cup \{n \in \mathbb{N}\colon |\phi(\xi_0 - \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\} \end{aligned}$$

and from (1.9), (1.10) following inequality

$$\begin{aligned} &\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) \leq \\ &\leq \delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\}) + \delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 - \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\}) \end{aligned}$$

holds. Hence,

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Hence, ϕ is statistical strong weakly symmetrically continuous at ξ_0. ■

Theorem 1.9. *Let $\phi : X \to \mathbb{R}$ be a function. If $\phi \in \mathscr{S}\mathscr{W}\mathscr{S}\mathscr{C}^{st}$, then $\phi \in \mathscr{W}\mathscr{S}\mathscr{C}^{st}$.*

Proof. Suggesting that ϕ is statistical strong weakly symmetrically continuous at ξ_0. Then, for sequence $\forall(\lambda_n) \in \mathbb{R}$ with $\xi_0 + \lambda_n, \xi_0 - \lambda_n \in X$ satisfying $\lim_{n\to\infty} \lambda_n = 0$ such that $\forall\epsilon > 0$,

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

We can choose a subsequence (λ_{n_k}) of (λ_n) such that $(\lambda_{n_k}) \in \mathbb{R}^+$ with $\xi_0 + \lambda_{n_k}, \xi_0 - \lambda_{n_k} \in X$ satisfying $\lambda_{n_k} \to 0$ $(n_k \to \infty)$.
Therefore, $\forall\epsilon > 0$

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_{n_k}) - \phi(\xi_0 - \lambda_{n_k})| \geq \epsilon\}) = 0.$$

Thus, ϕ is statistical weakly symmetrically continuous at ξ_0. ■

Following examples are related to Theorem 1.6, Theorem 1.7, Theorem 1.8 and Theorem 1.9.

Example 1.10. *Let* $K = \{\frac{1}{n^3} : n \in \mathbb{Z} - \{0\}\} \cup \{0\}$ *a set and define a* $\phi : \mathbb{R} \to \{-1, 0, 1\}$ *by*

$$\phi(\xi) := \begin{cases} 0, & \xi \in K, \\ 2, & \xi > 0 \wedge \xi \notin K, \\ -2, & \xi < 0 \wedge \xi \notin K. \end{cases}$$

If we consider (λ_n) *as*

$$\lambda_n := \begin{cases} \frac{1}{n^3}, & n \neq k^3, \\ \frac{1}{n^3+1}, & n = k^3, \end{cases}$$

then, it is clear that $(\lambda_n) \in U_0(\mathbb{R})$, $(-\lambda_n) \in L_0(\mathbb{R})$ *and*

$$|\phi(\lambda_n) - \phi(0)| = \begin{cases} 0, \; n \neq k^3, \\ 2, \; n = k^3, \end{cases}$$

holds. This implies that for all $\epsilon > 0$,

$$\{n \in \mathbb{N} \colon |\phi(\lambda_n) - \phi(0)| \geq \epsilon\} \subseteq \{k^3 : k \in \mathbb{N}\}.$$

Therefore, $\delta(\{n \in \mathbb{N} \colon |\phi(\lambda_n) - \phi(0)| \geq \epsilon\}) = 0$. *Similarly, we have*

$$|\phi(-\lambda_n) - \phi(0)| = \begin{cases} 0, \; n \neq k^3, \\ 2, \; n = k^3, \end{cases}$$

and $\delta(\{n \in \mathbb{N} \colon |\phi(-\lambda_n) - \phi(0)| \geq \epsilon\}) = 0$. *Therefore,* ϕ *is statistical weakly continuous at* 0. *Now, let us consider following sequence*

$$\lambda_t := \begin{cases} \frac{1}{t^3}, \; t \neq k^2, \\ \frac{1}{t^2}, \; t = k^2. \end{cases}$$

It is clear that $(\lambda_t) \in S_0(\mathbb{R})$ *and*

$$|\phi(0 + \lambda_t) - \phi(0 - \lambda_t)| = \begin{cases} 0, \; t \neq k^2, \\ 4, \; t = k^2. \end{cases}$$

So, for any $\epsilon > 0$ *we have*

$$\{t \in \mathbb{N} \colon |\phi(0 + \lambda_t) - \phi(0 - \lambda_t)| \geq \epsilon\} \subseteq \{k^2 : k \in \mathbb{N}\}$$

and this inclusion implies that

$$\delta(\{t \in \mathbb{N} \colon |\phi(\lambda_t) - \phi(-\lambda_t)| \geq \epsilon\}) = 0.$$

Therefore, ϕ *is statistical weakly symmetrically continuous at* 0.
Now, let define

$$\lambda_m := \begin{cases} \frac{1}{m^3}, \; m \neq 3k - 1, \\ \frac{1}{m^2}, \; m = 3k - 1. \end{cases}$$

such that $\lambda_m \to 0$ $(m \to \infty)$. *Then,*

$$|\phi(0 + \lambda_m) - \phi(0 - \lambda_m)| = \begin{cases} 0, \; m \neq 3k - 1, \\ 4, \; m = 3k - 1. \end{cases}$$

Let $S \subset \mathbb{N}$ be a finite set and for any $\epsilon > 0$, we have

$$\{m \in \mathbb{N}: |\phi(0+\lambda_m) - \phi(0-\lambda_m)| \geq \epsilon\} \supseteq \{3k-1 : k \in \mathbb{N}\} \backslash S$$

and

$$\delta(\{m \in \mathbb{N}: |\phi(\lambda_m) - \phi(-\lambda_m)| \geq \epsilon\}) \geq \frac{1}{3}.$$

Hence, ϕ is not statistical strong weakly symmetrically continuous at 0.
Also, ϕ is not statistical continuous at 0. Because $\lambda_m \to 0$ $(m \to \infty)$ for $\forall m \in \mathbb{N}$, we have

$$|\phi(\lambda_m) - \phi(0)| = \begin{cases} 0, \ m \neq 3k-1, \\ 2, \ m = 3k-1. \end{cases}$$

There exists $S \subset \mathbb{N}$ finite set and for $\forall \epsilon > 0$ such that

$$\{m \in \mathbb{N}: |\phi(\lambda_m) - \phi(0)| \geq \epsilon\} \supseteq \{3k-1 : k \in \mathbb{N}\} \backslash S$$

satisfies. So,

$$\delta(\{m \in \mathbb{N}: |\phi(\lambda_m)| \geq \epsilon\}) \geq \frac{1}{3} \neq 0.$$

Example 1.11. *Let $K = \{\frac{1}{n} : n \in \mathbb{N}\}$, $L = \{\frac{\sqrt{2}}{n+\sqrt{n}} : n \in \mathbb{N}\}$, $M = \{-\frac{1}{n} : n \in \mathbb{N}\}$, $P = \{-\frac{\sqrt{2}}{n+\sqrt{n}} : n \in \mathbb{N}\}$ and $X = K \cup L \cup M \cup P \cup \{0\}$. Define a function $\phi : \mathbb{X} \to \mathbb{R}$ by*

$$\phi(\xi) := \begin{cases} 1, \ \xi \in K \cup P \cup \{0\}, \\ \xi, \quad \ \xi \in L \cup M. \end{cases}$$

For all sequence $(\lambda_n) \in S_0(X)$, we have

$$|\phi(0+\lambda_n) - \phi(0-\lambda_n)| = \begin{cases} \left|1 + \frac{1}{n}\right|, & (\lambda_n) \in K, \\ \left|\frac{\sqrt{2}}{n+\sqrt{n}} - 1\right|, & (\lambda_n) \in L. \end{cases}$$

So, for any $\epsilon > 0$, there exists finite set $S \subset \mathbb{N}$ such that

$$\{n \in \mathbb{N}: |\phi(0+\lambda_n) - \phi(0-\lambda_n)| \geq \epsilon\} = \begin{cases} \mathbb{N}, & (\lambda_n) \in K, \\ \mathbb{N} - S, & (\lambda_n) \in L, \end{cases}$$

is true. Hence, we have

$$\delta(\{n \in \mathbb{N}: |\phi(0+\lambda_n) - \phi(0-\lambda_n)| \geq \epsilon\}) > 0.$$

Thus, ϕ is not statistical weakly symmetrically continuous at 0. Also, it is known from Theorem 1.9 that the function ϕ is not statistical strong weakly symmetrically continuous at 0. Let $\eta_t \in U_0(X)$ and $\xi_m \in L_0(X)$ as follows

$$\eta_t := \begin{cases} \frac{1}{t}, & t \neq k^2, \\ \frac{\sqrt{2}}{t+\sqrt{t}}, & t = k^2, \end{cases} \quad \text{and} \quad \xi_m := \begin{cases} -\frac{\sqrt{2}}{m+\sqrt{m}}, & m \neq k^2, \\ -\frac{1}{m}, & m = k^2, \end{cases}$$

respectively. Then, we have

$$|\phi(\eta_t) - \phi(0)| = \begin{cases} 0, & t \neq k^2, \\ \left|\frac{\sqrt{2}}{t+\sqrt{t}} - 1\right|, & t = k^2, \end{cases}$$

and

$$\{t \in \mathbb{N}\colon |\phi(\eta_t) - \phi(0)| \geq \epsilon\} \subseteq \{k^2 : k \in \mathbb{N}\}$$

is satisfied for all $\epsilon > 0$. *Hence,*

$$\delta(\{t \in \mathbb{N}\colon |\phi(\eta_t) - \phi(0)| \geq \epsilon\}) = 0.$$

Similarly,

$$|\phi(\xi_m) - \phi(0)| = \begin{cases} 0, & m \neq k^2, \\ \left|\frac{1}{m} + 1\right|, & m = k^2, \end{cases}$$

and

$$\{m \in \mathbb{N}\colon |\phi(\xi_m) - \phi(0)| \geq \epsilon\} \subseteq \{k^2 : k \in \mathbb{N}\}$$

implies that

$$\delta(\{m \in \mathbb{N}\colon |\phi(\xi_m) - \phi(0)| \geq \epsilon\}) = 0.$$

Therefore, ϕ *is statistical weakly continuous at* 0.

Example 1.12. *Let* $K = \{\frac{1}{n} : n \in \mathbb{Z} - \{0\}\}$, $L = \{\frac{\sqrt{2}}{n+\sqrt{n}} : n \in \mathbb{N}\}$, $M = \{-\frac{\sqrt{2}}{n+\sqrt{n}} : n \in \mathbb{N}\}$ *and* $X = K \cup L \cup M \cup \{0\}$. *Define the function* $\phi : X \to \mathbb{R}$ *by*

$$\phi(\xi) := \begin{cases} 1, & \xi \in K, \\ \xi, & \xi \in X - K. \end{cases}$$

Let $(\lambda_n) \in S_0(X)$ *as*

$$\lambda_n := \begin{cases} \frac{1}{n}, & n \neq k^2, \\ \frac{\sqrt{2}}{n+\sqrt{n}}, & n = k^2. \end{cases}$$

So, we have

$$|\phi(0 + \lambda_n) - \phi(0 - \lambda_n)| = \begin{cases} 0, & n \neq k^2, \\ \left|\frac{2\sqrt{2}}{n+\sqrt{n}}\right|, & n = k^2, \end{cases}$$

and for every $\epsilon > 0$,

$$\{n \in \mathbb{N}\colon |\phi(0 + \lambda_n) - \phi(0 - \lambda_n)| \geq \epsilon\} \subseteq \{k^2 : k \in \mathbb{N}\}$$

imply that

$$\delta(\{n \in \mathbb{N}\colon |\phi(\lambda_n) - \phi(-\lambda_n)| \geq \epsilon\}) = 0.$$

Therefore, ϕ *is statistical weakly symmetrically continuous at* 0. *For all* $(\eta_m) \in U_0(X)$,

$$|\phi(\eta_m) - \phi(0)| = \begin{cases} 1, & \eta_m \in K, \\ \frac{\sqrt{2}}{m+\sqrt{m}}, & \eta_m \in L. \end{cases}$$

Hence, for $\forall \epsilon > 0$, *there exists* $S \subset \mathbb{N}$ *finite set such that*

$$\{m \in \mathbb{N}\colon |\phi(\eta_m) - \phi(0)| \geq \epsilon\} = \begin{cases} \mathbb{N}, & \eta_m \in K, \\ \mathbb{N}\backslash S, & \eta_m \in L. \end{cases}$$

Therefore,

$$\delta(\{m \in \mathbb{N}\colon |\phi(\eta_m) - \phi(0)| \geq \epsilon\}) > 0.$$

Thus, ϕ *is not statistical weakly continuous at* 0.

As a summary of the Theorems and Examples given above, we can provide the following inclusions:

(i) $\mathscr{SWSC}^{st} \subseteq \mathscr{WSC}^{st}$ and $\mathscr{WSC}^{st} \nsubseteq \mathscr{SWSC}^{st}$

(ii) $\mathscr{SWSC}^{st} \nsubseteq \mathscr{WC}^{st}$ and $\mathscr{WC}^{st} \nsubseteq \mathscr{SWSC}^{st}$

(iii) $\mathscr{WC}^{st} \nsubseteq \mathscr{WSC}^{st}$ and $\mathscr{WSC}^{st} \nsubseteq \mathscr{WC}^{st}$

(iv) $\mathscr{SWSC}^{st} \nsubseteq \mathscr{C}^{st}$, $\mathscr{WSC}^{st} \nsubseteq \mathscr{C}^{st}$ and $\mathscr{WC}^{st} \nsubseteq \mathscr{C}^{st}$

(v) $\mathscr{C}^{st} \subseteq \mathscr{SWSC}^{st}$, $\mathscr{C}^{st} \subseteq \mathscr{WSC}^{st}$ and $\mathscr{C}^{st} \subseteq \mathscr{WC}^{st}$

2. Some algebraic properties of new continuities

This section examines the algebraic properties of the set of $\mathscr{WSC}^{st}$. The results concluded that the set $\mathscr{WSC}^{st}$ does not form a linear space over real numbers.

Theorem 2.1. *Let $\phi : X \to \mathbb{R}$ be a function. If $\phi \in \mathscr{WSC}^{st}$ and $c \in \mathbb{R}$ then, $|\phi|$, $c\phi \in \mathscr{WSC}^{st}$.*

Proof. Suppose that $S_{\xi_0}(X) \neq \emptyset$. Then, there exists a sequence $(\lambda_n) \in S_{\xi_0}(X)$ such that

$$\delta(\{n\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0$$

holds for all $\epsilon > 0$. So, the following inclusion

$$\{n \in \mathbb{N}\colon ||\phi|\,(\xi_0 + \lambda_n) - |\phi|\,(\xi_0 - \lambda_n)| \geq \epsilon\} \subseteq$$
$$\subseteq \{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}$$

implies that

$$\delta(\{n \in \mathbb{N}\colon ||\phi|\,(\xi_0 + \lambda_n) - |\phi|\,(\xi_0 - \lambda_n)| \geq \epsilon\}) \leq$$
$$\leq \delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\})$$

is true. Then,

$$\delta(\{n \in \mathbb{N}\colon ||\phi|\,(\xi_0 + \lambda_n) - |\phi|\,(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Therefore, $|\phi|$ is statistical weakly symmetrically continuous at ξ_0.
Additionally, $c \in \mathbb{R}$ and $\forall \epsilon > 0$ the following inclusion

$$\{n \in \mathbb{N}\colon |(c\phi)(\xi_0 + \lambda_n) - (c\phi)(\xi_0 - \lambda_n)| \geq \epsilon\} \subseteq$$
$$\subseteq \{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{|c|}\}$$

and related inequality

$$\delta(\{n \in \mathbb{N}\colon |(c\phi)(\xi_0 + \lambda_n) - (c\phi)(\xi_0 - \lambda_n)| \geq \epsilon\}) \leq$$
$$\leq \delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{|c|}\})$$

hold.

So, we have

$$\delta(\{n \in \mathbb{N}\colon |(c\phi)(\xi_0 + \lambda_n) - (c\phi)(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Hence, $c\phi$ is statistical weakly symmetrically continuous at ξ_0. ■

Theorem 2.2. *Let $\phi : X \to \mathbb{R}$ and $\psi : X \to \mathbb{R}$ be functions. If $\phi \in \mathscr{WSC}^{st}$ and $\psi \in \mathscr{SWSC}^{st}$ then, $\phi + \psi$, $\phi - \psi$, $\max\{\phi, \psi\}$ and $\min\{\phi, \psi\} \in \mathscr{WSC}^{st}$.*

Proof. Suppose that ϕ is statistical weakly symmetrically continuous function at the point ξ_0 and ψ is statistical strong weakly symmetrically continuous function at the point ξ_0. Then, $S_{\xi_0}(X) \neq \emptyset$ implies that there exists a sequence $(\lambda_n) \in S_{\xi_0}(X)$ such that $\forall \epsilon > 0$,

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

From Theorem 1.9, ψ is statistical weakly symmetrically continuous function at the point ξ_0. Then,

$$\delta(\{n \in \mathbb{N}\colon |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

holds. Therefore, following equality

$$\{n \in \mathbb{N}\colon |(\phi + \psi)(\xi_0 + \lambda_n) - (\phi + \psi)(\xi_0 - \lambda_n)| \geq \epsilon\} =$$
$$= \{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\} \cup$$
$$\cup \{n \in \mathbb{N}\colon |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\}$$

implies that

$$\delta(\{n \in \mathbb{N}\colon |(\phi + \psi)(\xi_0 + \lambda_n) - (\phi + \psi)(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Similarly, we have

$$\delta(\{n \in \mathbb{N}\colon |(\phi - \psi)(\xi_0 + \lambda_n) - (\phi - \psi)(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Consequently, $\phi + \psi$ and $\phi - \psi$ are statistical weakly symmetrically continuous at the point ξ_0.

Now, the following inequality

$$|\max\{\phi, \psi\}(\xi_0 + \lambda_n) - \max\{\phi, \psi\}(\xi_0 - \lambda_n)| \leq$$
$$\leq \frac{|\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)|}{2} + \frac{|\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)|}{2} +$$
$$+ \frac{||\phi - \psi|\,(\xi_0 + \lambda_n) - |\phi - \psi|\,(\xi_0 - \lambda_n)|}{2} \leq$$
$$\leq \frac{|\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)|}{2} + \frac{|\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)|}{2} +$$
$$+ \frac{|(\phi - \psi)(\xi_0 + \lambda_n) - (\phi - \psi)(\xi_0 - \lambda_n)|}{2} \leq$$
$$\leq |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| + |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)|$$

implies that

$$\{n \in \mathbb{N}\colon |\max\{\phi, \psi\}(\xi_0 + \lambda_n) - \max\{\phi, \psi\}(\xi_0 - \lambda_n)| \geq \epsilon\} \subseteq$$
$$\subseteq \{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\}$$
$$\cup \{n \in \mathbb{N}\colon |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\}$$

holds. So, we have

$$\delta(\{n \in \mathbb{N}\colon |\max\{\phi, \psi\}(\xi_0 + \lambda_n) - \max\{\phi, \psi\}(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Similarly, the following inequality

$$|\min\{\phi, \psi\}(\xi_0 + \lambda_n) - \min\{\phi, \psi\}(\xi_0 - \lambda_n)| \leq$$

$$\leq |\phi(\xi_0+\lambda_n)-\phi(\xi_0-\lambda_n)|+|\psi(\xi_0+\lambda_n)-\psi(\xi_0-\lambda_n)|$$

implies that

$$\{n\in\mathbb{N}\colon |\min\{\phi,\psi\}(\xi_0+\lambda_n)-\min\{\phi,\psi\}(\xi_0-\lambda_n)|\geq\epsilon\}\subseteq$$

$$\subseteq\{n\in\mathbb{N}\colon |\phi(\xi_0+\lambda_n)-\phi(\xi_0-\lambda_n)|\geq\frac{\epsilon}{2}\}$$

$$\cup\{n\in\mathbb{N}\colon |\psi(\xi_0+\lambda_n)-\psi(\xi_0-\lambda_n)|\geq\frac{\epsilon}{2}\}$$

holds. Hence,

$$\delta(\{n\in\mathbb{N}\colon |\min\{\phi,\psi\}(\xi_0+\lambda_n)-\min\{\phi,\psi\}(\xi_0-\lambda_n)|\geq\epsilon\})=0.$$

Thus, the functions $\max\{\phi,\psi\}$ and $\min\{\phi,\psi\}$ are statistical weakly symmetrically continuous at ξ_0. ■

Example 2.3. *(Exp.3.3. in [23]) Let $A=\{\frac{1}{n}:n\in\mathbb{Z}-\{0\}\}$ and $B=\{\frac{\sqrt{2}}{n}:n\in\mathbb{Z}-\{0\}\}$. Consider the functions $\phi,\psi:\mathbb{R}\to\mathbb{R}$ as follows:*

$$\phi(\xi)=\begin{cases}\xi, & \xi\in A\cup\{0\},\\ -1, & \xi>0\wedge\xi\notin A,\\ 1, & \xi<0\wedge\xi\notin A,\end{cases}\quad \textit{and}\quad \psi(\xi)=\begin{cases}\xi, & \xi\in B\cup\{0\},\\ -2, & \xi>0\wedge\xi\notin B,\\ 2, & \xi<0\wedge\xi\notin B.\end{cases}$$

The functions ϕ and ψ are weakly symmetrically continuous at 0 (see in [23]). By Lemma 1.5 the functions ϕ and ψ are also statistical weakly symmetrically continuous at 0.

$$(\phi+\psi)(\xi)=\begin{cases}-3, & \xi>0\wedge\xi\notin A\cup B,\\ 3, & \xi<0\wedge\xi\notin A\cup B,\\ \xi-2, & \xi>0\wedge\xi\in A,\\ \xi+2, & \xi<0\wedge\xi\in A,\\ \xi-1, & \xi>0\wedge\xi\in B,\\ \xi+1, & \xi<0\wedge\xi\in B,\\ 0, & \xi=0,\end{cases}$$

$$(\phi-\psi)(\xi)=\begin{cases}1, & \xi>0\wedge\xi\notin A\cup B,\\ -1, & \xi<0\wedge\xi\notin A\cup B,\\ \xi+2, & \xi>0\wedge\xi\in A,\\ \xi-2, & \xi<0\wedge\xi\in A,\\ -\xi-1, & \xi>0\wedge\xi\in B,\\ -\xi+1, & \xi<0\wedge\xi\in B,\\ 0, & \xi=0,\end{cases}$$

$$\max\{\phi,\psi\}(\xi)=\begin{cases}-1, & \xi>0\wedge\xi\notin A\cup B,\\ 2, & \xi<0\wedge\xi\notin A\cup B,\\ \xi, & \xi>0\wedge\xi\in A\cup B,\\ 2, & \xi<0\wedge\xi\in A,\\ 1, & \xi<0\wedge\xi\in B,\\ 0, & \xi=0,\end{cases}$$

$$\min\{\phi,\psi\}(\xi) = \begin{cases} -2, & \xi > 0 \wedge \xi \notin A \cup B, \\ 1, & \xi < 0 \wedge \xi \notin A \cup B, \\ -2, & \xi > 0 \wedge \xi \in, \\ \xi, & \xi < 0 \wedge \xi \in A \cup B, \\ -1, & \xi > 0 \wedge \xi \in B, \\ 0, & \xi = 0. \end{cases}$$

For $\forall(\lambda_n) \in S_0(\mathbb{R})$ *and* $\forall\epsilon > 0$,

$$|(\phi+\psi)(0+\lambda_n) - (\phi+\psi)(0-\lambda_n)| = \begin{cases} 6, & \lambda_n \notin A \cup B, \\ |2\lambda_n - 4|, & \lambda_n \in A, \\ |2\lambda_n - 2|, & \lambda_n \in B. \end{cases}$$

There exists a finite subset of natural numbers S *such that*

$$\{n \in \mathbb{N}: |(\phi+\psi)(\lambda_n) - (\phi+\psi)(-\lambda_n)| \geq \epsilon\} = \begin{cases} \mathbb{N}, & \lambda_n \notin A \cup B, \\ \mathbb{N} \backslash S, & \lambda_n \in A \cup B, \end{cases}$$

Hence,

$$\delta(\{n \in \mathbb{N}: |(\phi+\psi)(\lambda_n) - (\phi+\psi)(-\lambda_n)| \geq \epsilon\}) > 0.$$

Therefore $(\phi+\psi)$ *is not statistical weakly symmetrically continuous at* 0. *Similarly, for* $\forall n \in \mathbb{N}$,

$$|(\phi-\psi)(0+\lambda_n) - (\phi-\psi)(0-\lambda_n)| = \begin{cases} 2, & \lambda_n \notin A \cup B, \\ |2\lambda_n + 4|, & \lambda_n \in A, \\ |-2\lambda_n - 2|, & \lambda_n \in B, \end{cases}$$

$$|\max\{\phi,\psi\}(0+\lambda_n) - \max\{\phi,\psi\}(0-\lambda_n)| = \begin{cases} 3, & \lambda_n \notin A \cup B, \\ |\lambda_n - 2|, & \lambda_n \in A, \\ |\lambda_n - 1|, & \lambda_n \in B, \end{cases}$$

$$|\min\{\phi,\psi\}(0+\lambda_n) - \min\{\phi,\psi\}(0-\lambda_n)| = \begin{cases} 3, & \lambda_n \notin A \cup B, \\ |-\lambda_n - 2|, & \lambda_n \in A, \\ |-\lambda_n - 1|, & \lambda_n \in B, \end{cases}$$

For $\forall\epsilon > 0$,

$$\delta(\{n \in \mathbb{N}: |(\phi-\psi)(\lambda_n) - (\phi-\psi)(-\lambda_n)| \geq \epsilon\}) > 0,$$

$$\delta(\{n \in \mathbb{N}: |\max\{\phi,\psi\}(\lambda_n) - \max\{\phi,\psi\}(-\lambda_n)| \geq \epsilon\}) > 0,$$

$$\delta(\{n \in \mathbb{N}: |\min\{\phi,\psi\}(\lambda_n) - \min\{\phi,\psi\}(-\lambda_n)| \geq \epsilon\}) > 0.$$

Hence, the functions $\phi - \psi$, $\max\{\phi,\psi\}$ *and* $\min\{\phi,\psi\}$ *are not statistical weakly symmetrically continuous at* 0.

Theorem 2.4. *Let* $\phi : X \to \mathbb{R}$ *be a statistical weakly symmetrically continuous function at the point* ξ_0 *and let* $\psi : X \to \mathbb{R}$ *be a statistical strong weakly symmetrically continuous function at the point* ξ_0. *If* ϕ *and* ψ *are locally bounded at* ξ_0, *then* $\phi\psi$ *is statistical weakly symmetrically continuous at* ξ_0.

Proof. Suppose that ϕ is statistical weakly symmetrically continuous and ψ is statistical strong weakly symmetrically continuous at the point ξ_0. Then, $S_{\xi_0}(X) \neq \emptyset$ implies that there exists a sequence $(\lambda_n) \in S_{\xi_0}(X)$ such that $\forall \epsilon > 0$,

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\}) = 0$$

holds. Also from Theorem 1.9, ψ is statistical weakly symmetrically continuous function at the point ξ_0. Then,

$$\delta(\{n \in \mathbb{N}\colon |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\}) = 0.$$

holds for $\forall \epsilon > 0$.

Because of ϕ and ψ are locally bounded at ξ_0, there exists $K, M > 0$ and $\delta > 0$ such that $|\phi(\xi)| \leq M$ and $|\psi(\xi)| \leq K$ for all $\xi \in (\xi_0 - \delta, \xi_0 + \delta) \cap X$.

Since $(\lambda_n) \in S_{\xi_0}(X)$, we can pick $N \in \mathbb{N}$ such that $\xi_0 + \lambda_n, \xi_0 - \lambda_n \in (\xi_0 - \delta, \xi_0 + \delta) \cap X$ for $\forall n \geq N$ such that

$$\begin{aligned} &|(\phi\psi)(\xi_0 + \lambda_n) - (\phi\psi)(\xi_0 - \lambda_n)| = \\ &= |\phi(\xi_0 + \lambda_n)\psi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)\psi(\xi_0 - \lambda_n)| \leq \\ &\leq |\phi(\xi_0 + \lambda_n)|\, |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)| + \\ &+ |\psi(\xi_0 - \lambda_n)|\, |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \leq \\ &\leq M\, |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)| + K\, |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \end{aligned}$$

holds. So, following inclusion

$$\begin{aligned} &\{n \in \mathbb{N}\colon |(\phi\psi)(\xi_0 + \lambda_n) - (\phi\psi)(\xi_0 - \lambda_n)| \geq \epsilon\} \subseteq \\ &\subseteq \{n \in \mathbb{N}\colon M.\, |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\} \cup \\ &\cup \{n \in \mathbb{N}\colon K.\, |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\} \end{aligned}$$

implies that

$$\delta(\{n \in \mathbb{N}\colon |(\phi\psi)(\xi_0 + \lambda_n) - (\phi\psi)(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Therefore, $\phi\psi$ is statistical weakly symmetrically continuous at ξ_0. ■

The following example shows that if $\phi \in \mathscr{W}\mathscr{S}\mathscr{C}^{st}$ and $\psi \in \mathscr{S}\mathscr{W}\mathscr{S}\mathscr{C}^{st}$ but at least one of ϕ or ψ is not locally bounded, then $\phi\psi \notin \mathscr{W}\mathscr{S}\mathscr{C}^{st}$.

Example 2.5. *Consider the functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$ defined by*

$$\phi(\xi) = \xi \quad \textit{and} \quad \psi(\xi) = \begin{cases} \frac{1}{\ln(|\xi|+1)}, & \xi \notin [-\frac{1}{e}, \frac{1}{e}] \\ 0, & otherwise, \end{cases}$$

For every $(\lambda_n) \in \mathbb{R}$ with $\lambda_n \to 0$, we have for every $\epsilon > 0$

$$\delta(\{n \in \mathbb{N}\colon |\phi(\lambda_n) - \phi(-\lambda_n)| \geq \epsilon\}) = 0$$

and

$$\delta(\{n \in \mathbb{N}\colon |\psi(\lambda_n) - \psi(-\lambda_n)| \geq \epsilon\}) = 0.$$

Then, ϕ and ψ are statistical strong weakly symmetrically continuous at 0. The function ϕ is locally bounded at 0 however ψ is not. By Theorem 1.9, the function ϕ is statistical weakly symmetrically continuous at 0. Additionally,

$$(\phi\psi)(\xi) = \begin{cases} \frac{\xi}{\ln(|\xi|+1)}, & \xi \notin [-\frac{1}{e}, \frac{1}{e}] \\ 0, & otherwise. \end{cases}$$

Hence, $\forall\lambda_n \in S_0(\mathbb{R})$ and $\forall\epsilon > 0$ we have

$$|(\phi\psi)(0+\lambda_n) - (\phi\psi)(0-\lambda_n)| = \frac{2\lambda_n}{\ln(\lambda_n+1)}$$

$$\{n \in \mathbb{N}: |(\phi\psi)(0+\lambda_n) - (\phi\psi)(0-\lambda_n)| \geq \epsilon\} = \mathbb{N}$$

$$\delta(\{n \in \mathbb{N}: |(\phi\psi)(\lambda_n) - (\phi\psi)(-\lambda_n)| \geq \epsilon\}) > 0.$$

Hence, $\phi\psi$ is not statistical weakly symmetrically continuous at 0.

Theorem 2.6. *Let $\phi : X \to \mathbb{R}$ be a statistical weakly symmetrically continuous function at ξ_0. Suppose that $\phi(\xi) \neq 0$ for $\forall\xi \in X$ and $\frac{1}{\phi}$ is locally bounded at ξ_0. Then, $\frac{1}{\phi}$ is statistical weakly symmetrically continuous at ξ_0.*

Proof. Suppose that ϕ be a statistical weakly symmetrically continuous at a point ξ_0 and let $\phi(\xi) \neq 0$ for $\forall\xi \in X$ and $\frac{1}{\phi}$ is locally bounded at ξ_0. Let $S_{\xi_0}(X) \neq \emptyset$ then, there exists a sequence $(\lambda_n) \in S_{\xi_0}(X)$ such that $\forall\epsilon > 0$,

$$\delta(\{n \in \mathbb{N}: |\phi(\xi_0+\lambda_n) - \phi(\xi_0-\lambda_n)| \geq \epsilon\}) = 0$$

and $\exists\delta, M > 0$ such that $\left|\frac{1}{\phi(\xi)}\right| \leq M$, for $\forall\xi \in (\xi_0-\delta, \xi_0+\delta) \cap X$. Since $(\lambda_n) \in S_{\xi_0}(X)$, then we can pick $N \in \mathbb{N}$ such that $\xi_0+\lambda_n, \xi_0-\lambda_n \in (\xi_0-\delta, \xi_0+\delta) \cap X$ for $\forall n \geq N$.

So, following inequality

$$\begin{aligned} \left|\frac{1}{\phi(\xi_0+\lambda_n)} - \frac{1}{\phi(\xi_0-\lambda_n)}\right| &= |\phi(\xi_0+\lambda_n) - \phi(\xi_0-\lambda_n)| \left|\frac{1}{\phi(\xi_0+\lambda_n)\phi(\xi_0-\lambda_n)}\right| \\ &\leq M^2 |\phi(\xi_0+\lambda_n) - \phi(\xi_0-\lambda_n)| \end{aligned}$$

and related inclusion

$$\begin{aligned} &\{n \in \mathbb{N}: \left|\frac{1}{\phi(\xi_0+\lambda_n)} - \frac{1}{\phi(\xi_0-\lambda_n)}\right| \geq \epsilon\} \subseteq \\ &\subseteq \{n \in \mathbb{N} : M^2 |\phi(\xi_0+\lambda_n) - \phi(\xi_0-\lambda_n)| \geq \epsilon\} \end{aligned}$$

holds. Then,

$$\delta(\{n \in \mathbb{N}: \left|\frac{1}{\phi(\xi_0+\lambda_n)} - \frac{1}{\phi(\xi_0-\lambda_n)}\right| \geq \epsilon\}) = 0.$$

Therefore, $\frac{1}{\phi}$ is statistical weakly symmetrically continuous at ξ_0. ■

Theorem 2.7. *Let $\phi : X \to \mathbb{R}$ be a statistical weakly symmetrically continuous function at a point ξ_0 and locally bounded at ξ_0. Let $\psi : X \to \mathbb{R}$ be a statistical strong weakly symmetrically continuous function at a point ξ_0. If $\psi(\xi) \neq 0$ for all $\xi \in X$ and $\frac{1}{\psi}$ is locally bounded at ξ_0 then, $\frac{\phi}{\psi}$ is statistical weakly symmetrically continuous at ξ_0.*

Proof. It is omitted because of similarity with Theorem 2.6. ■

Theorem 2.8. *Let $\phi : X \to Y$ and $\psi : Y \to \mathbb{R}$. Suppose that $\phi \in \mathscr{W}\mathscr{S}\mathscr{C}^{st}$ and ψ be a uniformly continuous on Y. Then, $\psi \circ \phi \in \mathscr{W}\mathscr{S}\mathscr{C}^{st}$.*

Proof. Suppose that ϕ is statistical weakly symmetrically continuous at ξ_0 and ψ is uniformly continuous on Y. Then, $S_{\xi_0}(X) \neq \emptyset$ implies that there exists a sequence $(\lambda_n) \in S_{\xi_0}(X)$ such that $\forall \epsilon > 0$,

$$\delta(\{n \in \mathbb{N}: |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0$$

and $\forall \epsilon > 0$, $\exists \delta \equiv \delta(\epsilon) > 0 \ni |\zeta_0 - \zeta_1| \leq \delta$ implies that for $\forall \zeta_0, \zeta_1 \in Y$

$$|\psi(\zeta_0) - \psi(\zeta_1)| < \epsilon \tag{2.1}$$

There is $N \in \mathbb{N}$ such that for all $n \geq N$

$$|\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| < \delta. \tag{2.2}$$

By equation (2.1) and (2.2),

$$|(\psi \circ \phi)(\xi_0 + \lambda_n) - (\psi \circ \phi)(\xi_0 - \lambda_n)| = |\psi(\phi(\xi_0 + \lambda_n)) - \psi(\phi(\xi_0 + \lambda_n))| < \epsilon$$

So, we have below inclusion

$$\{n \in \mathbb{N}: |(\psi \circ \phi)(\xi_0 + \lambda_n) - (\psi \circ \phi)(\xi_0 - \lambda_n)| \geq \epsilon\} \subseteq \{1, 2, ...N\}$$

and

$$\delta(\{n \in \mathbb{N}: |(\psi \circ \phi)(\xi_0 + \lambda_n) - (\psi \circ \phi)(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Consequently, $\psi \circ \phi$ is statistical weakly symmetrically continuous at ξ_0. ■

The following example shows that when $\phi \in \mathscr{W}\mathscr{S}\mathscr{C}^{st}$ but ψ is not uniformly continuous on the domain, it will be $\psi \circ \phi \notin \mathscr{W}\mathscr{S}\mathscr{C}^{st}$

Example 2.9. *Define* $\phi, \psi : \mathbb{R} \to \mathbb{R}$ *by*

$$\psi(\xi) = \begin{cases} \frac{1}{\xi}, & \xi \neq 0, \\ 0, & \xi = 0. \end{cases} \quad \textit{and} \quad \phi(\xi) = \xi \cos \xi$$

The function ϕ *is statistical weakly symmetrically continuous at* 0 *and* ψ *is not uniformly continuous on* $\mathbb{R}$.

$$(\psi \circ \phi)(\xi) = \begin{cases} \frac{1}{\xi \cos \xi}, & \xi \neq 0 \wedge \xi \neq (k\pi + \frac{\pi}{2}), \\ 0, & otherwise, \end{cases}$$

for all $k \in \mathbb{Z}$. *For* $\forall (\lambda_n) \in S_0(\mathbb{R})$ *and* $\epsilon > 0$,

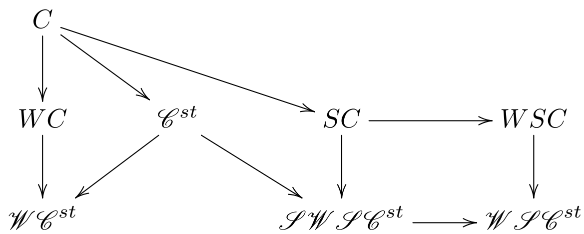
$$|(\psi \circ \phi)(0 + \lambda_n) - (\psi \circ \phi)(0 - \lambda_n)| = \frac{2}{\lambda_n \cos(\lambda_n)}$$

$$\{n \in \mathbb{N}: |(\psi \circ \phi)(\lambda_n) - (\psi \circ \phi)(-\lambda_n)| \geq \epsilon\} = \mathbb{N}$$
$$\delta(\{n \in \mathbb{N}: |(\psi \circ \phi)(\lambda_n) - (\psi \circ \phi)(-\lambda_n)| \geq \epsilon\}) = 1 > 0.$$

Hence, $\psi \circ \phi$ *is not statistical weakly symmetrically continuous at* 0.

3. Conclusion and some Remarks

P. Pongsriim-T. Thongsiri in [23] classified functions with removable discontinuity, and SC, WC and WSC classes were created. In this study, functions with removable discontinuities were subjected to a new classification with the help of natural density, and the following inclusions diagram was obtained. (Note that $E \to D$ means that $E \subseteq D$)



As a continuation of this study, the first question that comes to mind is to make a similar extension by taking a different kinds of densities instead of natural density, for example, logarithmic density, uniform density, and density produced by a regular matrix, generalized density, etc.

Maybe the other problem is determining whether there is any class of functions between X and Y where $X \in \{SC, WC, WSC\}$ and $Y \in \{\mathscr{S}\mathscr{W}\mathscr{S}\mathscr{C}^{st}, \mathscr{W}\mathscr{C}^{st}, \mathscr{W}\mathscr{S}\mathscr{C}^{st}\}$.

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On nearly recurrent Riemannian manifolds

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Abstract. The object of the present paper is to introduce a type of recurrent Riemannian manifold called nearly recurrent Riemannian manifold . The existence of nearly recurrent Riemannian manifold have been proved by non trivial example.

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Keywords: Nearly recurrent manifold, cyclic Ricci tensor, codazzi type Ricci tensor, concurrent vector field.

Contents

1. Introduction

Recurrent spaces have been of great importance and were studied by a large number of authors such as Ruse [1], Patterson [2] , Walker [3], Singh and Khan([4] and [5]) etc. In 1991, De and Guha [6] introduced and studied generalized recurrent manifold whose curvature tensor $R(X,Y)Z$ of type (1,3) satisfies the condition:

$$(D_U R)(X,Y)Z = A(U)R(X,Y)Z + B(U)[g(Y,Z)X - g(X,Z)Y], \tag{1.1}$$

where A and B are two non-zero 1-forms and D denotes the operator of covariant differentiation with respect to metric tensor g. Such a space has been denoted by GK_n. In recent papers Bandyopadhyay [7], Prakasha and Yildiz [8], Khan [9] etc explored various geometrical properties by using generalized recurrent manifold on Sasakian manifold and Lorentzian α-Sasakian manifold.

Further one of the author Prasad [10] considered a non-flat Riemannian manifold $(M^n, g)(n > 3)$ whose curvature tensor R satisfies the following condition

$$(D_U R)(X,Y)Z = A(U)R(X,Y)Z + B(U)g(Y,Z)X, \tag{1.2}$$

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where A and B are two non-zero 1-forms and D has the meaning already mentioned. Such a manifold called by the author as semi-generalized recurrent manifold and denoted by $(SGK)_n$. Singh, Singh and Kumar[11],[12] and Chaudhary, Kumar and Singh [13] extended this notation to Lorentzian α-Sasakian manifold, P-Sasakian manifold and trans-Sasakian manifold.

The object of the present paper is to study a type of non-flat recurrent Riemannian manifold $(M^n, g)(n > 2)$ whose curvature tensor $R(X,Y)Z$ of the type (1,3) satisfies the condition

$$(D_U R)(X,Y)Z = [A(U) + B(U)]R(X,Y)Z + B(U)[g(Y,Z)X - g(X,Z)Y], \tag{1.3}$$

where A and B are two non-zero 1-forms and ρ_1 and ρ_2 are two vector fields such that

$$g(U, \rho_1) = A(U)\ and\ g(U, \rho_2) = B(U). \tag{1.4}$$

Such a manifold shall be called as a nearly recurrent Riemannian manifold and 1-forms A and B shall be called its associated 1-forms and n-dimensional recurrent manifold of this kind shall be denoted by $(NR)_n$.
If in particular $B = 0$, then the space reduced to a recurrent space according to Ruse [14] and Walker [3] which is denoted by K_n.
Moreover, in particular if $A = B = 0$ then (1.3) becomes $(D_U R)(X,Y)Z = 0$. That is , a Riemannian manifold is symmetric accordingly Kobayashi and Nomizu [15] and Desai and Amer [16]. The name nearly recurrent Riemannian manifold was chosen because if $B = 0$ in (1.3) then the manifold reduces to a recurrent manifold which is very close to recurrent space. This justifies the name $Nearly\ recurrent\ Riemannian\ manifold$ for the manifold defined by (1.3) and the use of the symbol $(NR)_n$ for it.

In this paper, after preliminaries, a necessary and sufficient condition for constant scaler curvature of $(NR)_n$ is obtained. Nearly recurrent manifold with cyclic Ricci tensor and Codazzi type Ricci tensor are studied. Finally, we give examples of $(NR)_n$.

2. Preliminaries

Let S and r denote the Ricci tensor of type (0,2) and scalar curvature respectively and Q denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor, i.e.

$$S(X,Y) = g(QX,Y), \tag{2.1}$$

for any vector field X and Y.
From(1.3), we get

$$(D_U S)(Y,Z) = [A(U) + B(U)]S(Y,Z) + (n-1)B(U)g(Y,Z). \tag{2.2}$$

Contracting (2.2), we have

$$dr(U) = Ur = [A(U) + B(U)]r + n(n-1)B(U). \tag{2.3}$$

3. Nature of the 1-forms A and B on a nearly recurrent space

From (2.3) suppose $r = 0$,then

$$B(U) = 0$$

which is not possible. Hence we have the following theorem:

Theorem 3.1. *The scalar curvature tensor of $(NR)_n$ can not be zero.*

Now we consider $(NR)_n$ is of constant scalar curvature then from (2.3), we have

$$[A(U) + B(U)]r + n(n-1)B(U) = 0. \tag{3.1}$$

Again if (3.1) holds, then from (2.3), we get

$$dr(U) = 0,$$
$$r = constant$$

Hence, we can state the following theorem:

Theorem 3.2. *A $(NR)_n$ is of constant curvature if and only if (3.1) holds.*

Now, taking covariant derivative of (3.1) with respect to V, we get

$$[(D_V A)(U) + (D_V B)(U)]r + n(n-1)(D_V B)U = 0. \tag{3.2}$$

Interchanging U and V in (3.2) and then subtracting, we get

$$[(dA(U,V) + dB(U,V)]r + n(n-1)dB(U,V) = 0. \tag{3.3}$$

Thus we have the following theorem:

Theorem 3.3. *In a nearly recurrent space of non-zero constant scalar curvature r, if the 1-forms B is closed then A is closed , if A is closed then B is also closed.*

From (1.3), we have

$$(D_V R)(X,Y)Z = [A(V) + B(V)]R(X,Y)Z + B(V)[g(Y,Z)X - g(X,Z)Y].$$

This gives

$$\begin{aligned}(D_U D_V R)(X,Y)Z =& [(D_U A)(V) + A(D_U V) + (D_U B)(V) + B(D_U V)]R(X,Y)Z \\ &+ [A(U) + B(U)][A(V) + B(V)]R(X,Y)Z + \\ &[A(V) + B(V)]B(U)[g(Y,Z)X - g(X,Z)Y]\end{aligned} \tag{3.4}$$

Therefore from(3.4), we have

$$\begin{aligned}(D_V D_U R)(X,Y)Z =& [(D_V A)(U) + A(D_V U) + (D_V B)(U) + B(D_V U)]R(X,Y)Z \\ &+ [A(U) + B(U)][A(V) + B(V)]R(X,Y)Z + \\ &[A(U) + B(U)]B(V)[g(Y,Z)X - g(X,Z)Y]\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}(D_{[U,V]}R)(X,Y)Z =& [A([U,V]) + B([U,V])]R(X,Y)Z + \\ &B([U,V])[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{3.6}$$

Now, subtracting (3.5) and (3.6) from (3.4), we get

$$\begin{aligned}(R(U,V).R)(X,Y)Z =& [(dA(U,V) + dB(U,V)]R(X,Y)Z + \\ &dB(U,V)[g(Y,Z)X - g(X,Z)Y] + \\ &[A(V)B(U) - A(U)B(V)].\end{aligned} \tag{3.7}$$

Thus, we can state the following theorem:

Theorem 3.4. *In a $(NR)_n$ with constant scalar curvature , R(X,Y).R=0 if and only if*

$$[(dA(U,V)+dB(U,V)]R(X,Y)Z+dB(U,V)[g(Y,Z)X-g(X,Z)Y]+$$
$$[A(V)B(U)-A(U)B(V)]=0.$$

Next, we consider the case when the scalar curvature r is not constant.
From (2.3) it follows that

$$VUr=(D_VA)(U)r+A(U)(Vr)+n(n-1)(D_VB)(U). \tag{3.8}$$

Interchanging U and V in (3.8) and then subtracting, we get

$$[(D_VA)(U)-(D_UA)(V)+(D_VB)(U)-(D_UB)(V)]r+$$
$$n(n-1)\{(D_VB)(U)-(D_UB)(V)\}+[r+n(n-1)][A(U)B(V)-A(V)B(U)]=0.$$

which gives

$$\begin{aligned}&[dA(V,U)+dB(V,U)]r+n(n-1)dB(V,U)+\\&[r+n(n-1)][A(U)B(V)-A(V)B(U)]=0.\end{aligned} \tag{3.9}$$

Thus we have the following theorem:

Theorem 3.5. *In a nearly recurrent space of non-zero constant scalar curvature r, the 1-forms A and B are closed if and only if the 1-forms A and B are co-directional.*

4. $(NR)_n$ with cyclic Ricci tensor

In this section we consider a $(NR)_n$ in which the Ricci tensor is a cyclic tensor, i.e.

$$(D_XS)(Y,Z)+(D_YS)(Z,X)+(D_ZS)(X,Y)=0, \tag{4.1}$$

which implies

$$dr(X)=0. \tag{4.2}$$

From (1.3), we have

$$dr(X)=[A(X)+B(X)]r+n(n-1)B(X). \tag{4.3}$$

Therefore from (4.2) and (4.3), we get

$$[A(X)+B(X)]r+n(n-1)B(X)=0. \tag{4.4}$$

From (4.1), we have

$$[A(X)+B(X)]S(Y,Z)+[A(Y)+B(Y)]S(Z,X)+[A(Z)+B(Z)]S(X,Y)$$
$$+(n-1)[B(X)g(Y,Z)+B(Y)g(X,Z)+B(Z)g(X,Y)]=0,$$

which yields on contraction

$$\begin{aligned}A(QX)+B(QX)&=\frac{r}{n}[A(X)+B(X)]\\ or\ \ S(X,\rho_1)+S(X,\rho_2)&=\frac{r}{n}[g(X,\rho_1)+g(X,\rho_2)]\\ or\ \ S(X,\rho_1+\rho_2)&=\frac{r}{n}[g(X,\rho_1+\rho_2)\end{aligned}$$

Above can be written as

$$S(X, \mu) = \frac{r}{n} g(X, \mu), \tag{4.5}$$

where $\mu = \rho_1 + \rho_2$.
Hence we have the following theorem:

Theorem 4.1. *If $(NR)_n$ has cyclic Ricci tensor, then $\frac{r}{n}$ is an eigen value of Ricci tensor S and μ is an eigen vector corresponding to the eigen value.*

5. $(ER)_n$ with Codazzi type of Ricci tensor

In this section, we consider an $(NR)_n$ in which the Ricci tensor is a Codazzi type of Ricci tensor Ferus [17]

$$(D_X S)(Y, Z) = (D_Z S)(Y, X). \tag{5.1}$$

By view of Bianchi identity and (5.1), we have

$$(divR)(X, Y)Z = 0. \tag{5.2}$$

In view of (1.3), we get on contraction

$$(divR)(X, Y)Z = A(R(X, Y)Z) + B(R(X, Y)Z) + B(X)g(Y, Z) - B(Y)g(X, Z). \tag{5.3}$$

Now using (5.2) in (5.3), we get

$$A(R(X, Y)Z) + B(R(X, Y)Z) + B(X)g(Y, Z) - B(Y)g(X, Z) = 0. \tag{5.4}$$

In view of (5.4), we get

$$A(QX) + B(QX) = -(n - 1)B(X). \tag{5.5}$$

From (2.2) and (5.1), we have

$$\begin{aligned} &[A(X) + B(X)]S(Y, Z) - [A(Z) + B(Z)]S(Y, X) \\ &+ (n - 1)[B(X)g(Y, Z) - B(Z)g(X, Y)] = 0. \end{aligned} \tag{5.6}$$

On contracting of (5.6), we have

$$[A(X) + B(X)]r = [A(QX) + B(QX)] - (n - 1)^2 B(X). \tag{5.7}$$

Using (5.5) and (5.7) in (2.3), we have

$$dr(X) = 0. \tag{5.8}$$

Again it is known [18] that in a Riemannian manifold $(M^n, g)(n > 3)$

$$\begin{aligned} (divC)(X, Y)Z = &\frac{n-3}{n-2}[(D_X S)(Y, Z) - (D_Z S)(Y, X)] + \\ &\frac{1}{2(n-1)}[g(X, Y)dr(Z) - g(Y, Z)dr(X)], \end{aligned} \tag{5.9}$$

where C denotes the conformal curvature.
As a consequences of (5.1) and (5.8), (5.9) reduces to

$$(divC)(X, Y)Z = 0,$$

which shows that the tensor is conservative [19].
Hence we can state the following theorem:

Theorem 5.1. *If in a $(NR)_n$ the Ricci tensor is a Codazzi type tensor then its conformal curvature tensor is conservative.*

6. Nearly recurrent with concurrent vector field

In this section first we suppose that the $(NR)_n$ admits a concurrent unit vector fields $\widetilde{V}$,

$$D_X\widetilde{V} = \rho X, \tag{6.1}$$

where ρ is a non-zero constant .
By Ricci-identity

$$R(X,Y)\widetilde{V} = 0. \tag{6.2}$$

Taking covariant derivative of (6.2), we get

$$(D_W R)(X,Y)\widetilde{V} = -\rho R(X,Y)W \tag{6.3}$$

Also by definition of $(NR)_n$,we find

$$(D_W R)(X,Y)\widetilde{V} = [A(W)+B(W)]R(X,Y)\widetilde{V} + B(W)[g(Y,\widetilde{V})X - g(X,\widetilde{V})Y]. \tag{6.4}$$

In view of (6.2),(6.3)and (6.4),we get

$$-\rho R(X,Y)W = B(W)[g(Y,\widetilde{V})X - g(X,\widetilde{V})Y].$$

On contraction, we find

$$-\rho S(Y,W) = (n-1)B(W)g(Y,\widetilde{V}). \tag{6.5}$$

Again on contraction of (6.5), we get

$$-\rho r = (n-1)B(\widetilde{V}) = (n-1)g(\rho_2,\widetilde{V}), \tag{6.6}$$

Since $\rho \neq 0$ and $r \neq 0$,then from (6.6),we get

$$g(\rho_2,\widetilde{V}) \neq 0. \tag{6.7}$$

Hence we have the following theorem:

Theorem 6.1. *If a $(NR)_n$ the associated vector field ρ_2 cannot concurrent vector field .*

7. Example

Example (7.1) Let us consider $M^4 = \left\{(x^1,x^2,x^3,x^4) \in R^4\right\}$ be an open subset of R^4 endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{3}{2}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2 \tag{7.1}$$

where $i, j = 1, 2, 3, 4$.
Then the only non-vanishes components of the Christoffel symbols and curvature tensor are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{3}{4(x^4)}, \quad \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = -\frac{3}{4}(x^4)^{\frac{1}{2}}$$

$$R_{1441} = R_{2442} = R_{3443} = -\frac{3}{16(x^4)^{\frac{1}{2}}} \tag{7.2}$$

The non-vanishing components of the Ricci tensor are

$$R_{11} = R_{22} = R_{33} = -\frac{3}{16(x^4)^{\frac{1}{2}}}, \quad R_{44} = -\frac{3}{16(x^4)^2}$$

and scalar curvature is

$$R = g^{ii}R_{ii} = -\frac{3}{16(x^4)^2}$$

Taking covariant derivative of (7.2), we get

$$R_{1441,4} = \frac{3}{32(x^4)^{\frac{3}{2}}},\ \ R_{2442,4} = \frac{3}{32(x^4)^{\frac{3}{2}}},\ \ R_{3443,4} = \frac{3}{32(x^4)^{\frac{3}{2}}} \tag{7.3}$$

Consequently, the manifold under consideration is not recurrent .
Let us choose the associated 1-form as

$$A_i = \begin{cases} \frac{3}{32}\cdot\frac{64(x^4)^2-1}{(x^4)^3-4(x^4)^2}, & i=4 \\ 0, & \text{otherwise} \end{cases} \tag{7.4}$$

$$B_i = \begin{cases} \frac{3}{32}\cdot\frac{1}{(x^4)^3-4(x^4)^2}, & i=4 \\ 0, & \text{otherwise} \end{cases} \tag{7.5}$$

From (1.3), we have

$$R_{hiih,i} = (A_i + B_i)R_{hiih} + B_i[g_{ii}g_{hh} - g_{hi}g_{ih}] \tag{7.6}$$

By virtue of (7.2), (7.3),(7.4) and (7.5), it can be easily seen that the Riemannian manifold satisfies relation (7.6). Hence the manifold under consideration is a nearly recurrent Riemannian manifold (M^4, g), which is neither recurrent nor symmetric.
This leads to the following theorem:

Theorem 7.1. *There exist a nearly recurrent Riemannian manifold (M^4, g), which is neither recurrent nor symmetric.*

Example (7.2) Let us consider the 3-dimensional manifold $M = \{(x,y,z) \in R^3, z \neq 0\}$, where (x,y,z) are standard co-ordinate of R^3.
We choose the vector fields

$$e_1 = \frac{1}{2}\frac{\partial}{\partial y},\ e_2 = \frac{\partial}{\partial x} - z\frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z} \tag{7.7}$$

which is linearly independently at each point of M.
Let g be the Riemannian metric denoted by

$$g(e_i, e_j) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases} \tag{7.8}$$

Let D be the Levi-Civita connection with respect to metric g. Then from equation (7.7), we have

$$[e_1, e_2] = 0, [e_1, e_3] = 2e_1, [e_2, e_3] = 0. \tag{7.9}$$

The Riemannian connection D of the metric g is given by

$$\begin{aligned} 2g\left(D_X Y, Z\right) =& Xg\left(Y,Z\right) + Yg\left(X,Z\right) - Zg\left(X,Y\right) - g\left(X,[Y,Z]\right) \\ & - g\left(Y,[X,Z]\right) + g\left(Z,[X,Y]\right), \end{aligned} \tag{7.10}$$

which is known as Koszul's formula. Using (7.8) and (7.9) in (7.10), we get

$$\begin{aligned} &D_{e_1}e_3 = -e_2, \quad D_{e_1}e_2 = e_3, \quad D_{e_1}e_1 = 0, \\ &D_{e_2}e_3 = e_1, \quad D_{e_2}e_2 = 0, \quad D_{e_2}e_1 = -e_3, \\ &D_{e_3}e_3 = 0, \quad D_{e_3}e_2 = -e_1, \quad D_{e_3}e_1 = e_2. \end{aligned} \tag{7.11}$$

The curvature tensor is given by

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \tag{7.12}$$

Using (7.9) and (7.11) in (7.12) , we get

$$\begin{aligned}
&R(e_1,e_2)e_1 = e_2, \quad R(e_1,e_2)e_2 = e_1, \quad R(e_1,e_2)e_3 = 0\\
&R(e_2,e_3)e_1 = 0, \quad R(e_2,e_3)e_2 = 3e_3, \quad R(e_2,e_3)e_3 = e_2\\
&R(e_1,e_3)e_1 = -e_3, \quad R(e_1,e_3)e_2 = 0 \qquad R(e_1,e_3)e_3 = -e_1\\
&R(e_1,e_1)e_1 = R(e_1,e_1)e_2 = R(e_1,e_1)e_3 = 0\\
&R(e_2,e_2)e_1 = R(e_2,e_2)e_2 = R(e_2,e_2)e_3 = 0\\
&R(e_3,e_3)e_1 = R(e_3,e_3)e_2 = R(e_3,e_3)e_3 = 0.
\end{aligned} \tag{7.13}$$

The Ricci tensor is given by

$$S(e_i,e_i) = \sum_{i=1}^{3} g(R(e_i,X)Y,e_i) \tag{7.14}$$

From (7.13) and (7.14), we get

$$S(e_1,e_1) = 0, \quad S(e_2,e_2) = 2, \quad S(e_3,e_3) = 0 \tag{7.15}$$

and the scalar curvature is 2.
Since $\{e_1,e_2,e_3\}$ forms a basis of Riemannian manifold any vector field $X,Y,Z \in \chi(M)$ can be written as

$$X = a_1e_1 + b_1e_2 + c_1e_3, \;\; Y = a_2e_1 + b_2e_2 + c_2e_3, \;\; Z = a_3e_1 + b_3e_2 + c_3e_3,$$

where $a_i, b_i, c_i \in R^+$ (the set of all positive real numbers), $i = 1,2,3$.
Hence

$$R(X,Y)Z = l_1e_1 + m_1e_2 + n_1e_3 \tag{7.16}$$

$$g(Y,Z)X - g(X,Z)Y = l_2e_1 + m_2e_2 + n_2e_3 \tag{7.17}$$

By view of (7.16), we get

$$(D_{e_i}R)(X,Y)Z = u_ie_1 + v_ie_2 + w_ie_3 \;\; for\; i = 1,2,3. \tag{7.18}$$

where

$$\begin{aligned}
l_1 &= a_1b_2b_3 + a_2c_1c_3 - c_1c_2c_3,\\
m_1 &= a_1b_2a_3 + a_3b_1b_2 - b_1a_2a_3 + b_1c_2c_3,\\
n_1 &= 3b_1b_3c_2 - 3b_3c_1b_2 - a_1a_3c_2,\\
l_2 &= a_1b_2b_3 + a_1c_2c_3 - a_2b_1b_3 - a_2c_1c_3,\\
m_2 &= a_2a_3b_1 + b_1c_2c_3 - a_1a_3b_2 - b_2c_1c_3,\\
n_2 &= a_2a_3c_1 + b_2b_3c_1 - a_1a_3c_2 - b_1b_3c_2,\\
u_1 &= a_2a_3c_1 - a_2b_3c_1 - a_1b_3c_2,\\
v_1 &= 2b_2b_3c_1 - 2b_1b_3c_2 + a_1a_3c_2 - a_2a_3c_1,\\
w_1 &= 2a_2a_3b_1 - 2a_1a_3b_2 - a_3b_1b_2 + 2b_1c_2c_3,\\
u_2 &= 4b_1b_3c_2 - 3b_2b_3c_1 - 2a_1a_3c_2 - c_1c_2c_3 + a_2a_3c_1,\\
v_2 &= -2a_1b_2c_3 + 2a_3b_1c_2 + a_2b_1c_3 + a_2b_2c_3,\\
w_2 &= -4a_1b_2b_3 - 2a_2c_1c_3 + c_1c_2c_3 + 3a_2b_2b_3 - a_3c_1c_2 - 3a_3b_2c_1 + a_1a_2c_3,\\
u_3 &= -2a_1a_3b_2 - a_3b_1b_2 + 2a_2a_3b_1 - 2b_1c_2c_3 - 2a_1a_2b_3 + b_2c_1c_3,\\
v_3 &= 2a_1b_2b_3 + a_2c_1c_3 - c_1c_2c_3 - a_1a_2c_3 + a_3c_1c_3 + a_2b_1b_3,\\
w_3 &= -3a_1b_3c_2 - a_3b_1c_2 + 3a_2b_3c_1 + 2a_3b_2c_1.
\end{aligned}$$

Consequently, the manifold under consideration is not recurrent. Let us now consider 1-form non vanishes

$$A(e_i) = \frac{4(u_i + v_i + w_i)}{3(l_1 + m_1 + n_1) - (l_2 + m_2 + n_2)}$$
$$B(e_i) = \frac{-(u_i + v_i + w_i)}{3(l_1 + m_1 + n_1) - (l_2 + m_2 + n_2)} \tag{7.19}$$

such that

$$3(l_1 + m_1 + n_1) - (l_2 + m_2 + n_2) \neq 0.$$

From (1.3), we have

$$(D_{e_i} R)(X, Y)Z = [A(e_i) + B(e_i)]R(X, Y)Z + B(e_i)[g(Y, Z)X - g(X, Z)Y]. \tag{7.20}$$

By virtue of (7.16), (7.17), (7.18) and (7.19), it can be easily seen that the Riemannian manifold satisfies relation (7.20). Hence the manifold under consideration is a nearly recurrent Riemannian manifold (M^3, g), which is neither recurrent nor symmetric. Thus we have the following theorem:

Theorem 7.2. *There exist a nearly recurrent Riemannian manifold (M^3, g), which is neither recurrent nor symmetric.*

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Local isometry of the generalized helicoidal surfaces family in 4-space

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Abstract. In this paper, we consider the helicoidal surfaces family in four dimensional Euclidean space $\mathbb{E}^4$. We calculate normal pair and the curvatures of the surface family. Moreover, we find the local isometry from helicoidal surface family to the rotational surface family by using Bour's theorem in $\mathbb{E}^4$.

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Keywords: Euclidean 4-space, isometric deformation, Bour's theorem, helicoidal surface, mean curvature.

Contents

1. Introduction

The deformation of parametric surfaces family is determined by

$$\mathtt{X}_\beta(u,v)=\begin{pmatrix}\cos\beta\sin u\sinh v+\sin\beta\cos u\cosh v\\ -\cos\beta\cos u\sinh v+\sin\beta\sin u\cosh v\\ u\cos\beta+v\sin\beta\end{pmatrix}.$$

Here, $u,\beta\in(-\pi,\pi]$, $v\in(-\infty,\infty)$, β is the parameter of deformation. $\mathtt{X}_\beta$ is minimal, i.e., has zero mean curvature. $\mathtt{X}_0$ is the helicoid, $\mathtt{X}_{\pi/2}$ is the catenoid. Therefore, the surfaces are locally isometric, have the same Gauss map.

In addition, helices of $\mathtt{X}_0$ match to parallel circles of $\mathtt{X}_{\pi/2}$. Finally, we meet the classical theorem of the French mathematician Edmond Bour.

Bour's Theorem [1]. *A helicoidal surface is locally isometric to a rotational surface so that helices of the helicoidal surface match to parallel circles of the rotational surface.*

Some other Euclidean and also Lorentz-Minkowski versions of it were studied by [2]-[14].

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Next, we present some fundamental geometric and differential facts of four dimensional Euclidean space. Let

$$\overrightarrow{x}\cdot\overrightarrow{y}=x_1y_1+x_2y_2+x_3y_3+x_4y_4$$

be a Euclidean inner product, and let $\mathtt{X}$: $D\subset\mathbb{E}^2\to\mathbb{E}^4$ be a parametric representation of surface M in Euclidean 4-space $\mathbb{E}^4$. The tangent space of M at a point $\mathbf{p}=\mathtt{X}(u,v)$ is spanned by $\mathtt{X}_u$ and $\mathtt{X}_v$, where $\mathtt{X}_u=\frac{\partial\mathtt{X}}{\partial u}$, $\mathtt{X}_v=\frac{\partial\mathtt{X}}{\partial v}$.

The first fundamental form matrix of M is obtained by

$$\mathbf{I}=\begin{pmatrix}\mathtt{E} & \mathtt{F}\\ \mathtt{F} & \mathtt{G}\end{pmatrix}$$

where

$$\mathtt{E}=\mathtt{X}_u\cdot\mathtt{X}_u,\ \mathtt{F}=\mathtt{X}_u\cdot\mathtt{X}_v,\ \mathtt{G}=\mathtt{X}_v\cdot\ \mathtt{X}_v.$$

We assume the surface M is regular. That is, $W^2=\det\mathbf{I}=\mathtt{EG}-\mathtt{F}^2>0$. Let $\{\eta_1,\eta_2,\zeta_1,\zeta_2\}$ be a orthonormal frame of M where η_1, η_2 are tangent to M, ζ_1,ζ_2 are normal to M. The second fundamental form matrix of M w.r.t. the unit normal vector ζ_i, $i=1,2$, is described by

$$\mathbf{II}^i=\begin{pmatrix}\mathtt{L}^i & \mathtt{M}^i\\ \mathtt{M}^i & \mathtt{N}^i\end{pmatrix}$$

where

$$\mathtt{L}^i=\mathtt{X}_{uu}\cdot\zeta_i,\ \mathtt{M}^i=\mathtt{X}_{uv}\cdot\zeta_i,\ \mathtt{N}^i=\mathtt{X}_{vv}\cdot\zeta_i,$$

and $\mathtt{X}_{uu}=\frac{\partial^2\mathtt{X}}{\partial u^2}$, $\mathtt{X}_{uv}=\frac{\partial^2\mathtt{X}}{\partial u\partial v}$, $\mathtt{X}_{vv}=\frac{\partial^2\mathtt{X}}{\partial v^2}$.

We determine by

(a) $H_i=\frac{(\mathtt{E})(\mathtt{N}^i)+(\mathtt{G})(\mathtt{L}^i)-2(\mathtt{F})(\mathtt{M}^i)}{2W^2}$, the mean curvature of M w.r.t. n_i, $i=1,2$,

(b) $\overrightarrow{H}=H_1n_1+H_2n_2$, the mean curvature vector of M,

(c) $\overrightarrow{H}=0$, the surface M is minimal,

(d) $K=\frac{(\mathtt{L}^1)(\mathtt{N}^1)-(\mathtt{M}^1)^2+(\mathtt{L}^2)(\mathtt{N}^2)-(\mathtt{M}^2)^2}{W^2}=\frac{\det(\mathbf{II}^1)+\det(\mathbf{II}^2)}{W^2}$, the Gaussian curvature of M, respectively.

An orthonormal tangent frame field $\{\eta_1,\eta_2\}$ of M is choosen by

$$\eta_1=\frac{1}{\sqrt{\mathtt{E}}}\mathtt{X}_u,\quad \eta_2=\frac{1}{W\sqrt{\mathtt{E}}}\left(\mathtt{E}\mathtt{X}_v-\mathtt{F}\mathtt{X}_u\right),$$

with its Gauss map

$$\mathcal{G}=\frac{1}{W}\left(\mathtt{X}_u\wedge\mathtt{X}_v\right).$$

In this paper, we generalized the work of The Hieu and Ngoc Thang [14].

2. Generalized helicoidal surfaces family in $\mathbb{E}^4$

A vector (a,b,c,d) of $\mathbb{E}^4$ will be identified with its transpose in the rest of this work.

Let $\gamma: I\subset\mathbb{R}\longrightarrow\Pi$ be a curve in a plane Π in $\mathbb{E}^4$, ℓ be a line in Π. A generalized rotational surface family in $\mathbb{E}^4$ is described by rotating a profile curve γ about a line (i.e., axis) ℓ.

When γ rotates about ℓ, it simultaneously matches parallel lines perpendicular to the ℓ, so the displacement speed is proportional to the rotation speed. Therefore, the final surface is named the *generalized helicoidal surface family* with axis ℓ and pitch $a\in\mathbb{R}\backslash\{0\}$.

Parametrization of the profile curve is given by

$$\gamma(u) = (f(u), 0, g(u), h(u)),$$

where $f, g, h : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ are the differentiable functions for all $u \in I$. So, in $\mathbb{E}^4$, a generalized helicoidal surface family with pitch $a \in \mathbb{R} \backslash \{0\}$ is defined by

$$\mathcal{H}(u,v) = \begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ g(u) + av \\ h(u) \end{pmatrix}, \tag{2.1}$$

where f, g, h are the differentiable functions, $u, a \in \mathbb{R} \setminus \{0\}$, and $0 \leq v < 2\pi$. When $a = 0$, it is just a rotational surface in $\mathbb{E}^4$.

By taking the first derivatives w.r.t. u and v, respectively, of the generalized helicoidal surfaces family defined by Eq. (2.1), we find the following first quantities of the family

$$\mathtt{E} = f'^2 + g'^2 + h'^2, \quad \mathtt{F} = ag', \quad \mathtt{G} = f^2 + a^2,$$

where $f'^2 = \left(\frac{df}{du}\right)^2$, $g'^2 = \left(\frac{dg}{du}\right)^2$, $h'^2 = \left(\frac{dh}{du}\right)^2$.

We compute two normals of the generalized helicoidal surface family described by Eq. (2.1) as follows

$$\zeta_1 = \frac{1}{T}\begin{pmatrix} h'\cos v \\ h'\sin v \\ 0 \\ -f' \end{pmatrix}, \tag{2.2}$$

$$\zeta_2 = \frac{1}{WT}\begin{pmatrix} -ff'g'\cos v + a\left(f'^2 + h'^2\right)\sin v \\ -a\left(f'^2 + h'^2\right)\cos v - ff'g'\sin v \\ f\left(f'^2 + h'^2\right) \\ -fg'h' \end{pmatrix}, \tag{2.3}$$

respectively. Here, $T = \sqrt{f'^2 + h'^2}$, $W = \sqrt{a^2\left(f'^2 + h'^2\right) + f^2\left(f'^2 + g'^2 + h'^2\right)}$.

Using the second derivatives of the helicoidal surface defined by Eq. (2.1) w.r.t. u and v, respectively,

$$\begin{aligned} \mathcal{H}_{uu} &= (f''\cos v, f''\sin v, g'', h''), \\ \mathcal{H}_{uv} &= (-f'\sin v, f'\cos v, 0, 0), \\ \mathcal{H}_{vv} &= (-f\cos v, -f\sin v, 0, 0), \end{aligned}$$

where $f'' = \frac{\partial^2 f}{\partial u^2}$, $g'' = \frac{\partial^2 g}{\partial u^2}$, $h'' = \frac{\partial^2 h}{\partial u^2}$, and the normals determined by Eq. (2.2) and Eq. (2.3), we have the following second quantities of the generalized helicoidal surfaces family described by Eq. (2.1):

$$\mathtt{L}^1 = \frac{f''h' - f'h''}{T}, \ \mathtt{M}^1 = 0, \ \mathtt{N}^1 = -\frac{fh'}{T},$$

$$\mathtt{L}^2 = \frac{f\left(-\left(f'^2 + h'^2\right)g'' + g'\left(f'f'' + h'h''\right)\right)}{WT}, \ \mathtt{M}^2 = \frac{af'\left(f'^2 + h'^2\right)}{WT}, \ \mathtt{N}^2 = -\frac{f^2f'g'}{WT}.$$

Hence, the mean curvatures H_i $(i = 1, 2)$ and the Gaussian curvature K of the generalized helicoidal surfaces family defined by Eq. (2.1) are given by as follows

$$H_1 = \frac{\left(f^2+a^2\right)h'f'' - f\left(f'^2+g'^2\right)h' - fh'^3 - \left(a^2+f^2\right)f'h''}{2W^2\sqrt{f'^2+h'^2}},$$

$$H_2 = \frac{\left\{\begin{array}{c} f\left(a^2+f^2\right)\left\{f'g'f'' - \left(f'^2+h'^2\right)g'' + g'h'h''\right\} \\ -\left\{2a^2\left(f'^2+h'^2\right) + f^2\left(f'^2+g'^2+h'^2\right)\right\}f'g' \end{array}\right\}}{2W^3\sqrt{f'^2+h'^2}},$$

$$K = \frac{\left\{\begin{array}{c} -\left(f^3f'^2g'^2W^2 + fh'^2\right)f'' + \left(f^3f'g'\left(f'^2+h'^2\right)W^2\right)g'' \\ -\left(f^3f'g'^2h'W^2 - ff'h'\right)h'' - a^2f'^2\left(f'^2+h'^2\right)^2W^2 \end{array}\right\}}{W^4\left(f'^2+h'^2\right)}.$$

3. Bour's theorem on generalized helicoidal-rotational surfaces family in $\mathbb{E}^4$

Next, we generalize the Bour's theorem for the generalized helicoidal-rotational surfaces family in four dimensional Euclidean space.

Theorem 1. *Let $\mathcal{H}$ be the generalized helicoidal surfaces family described by Eq. (2.1), and let $p(u)$, $q(u)$, $u>0$ are the differentiable functions supplying the equation*

$$p^2+q^2 = \frac{a^2+f^2f'^2+\left(f^2+a^2\right)h'^2}{f^2}. \tag{3.1}$$

Therefore, the generalized helicoidal surface family $\mathcal{H}$ defined by Eq. (2.1) is locally isometric to the following generalized rotational surfaces family

$$\mathcal{R}(u,v) = \begin{pmatrix} \sqrt{f^2+a^2}\cos\left(v+\int\frac{ag'}{f^2+a^2}du\right) \\ \sqrt{f^2+a^2}\sin\left(v+\int\frac{ag'}{f^2+a^2}du\right) \\ \int\frac{f\,p(u)}{\sqrt{f^2+a^2}}du \\ \int\frac{f\,q(u)}{\sqrt{f^2+a^2}}du \end{pmatrix} \tag{3.2}$$

so that helices on the generalized helicoidal surface correspond to parallel circles on the generalized rotational surfaces.

Proof. The arc lenght element of the generalized helicoidal surface given by Eq. (2.1) is described by as follows

$$ds^2 = \left(f'^2+g'^2+h'^2\right)du^2 + 2ag'dudv + \left(a^2+f^2\right)dv^2.$$

Setting $\overline{u}=u$, $\overline{v}=v+\int\frac{ag'}{f^2+a^2}du$, the generalized helicoidal surface determined by Eq. (2.1) transforms to $\mathcal{H}(\overline{u},\overline{v})$. Considering the new parameters of the surface, its arc lenght element reduces to

$$ds^2 = \left(\left(f'^2+h'^2\right)+\frac{f^2g'^2}{a^2+f^2}\right)d\overline{u}^2 + \left(a^2+f^2\right)d\overline{v}^2.$$

On the other side, in $\mathbb{E}^4$, the following generalized rotational surfaces family

$$\mathcal{R}(\mathfrak{s},\mathfrak{t}) = \begin{pmatrix} \mathfrak{f}(\mathfrak{s})\cos\mathfrak{t} \\ \mathfrak{f}(\mathfrak{s})\sin\mathfrak{t} \\ \mathfrak{g}(\mathfrak{s}) \\ \mathfrak{h}(\mathfrak{s}) \end{pmatrix}$$

has the following arc lenght element

$$ds^2 = \left(\mathfrak{f}'^2 + \mathfrak{g}'^2 + \mathfrak{h}'^2\right) d\mathfrak{s}^2 + \mathfrak{f}^2 d\mathfrak{t}^2. \tag{3.3}$$

Again setting $\mathfrak{f} = \sqrt{a^2 + f^2}$, $p(u) = \mathfrak{g}'$, $q(u) = \mathfrak{h}'$, we get the following functions

$$\mathfrak{g} = \int \frac{f\ p(u)}{\sqrt{a^2 + f^2}} du, \quad \mathfrak{h} = \int \frac{f\ q(u)}{\sqrt{a^2 + f^2}} du.$$

Hence, differential Eq. (3.1) determines that the generalized helicoidal surfaces family given by Eq. (2.1) is locally isometric to the generalized rotational surfaces family determined by Eq. (3.2).

The helices of $\mathcal{H}$ are defined by $u = u_0$; where u_0 is a constant, those are match to the curves of $\mathcal{R}$ determined by $\mathfrak{f} = \sqrt{u_0^2 + a^2}$; that is, those are the circles of the plane $\{x_3 = \mathfrak{g}(\mathfrak{s}),\ x_4 = \mathfrak{h}(\mathfrak{s})\}$.

We now taking the isometric surfaces in Theorem 1, consider the following.

Theorem 2. *Let $\mathcal{H}$ and $\mathcal{R}$ be the generalized surfaces family related by Theorem 1. When the family have the same Gauss map, those are hyperplanar, minimal.*

Proof. Let $\{k_1, k_2, k_3, k_4\}$ be the canonical basis in $\mathbb{E}^4$ and denote $k_{ij} = k_i \wedge k_j$, $i, j = 1, 2, 3, 4$, $i < j$. So, the Gauss map of the generalized helicoidal surface (2.1) is

$$\mathcal{G}_{\mathcal{H}} = \frac{1}{W} \left\{ \begin{array}{c} ff'k_{12} \\ +\left(af'\cos v + fg'\sin v\right) k_{13} \\ +fh'\sin v k_{14} \\ +\left(af'\sin v - fg'\cos v\right) k_{23} \\ -fh'\cos v k_{24} \\ -ah'k_{34} \end{array} \right\}, \tag{3.4}$$

and also the Gauss map of the generalized rotational surface (3.2) is as follows

$$\mathcal{G}_{\mathcal{R}} = \frac{1}{W} \left\{ \begin{array}{c} ff'k_{12} \\ +fp\sin\left(v + \int \frac{ag'}{f^2+a^2} du\right) k_{13} \\ +fq\sin\left(v + \int \frac{ag'}{f^2+a^2} du\right) k_{14} \\ -fp\cos\left(v + \int \frac{ag'}{f^2+a^2} du\right) k_{23} \\ -fq\cos\left(v + \int \frac{ag'}{f^2+a^2} du\right) k_{24} \end{array} \right\}, \tag{3.5}$$

where

$$W = \sqrt{a^2\left(f'^2 + h'^2\right) + f^2\left(f'^2 + g'^2 + h'^2\right)}.$$

When $\mathcal{G}_{\mathcal{H}}$ is equal to $\mathcal{G}_{\mathcal{R}}$, identically, Eq. (3.4) and Eq. (3.5) give rise to the following

$$af'\cos v + fg'\sin v = fp\sin\left(v_{\mathcal{R}}\right), \tag{3.6}$$

$$af'\sin v - fg'\cos v = -fp\cos\left(v_{\mathcal{R}}\right), \tag{3.7}$$

$$fh'\sin v = fq\sin\left(v_{\mathcal{R}}\right), \tag{3.8}$$

$$-fh'\cos v = -fq\cos\left(v_{\mathcal{R}}\right), \tag{3.9}$$

$$-ah' = 0, \tag{3.10}$$

where $v_{\mathcal{R}} = v + \int \frac{ag'}{f^2+a^2} du$. Using Eqs. $(3.8) - (3.10)$, we have

$$h' = 0 \text{ and } q = 0.$$

That is, generalized surfaces determined by Eq. (2.1) and Eq. (3.2) are hyperplanar.

Now, we prove surfaces given by Eq. (2.1) and Eq. (3.2) are minimal. For this, since $q=0$, then $p\neq 0$ by using $((3.6)\,.\cos v+(3.7)\,.\sin v)$, it gives

$$af'=fp\sin\left(\int\frac{ag'}{f^2+a^2}du\right).$$

Also, $((3.6)\,.\sin v-(3.7)\,.\cos v)$ reduces to

$$fg'=fp\cos\left(\int\frac{ag'}{f^2+a^2}du\right).$$

Hence, we have

$$arc\cot\left(\frac{fg'}{af'}\right)=\int\frac{ag'}{f^2+a^2}du.$$

Derivativing the last equation w.r.t. u, we obtain the following

$$\left(f^2+a^2\right)\left(f'^2g'+ff'g''-ff''g'\right)+\left(a^2f'^2+f^2g'^2\right)g'=0. \tag{3.11}$$

The mean curvatures of the generalized helicoidal surfaces family given by Eq. (2.1) w.r.t. the following normals

$$\zeta_1=\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ \zeta_2=\frac{1}{\sqrt{(a^2+f^2)\,f'^2+f^2g'^2}}\begin{pmatrix}-fg'\cos v+af'\sin v\\-fg'\sin v-af'\cos v\\ff'\\0\end{pmatrix}$$

are described by, respectively,

$$\begin{aligned}H_1&=0,\\H_2&=\frac{\left(f^2+a^2\right)\left(fg'f''-ff'g''-f'^2g'\right)-g'\left(a^2f'^2+f^2g'^2\right)}{2\left(\left(a^2+f^2\right)f'^2+f^2g'^2\right)^{3/2}},\end{aligned}$$

And also, the mean curvatures of the generalized rotational surfaces family defined by Eq. (3.2) w.r.t. the following normals

$$\zeta_1=\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ \zeta_2=\frac{1}{\sqrt{(a^2+f^2)\,f'^2+f^2g'^2}}\begin{pmatrix}-\sqrt{f^2g'^2+a^2f'^2}\cos\left(v+\int\frac{ag'}{f^2+a^2}du\right)\\-\sqrt{f^2g'^2+a^2f'^2}\sin\left(v+\int\frac{ag'}{f^2+a^2}du\right)\\ff'\\0\end{pmatrix}$$

are determined by, respectively,

$$\begin{aligned}H_1&=0,\\H_2&=\frac{f^2g'\left[\left(f^2+a^2\right)\left(fg'f''-ff'g''-f'^2g'\right)-g'\left(a^2f'^2+f^2g'^2\right)\right]}{2\sqrt{f^2+a^2}\sqrt{f^2g'^2+a^2f'^2}\left(\left(a^2+f^2\right)f'^2+f^2g'^2\right)^{3/2}}.\end{aligned}$$

From Eq. (3.11), the helicoidal-rotational surfaces family have the mean curvatures $H_2=0$. Finally, the generalized helicoidal surface family determined by Eq. (2.1) and the generalized rotational surfaces family described by Eq. (3.2) are minimal. That is, $\overrightarrow{H}=0$.

Theorem 3. *Let the generalized helicoidal surfaces family defined by Eq.* (2.1), *and the generalized rotational surfaces family given by Eq.* (3.2) *having the same Gauss map be the locally isometric surfaces family related by Theorem 1. Therefore, the parametrizations of the family are described by*

$$\mathcal{H}(u,v)=\begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ g(u)+av \\ c \end{pmatrix},$$

$$\mathcal{R}(u,v)=\begin{pmatrix} \sqrt{f^2+a^2}\cos\left(v+\int\frac{ag'}{f^2+a^2}du\right) \\ \sqrt{f^2+a^2}\sin\left(v+\int\frac{ag'}{f^2+a^2}du\right) \\ b\ \arg\cosh\left(\frac{\sqrt{f^2+a^2}}{b}\right) \\ d \end{pmatrix},$$

respectively. Here,

$$g(u)=\sqrt{b^2-a^2}\ln\sqrt{\frac{\sqrt{f^2+a^2}+\sqrt{f^2+a^2-b^2}}{\sqrt{f^2+a^2}-\sqrt{f^2+a^2-b^2}}}-a\arctan\left(\sqrt{\frac{(b^2-a^2)(f^2+a^2)}{a^2(f^2+a^2-b^2)}}\right),$$

and $a,b,c,d\in\mathbb{R}$, $b\geq a$, $f>\sqrt{b^2-a^2}$.

Proof. Generalized surfaces $\mathcal{H}$ and $\mathcal{R}$ are the hyperplanar from Theorem 2. Assume $\mathcal{H}$ covered by the hyperplane $\mathfrak{f}(\mathfrak{s})=c$, and also $\mathcal{R}$ covered by the hyperplane $\mathfrak{f}(\mathfrak{s})=d$. Since $\mathcal{R}$ is minimal, it is just a catenoid. Thus, $\mathfrak{g}(\mathfrak{s})=b\arg\cosh\left(\frac{\mathfrak{s}}{b}\right)$, where $b\neq 0$. Therefore,

$$b\ \arg\cosh\left(\frac{\sqrt{f^2+a^2}}{b}\right)=\int\sqrt{\frac{f^2g'^2+a^2f'^2}{f^2+a^2}}du.$$

Then, we get

$$g'=\frac{\sqrt{b^2-a^2}\sqrt{f^2+a^2}}{f\sqrt{f^2+a^2-b^2}}. \tag{3.12}$$

Finally, after some computations, we obtain

$$g'=\sqrt{b^2-a^2}\ln\sqrt{\frac{w+1}{w-1}}-a\arctan\left(\frac{\sqrt{b^2-a^2}}{a}w\right),$$

where $w=\sqrt{\frac{f^2+a^2}{f^2+a^2-b^2}}>0$.

4. Conclusion

Considering the findings in the previous section, we obtain the following results.

Corollary 1. *When* $g'=0$, *generalized helicoidal surfaces family* $\mathcal{H}$ *describe the helicoid. The mean curvature of the generalized rotational surfaces family* $\mathcal{R}$ *is zero. That is, the generalized rotational surfaces family is transform to the catenoid. By using Eq.* (3.12), *the pitch* a *is equals to* b.

Example 1. *Taking $f(u) = u$, $c = d = 0$, $b = 2$, $a = 1$ in Theorem 3, the other function is described by*

$$g(u) = \sqrt{3}\ln\left(\sqrt{\frac{\sqrt{u^2+1}+\sqrt{u^2-3}}{\sqrt{u^2+1}-\sqrt{u^2-3}}}\right) - \arctan\left(\sqrt{\frac{3\left(u^2+1\right)}{u^2-3}}\right).$$

Then, we have the projection of the isometric helicoidal-rotational surfaces from dimension four to three. See Figure 1 for the graphics of the helicoidal surface, and also see Figure 2 for the rotational surface.

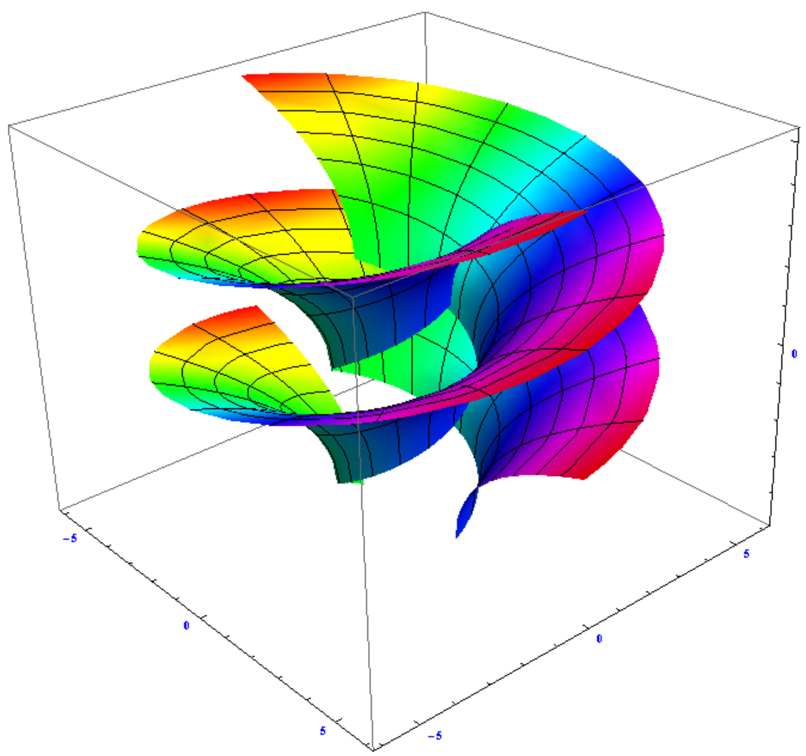

Figure 1: Helicoidal surface

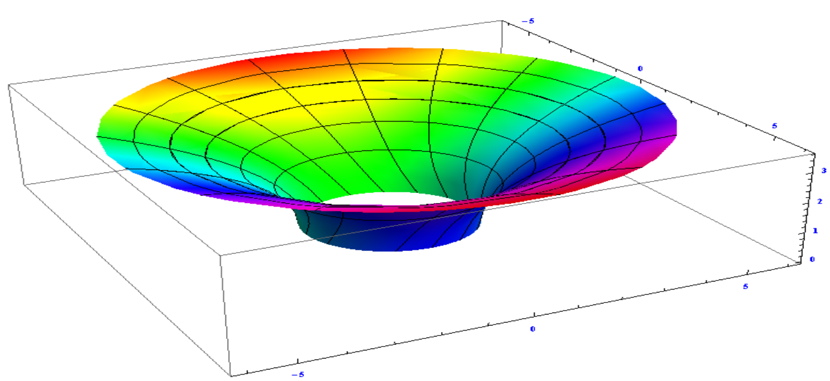

Figure 2: Rotational surface

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Initial coefficient estimates for subclasses of bi-univalent functions

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Abstract. The purpose of the present paper is to introduce a new subclasses of the function class Σ of normalized analytic and bi-univalent functions in the open disk $\mathbb{U}$. We obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions of this subclasses.

AMS Subject Classifications: 30C45.

Keywords: Analytic functions, bi-univalent functions, coefficient bounds, Taylor-Maclaurin series, q-derivative operator.

Contents

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k \qquad (a_k \geq 0, n \in \mathbb{N} = \{1,2,3,...\}) \tag{1.1}$$

which are analytic and univalent in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C}; |z| < 1\}.$$

We shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ by $\mathcal{S}$, for details (see [9]; see also the work [7], [8], [17]). For $0 < q < 1$, we introduce the family of new functions defined as follows:

$$Q^q_\lambda(\beta) = \left\{ f \in \mathcal{A} : Re\left((1-\lambda)\frac{f(z)}{z} + \lambda D_q f(z)\right) > \beta, \beta < 1,\ \lambda \geq 0 \right\} \tag{1.2}$$

where D_q stands for q-derivative of the function $f(z)$ introduced by Jackson [14]. For $q \to 1^-$ it reduces to class of analytic function introduced by Ding et al. [6].

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For function $f(z) \in \mathcal{A}$ given by (1.1) and $0 < q < 1$, the q-derivative of a function $f(z)$ is defined by (also refer [12], [21])

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}; \qquad (z \neq 0, q \neq 0), \tag{1.3}$$

from (1.3), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k \, z^{k-1} \tag{1.4}$$

where

$$[k]_q = \frac{1-q^k}{1-q}. \tag{1.5}$$

It is well known that every function $f \in \mathcal{S}$ has a inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{U})$$

and

$$f^{-1}(f(w)) = w, \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots .$$

A function $f(z) \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We denote by Σ the class of all functions $f(z)$ which are bi-univalent in $\mathbb{U}$ and are given by the Taylor-Maclaurin series expansion (1.1). The familiar Koebe function is not a member of Σ because it maps the unit disk $\mathbb{U}$ univalently onto the entire complex plane minus a slit along the line $-\frac{1}{4}$ to $-\infty$. Hence image domain does not contain in $\mathbb{U}$. A systematic study of the class Σ of bi-univalent function in $\mathbb{U}$, which is introduced in 1967 by Lewin [17]. Ever since then, several authors investigated various subclasses of the class Σ of bi-univalent functions. By using Grunsky inequalities Lewin showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [18], showed that $\max\limits_{f \in \sigma} |a_2| = \frac{4}{3}$. In 1985 Branges [1] proved Bieberbach conjecture which state that, for each $f(z) \in \mathcal{S}$ given by Taylor-Maclaurin expansion (1.1) the following coefficient inequality holds true:

$$|a_n| \leq n; \quad (n \in \mathbb{N} - 1),$$

$\mathbb{N}$ being positive integer.

Brannan and Taha [6](see also [5]) introduce certain subclass of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order $\alpha\,(0 < \alpha \leq 1)$, respectively. According to Brannan and Taha [6] (see also [3]) a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^*_\Sigma(\alpha)$ of strongly bi-univalent functions of order $\alpha\,(0 < \alpha \leq 1)$ if each of the following conditions is satisfied

$$f \in \Sigma \text{ and } \left| arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}; \; (0 < \alpha \leq 1, z \in \mathbb{U})$$

and

$$\left| arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2}; \; (0 < \alpha \leq 1, w \in \mathbb{U}),$$

where g is the extension of f^{-1} in $\mathbb{U}$. Recently, several researchers such as (see [2, 11, 13, 15, 16, 19]) obtained coefficients $|a_2|$ and $|a_3|$ of bi-univalent functions for the various subclasses of the function class Σ. For a further historical amount of functions of class Σ, see the recent pioneering work by Srivastava et al. [22, 23]. The coefficient estimate problem involving the bound of $|a_n|(n \in \mathbb{N} \setminus \{1, 2\})$ for each $f \in \Sigma$ given by (1.1) is still an open problem.

The main aim of the present investigation is to introduce and study two new subclasses of the function class Σ and find estimates on the initial coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ using q-differential operator.

2. Coefficient bounds for the function class $\mathcal{H}_{\Sigma}^{q}(\alpha, \lambda)$

We now introduce the following class of bi-univalent functions.

Definition 2.1. *: A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_{\Sigma}^{q}(\alpha, \lambda)$ if the following conditions satisfied:*

$$f \in \Sigma \text{ and } \left| arg\left((1-\lambda)\frac{f(z)}{z} + \lambda D_q f(z) \right) \right| < \frac{\alpha\pi}{2};\ (0 < \alpha \leq 1, \lambda \geq 1, z \in \mathbb{U}) \tag{2.1}$$

and

$$\left| arg\left((1-\lambda)\frac{g(w)}{w} + \lambda D_q g(w) \right) \right| < \frac{\alpha\pi}{2};\ (0 < \alpha \leq 1, \lambda \geq 1, w \in \mathbb{U}) \tag{2.2}$$

where the function g is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2.3}$$

We note that for $\lambda = 1$ and $q \to 1^-$, the class $\mathcal{H}_{\Sigma}^{q}(\alpha, \lambda)$ reduces to the class $\mathcal{H}_{\Sigma}^{\alpha}$ introduced and studied by Srivastava et al. [24] and for $q \to 1^-$, the class $\mathcal{H}_{\Sigma}^{q}(\alpha, \lambda)$ reduces to the class $\mathcal{B}_{\Sigma}(\alpha, \lambda)$ introduced and studied by Frasin and Aouf [11]. We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for function in the class $\mathcal{H}_{\Sigma}^{q}(\alpha, \lambda)$.

In order to derive our main results, we have to recall here the following lemma.

Lemma 2.2. *[9] If $p \in \mathcal{P}$ then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{U}$ for which $Re\{p(z)\} > 0$*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... \qquad for\ z \in \mathbb{U}.$$

For functions in the class $\mathcal{H}_{\Sigma}^{q}(\alpha, \lambda)$ the following result is obtained.

Theorem 2.3. *Let $f(z)$ be given by (1.1) be in the function class $\mathcal{H}_{\Sigma}^{q}(\alpha, \lambda)$, $0 < \alpha \leq 1$; $0 < q < 1$ and $\lambda \geq 1$.Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2(1-\lambda+[3]_q\lambda)\alpha + (1-\alpha)(1-\lambda+[2]_q\lambda)^2}} \tag{2.4}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(1-\lambda+[2]_q\lambda)^2} + \frac{2\alpha}{(1-\lambda+[3]_q\lambda)}. \tag{2.5}$$

Proof: It follows from (2.1) and (2.2) that

$$(1-\lambda)\frac{f(z)}{z}+\lambda D_q f(z)=[p(z)]^\alpha \tag{2.6}$$

and

$$(1-\lambda)\frac{g(w)}{w}+\lambda D_q g(w)=[q(w)]^\alpha \qquad (z,w\in\mathbb{U}), \tag{2.7}$$

respectively, where

$$p(z)=1+p_1z+p_2z^2+...$$
$$q(w)=1+q_1w+q_2w^2+...$$

in $\mathcal{P}$. Now, upon equating the coefficients of z and z^2 in (2.6) and (2.7), we get

$$(1-\lambda+[2]_q\lambda)\,a_2=\alpha p_1, \tag{2.8}$$

$$(1-\lambda+[3]_q\lambda)\,a_3=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2, \tag{2.9}$$

$$-(1-\lambda+[2]_q\lambda)\,a_2=\alpha q_1 \tag{2.10}$$

and

$$(1-\lambda+[3]_q\lambda)\,(2a_2^2-a_3)=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.11}$$

From (2.8) and (2.10), we obtain

$$p_1=-q_1 \tag{2.12}$$

and

$$2\,(1-\lambda+[2]_q\lambda)^2\,a_2^2=\alpha^2\left(p_1^2+q_1^2\right). \tag{2.13}$$

Also, from (2.9), (2.11) and (2.13), we find that

$$2\,(1-\lambda+[3]_q\lambda)\,a_2^2=\alpha\,(p_2+q_2)+\frac{(\alpha-1)\,(1-\lambda+[2]_q\lambda)^2\,a_2^2}{\alpha}. \tag{2.14}$$

Therefore, we obtain

$$a_2^2=\frac{\alpha^2}{2(1-\lambda+[3]_q\lambda)\alpha+(1-\alpha)(1-\lambda+[2]_q\lambda)^2}\,(p_2+q_2)\,. \tag{2.15}$$

Applying lemma 2.2 for the coefficients p_2 and q_2, yields

$$|a_2|\leq\frac{2\alpha}{\sqrt{2(1-\lambda+[3]_q\lambda)\alpha+(1-\alpha)(1-\lambda+[2]_q\lambda)^2}}, \tag{2.16}$$

which gives desired estimate on $|a_2|$ as asserted in (2.4).

Next, in order to find the bound on $|a_3|$, we subtract (2.11) from (2.9), We thus get

$$2\,(1-\lambda+[3]_q\lambda)\,a_3=\alpha\,(p_2-q_2)+\frac{\alpha^2\left(p_1^2+q_1^2\right)(1-\lambda+[3]_q\lambda)}{(1-\lambda+[2]_q\lambda)^2}. \tag{2.17}$$

Applying lemma 2.2 for the coefficients p_1, q_1, p_2 and q_2 in above equality, we get

$$|a_3|\leq\frac{4\alpha^2}{(1-\lambda+[2]_q\lambda)^2}+\frac{2\alpha}{(1-\lambda+[3]_q\lambda)}. \tag{2.18}$$

This completes the proof.

If we choose $\lambda=1$ and $q\to1^-$ in Theorem 2.3, we have the following result.

Corollary 2.4. *([24]). Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_{\Sigma}^{\alpha}$, $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{2+\alpha}} \tag{2.19}$$

and

$$|a_3| \leq \frac{\alpha(3\alpha+2)}{3}. \tag{2.20}$$

If we take $q \to 1^-$ in Theorem 2.3, we have the following result.

Corollary 2.5. *([11]). Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}(\alpha, \lambda)$, $(0 < \alpha \leq 1)$ and $(\lambda \geq 1)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda+1)^2 + \alpha(1+2\lambda-\lambda^2)}} \tag{2.21}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{2\lambda+1}. \tag{2.22}$$

3. Coefficient bounds for the function class $\mathcal{H}_{\Sigma}^{q}(\beta, \lambda)$

We now introduce the following class of bi-univalent functions.

Definition 3.1. *: A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_{\Sigma}^{q}(\beta, \lambda)$ if the following conditions satisfied:*

$$f \in \Sigma \text{ and } Re\left((1-\lambda)\frac{f(z)}{z} + \lambda D_q f(z)\right) > \beta;\ (0 \leq \beta < 1, \lambda \geq 1, z \in \mathbb{U}) \tag{3.1}$$

and

$$Re\left((1-\lambda)\frac{g(w)}{w} + \lambda D_q g(w)\right) > \beta;\ (0 \leq \beta < 1, \lambda \geq 1, w \in \mathbb{U}). \tag{3.2}$$

For functions in the class $\mathcal{H}_{\Sigma}^{q}(\beta, \lambda)$ the following result is obtained.

Theorem 3.2. *Let $f(z)$ be given by (1.1) be in the function class $\mathcal{H}_{\Sigma}^{q}(\beta, \lambda)$, $0 \leq \beta < 1$; $0 < q < 1$ and $\lambda \geq 1$.Then*

$$|a_2| \leq \min\left\{\frac{2(1-\beta)}{(1-\lambda+[2]_q\lambda)}, \sqrt{\frac{2(1-\beta)}{(1-\lambda+[3]_q\lambda)}}\right\} \tag{3.3}$$

and

$$|a_3| \leq \min\left\{\frac{2(1-\beta)}{(1-\lambda+[3]_q\lambda)}, \frac{4(1-\beta)^2}{(1-\lambda+[2]_q\lambda)^2} + \frac{2(1-\beta)}{(1-\lambda+[3]_q\lambda)}\right\}. \tag{3.4}$$

Proof: It follows from (3.1) and (3.2) that

$$(1-\lambda)\frac{f(z)}{z}+\lambda D_q f(z)=\beta+(1-\beta)p(z) \tag{3.5}$$

and

$$(1-\lambda)\frac{g(w)}{w}+\lambda D_q g(w)=\beta+(1-\beta)q(w) \qquad (z,w\in\mathbb{U}), \tag{3.6}$$

respectively, where

$$p(z)=1+p_1z+p_2z^2+...$$
$$q(w)=1+q_1w+q_2w^2+...$$

in $\mathcal{P}$. Now, upon equating the coefficients of (3.5) and (3.6), we obtain

$$\left(1-\lambda+[2]_q\lambda\right)a_2=(1-\beta)p_1, \tag{3.7}$$

$$\left(1-\lambda+[3]_q\lambda\right)a_3=(1-\beta)p_2, \tag{3.8}$$

$$-\left(1-\lambda+[2]_q\lambda\right)a_2=(1-\beta)q_1 \tag{3.9}$$

and

$$\left(1-\lambda+[3]_q\lambda\right)\left(2a_2^2-a_3\right)=(1-\beta)q_2. \tag{3.10}$$

From (3.7) and (3.9), we obtain

$$p_1=-q_1 \tag{3.11}$$

and

$$2\left(1-\lambda+[2]_q\lambda\right)^2a_2^2=(1-\beta)^2\left(p_1^2+q_1^2\right). \tag{3.12}$$

Also, from (3.8) and (3.10), we have

$$2\left(1-\lambda+[3]_q\lambda\right)a_2^2=(1-\beta)\left(p_2+q_2\right). \tag{3.13}$$

Applying lemma 2.2 for (3.12) and (3.13), we get

$$|a_2|\le\min\left\{\frac{2(1-\beta)}{(1-\lambda+[2]_q\lambda)},\sqrt{\frac{2(1-\beta)}{(1-\lambda+[3]_q\lambda)}}\right\}, \tag{3.14}$$

we get desired estimate on $|a_2|$ as asserted in (3.3).

Next, in order to find the bound on $|a_3|$, we subtract (3.10) and (3.8), we get

$$2(1-\lambda+[3]_q\lambda)\,a_3=(1-\beta)\left(p_2-q_2\right)+2(1-\lambda+[3]_q\lambda)\,a_2^2, \tag{3.15}$$

which, upon substitution of the value of a_2^2 from (3.12), yields

$$|a_3|=\frac{(1-\beta)^2}{2(1-\lambda+[2]_q\lambda)^2}(p_1^2+q_1^2)+\frac{(1-\beta)}{(1-\lambda+[3]_q\lambda)}(p_2-q_2). \tag{3.16}$$

On the other hand, by using (3.13) into (3.15), it follows that

$$|a_3|=\frac{(1-\beta)}{2(1-\lambda+[3]_q\lambda)^2}\,p_2. \tag{3.17}$$

Applying lemma 2.2 for (3.16) and (3.17), yields

$$|a_3|\le\min\left\{\frac{2(1-\beta)}{(1-\lambda+[3]_q\lambda)},\frac{4(1-\beta)^2}{(1-\lambda+[2]_q\lambda)^2}+\frac{2(1-\beta)}{(1-\lambda+[3]_q\lambda)}\right\}. \tag{3.18}$$

This completes the proof.

The next Corollary can be easily obtained from Theorem 3.2.

Corollary 3.3. . *Let $f(z)$ given by (1.1) be in the class $\mathcal{H}^{\alpha}_{\Sigma}$, $0 \leq \beta < 1$. Then*

$$|a_2| = \begin{cases} \sqrt{\frac{2(1-\beta)}{3}}, & \text{for } 0 \leq \beta \leq 1/3 \\ 1-\beta, & \text{for } 1/3 \leq \beta < 1 \end{cases} \tag{3.19}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{3}. \tag{3.20}$$

Remark 3.4. *Corollary (3.3) provides an improvement for the estimates obtained by Srivastava et al. ([24]).*

Corollary 3.5. *([24]). Let $f(z)$ given by (1.1) be in the class $\mathcal{H}^{\beta}_{\Sigma}$, $(0 \leq \beta < 1)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3}} \tag{3.21}$$

and

$$|a_3| \leq \frac{(1-\beta)(5-3\beta)}{3}. \tag{3.22}$$

If we choose $q \to 1^-$ in Theorem 3.2, we have the following result.

Corollary 3.6. *([11]). Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}(\beta, \lambda)$, $(0 \leq \beta < 1)$ and $(\lambda \geq 1)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{2\lambda+1}} \tag{3.23}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(\lambda+1)^2} + \frac{2(1-\beta)}{2\lambda+1}. \tag{3.24}$$

Remark 3.7. . *For $\lambda = 1$ the results obtained in this paper are coincides with the results discussed in ([2]).*

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Double domination number of the shadow (2,3)-distance graphs

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Abstract. Let $G = (V, E)$ be a graph with the vertex set $V(G)$ and S be a subset of $V(G)$. If every vertex of V is dominated by S at least twice, then the set S is called a double domination set of the graph. The number of elements of the double domination set with the smallest cardinality is called double domination number and denoted by $\gamma_{\times 2}(G)$ notation. In this paper, we discussed the double domination parameter on some types of shadow distance graphs such as cycle, path, star, complete bipartite and wheel graphs.

AMS Subject Classifications: 05C12, 05C69, 05C82, 68M10, 68R10.

Keywords: Domination, double domination, shadow distance graph.

Contents

1. Introduction and Background

Many real-life problems can be modeled mathematically by using differential equations, integral equations, algebraic relations, etc. However, the graphical representation of such problems, showing how the various components are related, appeals to anyone working on it. Although the beginning of these graphic representations dates back many years, its emergence as a concrete mathematical structure was shaped by the finding of a new branch of mathematics, graph theory. As one of the most important characterizations, graph domination, has been associated with various application areas such as analyzing chemical structures, electrical and communication networks, and database management. Thus, graph domination has attracted interest from many mathematicians due to its application potential to apply many problems such as design and analysis of communication networks as well as defense supervision [4, 14, 19].

Now, we provide some basic information and definitions that will form the basis of this study. In general, we follow [8, 15]. Let $G = (V(G), E(G))$ be a graph. The open neighborhood of a vertex $v \in V(G)$ is $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and its closed neighborhood $N[v] = N(v) \cup \{v\}$. The degree of v, denoted by $\deg(v)$, is the size of its open neighborhood. One degree vertex is called as a pendant vertex or a leaf, and its neighbor is called a support vertex. An edge incident to a leaf (or a pendant vertex) is called a pendant edge.

Let D be a subgraph of the vertex set of a graph G. If D is a dominating set in a graph G then every vertex in $V(G)\backslash D$ is adjacent to at least one vertex in D, and the number of elements of the minimum cardinality

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domination set is called the domination number of G, denoted by $\gamma(G)$ [15]. Many variants of domination parameter are studied in the literature [1–3, 6, 7, 11, 13, 16, 17].

In this paper, we focused on the double domination parameter. Double dominating set (abbreviated DDS) is introduced in [12]. A set $S \subseteq V$ is a double dominating set for G if each vertex in V is dominated by at least two vertices in S. The smallest cardinality of a double dominating set is called the double domination number $\gamma_{\times 2}(G)$. If S is a DDS of G of size as double domination number, then it is called as $\gamma_{\times 2}(G)$-set [12, 13]. Frankly, double domination is defined only for graphs without isolated vertices.

Let D be the set of all distances between distinct pairs of vertices in G and let $D_s \subseteq D$ is called the distance set. The distance graph of G denoted by $D\left(G, D_s\right)$ is the graph having the same vertex set with G and if $d(u, v) \in D_s$ then two vertices u and v are adjacent in $D\left(G, D_s\right)$. The shadow distance graph of G, denoted by $D_{\text{sd}}\ \left(G, D_s\right)$ is formed from G to satisfy the following properties [12, 18, 20] :

- *P1* : G has two copies say G itself and G'
- *P2* : if $u \in V(G)$ is first copy then the corresponding vertex as $u' \in V\left(G'\right)$ is second copy
- *P3* : the vertex set of shadow distance graph, $D_{sd}\left(G, D_s\right)$, is $V(G) \cup V\left(G'\right)$
- *P4* : the edge set of shadow distance graph , $D_{sd}\left(G, D_s\right)$, is $E(G) \cup E\left(G'\right) \cup E_{ds}$ where E_{ds} is the set of all edges between two distinct vertices $u \in V(G)$ and $v' \in V\left(G'\right)$ that satisfy the condition $d(u, v) \in D_s$ in G.

2. Main Results

We recall the following results related to the double domination number of a graph.

Theorem 2.1. *[10] Let G be a graph with no isolated vertices. Then* $2 \leq \gamma_{\times 2}(G) \leq n$.

Theorem 2.2. *[10] If G is any graph without isolated vertices, then* $\gamma(G) \leq \gamma_{\times 2}(G) - 1$.

Theorem 2.3. *[5, 10, 12]*

a) If $G \cong P_n$ *is a path graph for* $n \geq 2$, *then* $\gamma_{\times 2}\left(P_n\right) = \left\lceil \frac{2n+2}{3} \right\rceil$

b) If $G \cong C_n$ *is a cycle graph for* $n \geq 3$, *then* $\gamma_{\times 2}\left(C_n\right) = \left\lceil \frac{2n}{3} \right\rceil$

c) If $G \cong K_{1,m}$ *is a star graph for* $m > 1$, $\gamma_{\times 2}\left(K_{1,m}\right) = m + 1$.

Observation 2.4. *[9] Each* $DD - set$ *generated for any graph must contain all leaves and support vertices of the graph.*

We begin our results with the some distance shadow graphs.

Theorem 2.5. *If* $G \cong P_n$ *for* $n \geq 8$, *then*

$$\gamma_{\times 2}\left(D_{sd}\left(G, \{2\}\right)\right) = \begin{cases} \left\lceil \dfrac{4(n+1)}{5} \right\rceil & , n \equiv 3,\ 4 \ \textit{(mod 5)} \\ \left\lceil \dfrac{4(n+1)}{5} \right\rceil + 1 & , n \equiv 0\,, 2 \ \textit{(mod 5)} \\ \left\lceil \dfrac{4(n+1)}{5} \right\rceil + 2 & , n \equiv 1 \ \textit{(mod 5)} \end{cases}$$

Proof. Consider two copies of G, one G itself and the other denoted by G'. Let $V_1 = \{1, 2, \ldots, n\}$ be the vertices of G and let $V_2 = \{n+1, n+2, \ldots, 2n\}$ be the vertices of G'. We first establish upper bounds on $\gamma_{\times 2}(D_{sd}(G, \{2\}))$. Let

$D_1 = \bigcup_{i=0}^{\lfloor \frac{n}{5} \rfloor - 1} \{(5i+2), (5i+3)\}$, $D_2 = \bigcup_{i=0}^{\lfloor \frac{n}{5} \rfloor - 1} \{(n+5i+2), (n+5i+3)\}$ and $D = D_1 \cup D_2$.

If $n \equiv 0 \pmod 5$, let $S = D \cup \{(n-1), (2n-1)\}$. If $n \equiv i \pmod 5$ where $i \in \{1, 2, 3, 4\}$, let $S = D \cup \{(n-1), (n-2), (2n-1), (2n-2)\}$. In all cases, the set S is a $DD-set$ of $D_{sd}(G, \{2\})$. Further if $n \equiv 0, 2 \pmod 5$, then $|S| = \left\lceil \frac{4(n+1)}{5} \right\rceil + 1$, while if $n \equiv 1 \pmod 5$, then $|S| = \left\lceil \frac{4(n+1)}{5} \right\rceil + 2$. Finally, if $n \equiv 3, 4 \pmod 5$, then $|S| = \left\lceil \frac{4(n+1)}{5} \right\rceil$. Hence, $\gamma_{\times 2}(D_{sd}(G, \{2\})) \leq \left\lceil \frac{4(n+1)}{5} \right\rceil$ if $n \equiv 3, 4 \pmod 5$, $\gamma_{\times 2}(D_{sd}(G, \{2\})) \leq \left\lceil \frac{4(n+1)}{5} \right\rceil + 1$ if $n \equiv 0, 2 \pmod 5$ and $\gamma_{\times 2}(D_{sd}(G, \{2\})) \leq \left\lceil \frac{4(n+1)}{5} \right\rceil + 2$ if $n \equiv 1 \pmod 5$.

Now let's prove the lower bounds on $\gamma_{\times 2}(D_{sd}(G, \{2\}))$. Let's assume that the set $X = \{u_1, u_2, \ldots, u_i, \ldots, u_m, u_{m+1}, \ldots, u_j, \ldots, u_x\}$ is a $\gamma_{\times 2} - set$. Here; u_i and u_j are any two positive integers such that $u_1 < u_2 < \cdots < u_i < \cdots < u_m < u_{m+1} < \cdots < u_j < \cdots < u_x$, where $1 \leq u_i \leq n$ $i \in \{1, 2, \ldots, m\}$ and $n+1 \leq u_j \leq 2n$ $j \in \{n+1, \ldots, x\}$. We have $f_t = u_{t+2} - u_t$ for $t \in \{1, 2, \ldots, x-2\}$ and $t \neq m-1$. To show the inverse of the inequality, we need to show that $f_t \leq 5$.

Suppose $f_t \geq 6$ for at least one value of x. Without loss of generality, assume that $f_t = 6$. In accordance with this claim; the following sets are obtained.

$$D_1{}' = \{2, 3, 8, 9\} \cup \left\{ \bigcup_{i=0}^{\lceil \frac{n-12}{5} \rceil - 1} \{(5i+13), (5i+14)\} \right\} \text{ and}$$

$$D_2{}' = \{(n+2), (n+3), (n+4), (n+8), (n+9), (n+10)\} \cup \left\{ \bigcup_{i=0}^{\lceil \frac{n-12}{5} \rceil - 1} \{(n+5i+13), (n+5i+14)\} \right\}$$

In this case, $X = D_1{}' \cup D_2{}'$ and $|X| = 10 + 4\left\lceil \frac{n-12}{5} \right\rceil$. If $n \equiv 3 \pmod 5$, then $|X| = 10 + 4\left(\frac{n-8}{5}\right) = \frac{4n+18}{5}$. However, this value contradicts the upper value we found earlier as $|S| = \frac{4n+8}{5}$ for $n \equiv 3 \pmod 5$. A similar situation can easily be seen that the values obtained for $n \equiv 0, 1, 2, 4 \pmod 5$ according to the X set contradict the upper limits we obtained earlier. For all values of n according to $mod\, 5$, it is easily seen that $u_1 + u_2 + u_{m+1} + u_{m+2} = 2n + 10$ since $u_1 = 2$, $u_2 = 3$, $u_{m+1} = n+2$ and $u_{m+2} = n+3$.

If $n \equiv 0 \pmod 5$, then $\sum_{t_1=1}^{m-3} f_{t_1} + \sum_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} \leq 5(x-6) + 4$. Thus, we get $\sum_{t_1=1}^{m-3} f_{t_1} + \sum_{t_2=m+1}^{x-3} f_{t_2} = (u_{m-1} + u_{m-2} + u_{x-1} + u_{x-2}) - (2n+10) + f_{m-2} + f_{x-2}$. For $n \equiv 0 \pmod 5$, $u_{m-1} = n-2$, $u_{m-2} = n-3$, $u_{x-1} = 2n-2$ and $u_{x-2} = 2n-3$, $f_{m-2} = f_{x-2} = 2$. So, we have $6n - 10 - 2n - 10 + 4 \leq 5x - 30 + 4$ and $x \geq \left\lceil \frac{4n+10}{5} \right\rceil$. In this case, $|X| = x \geq \left\lceil \frac{4n+10}{5} \right\rceil = \left\lceil \frac{4(n+1)}{5} \right\rceil + 1$. This implies that $\gamma_{\times 2}(D_{sd}(G, \{2\})) \geq \left\lceil \frac{4(n+1)}{5} \right\rceil + 1$.

If $n \equiv 1 \pmod 5$, then $\sum_{t_1=1}^{m-4} f_{t_1} + \sum_{t_2=m+1}^{x-4} f_{t_2} + f_{m-3} + f_{m-2} + f_{x-3} + f_{x-2} \leq 5(x-8) + 8$. Thus, we get $\sum_{t_1=1}^{m-4} f_{t_1} + \sum_{t_2=m+1}^{x-4} f_{x_2} = (u_{m-2} + u_{m-3} + u_{x-2} + u_{x-3}) - (2n+10) + f_{m-3} + f_{m-2} + f_{x-3} + f_{x-2}$. For $n \equiv 1 \pmod 5$, $u_{m-2} = n-3$, $u_{m-3} = n-4$, $u_{x-2} = 2n-3$, $u_{x-3} = 2n-4$ and $f_{m-3} = f_{m-2} = f_{x-3} = f_{x-2} = 2$. So, we have $6n - 14 - 2n - 10 \leq 5x - 40$ and $x \geq \left\lceil \frac{4n+16}{5} \right\rceil$. In this case, $|X| = x \geq \left\lceil \frac{4n+16}{5} \right\rceil = \left\lceil \frac{4(n+1)}{5} \right\rceil + 2$. This implies that $\gamma_{\times 2}(D_{sd}(G, \{2\})) \geq \left\lceil \frac{4(n+1)}{5} \right\rceil + 2$.

If $n \equiv 2 \pmod 5$, then $\sum_{t_1=1}^{m-4} f_{t_1} + \sum_{t_2=m+1}^{x-4} f_{t_2} + f_{m-3} + f_{m-2} + f_{x-3} + f_{x-2} \le 5(x-8) + 16$. Thus, we get $\sum_{t_1=1}^{m-4} f_{t_1} + \sum_{t_2=m+1}^{x-4} f_{t_2} = (u_{m-2} + u_{m-3} + u_{x-2} + u_{x-3}) - (2n+10) + f_{m-3} + f_{m-2} + f_{x-3} + f_{x-2}$. For $n \equiv 2 \pmod 5$, $u_{m-2} = n-4,\ u_{m-3} = n-5,\ u_{x-2} = 2n-4$, $u_{x-3} = 2n-5$ and $f_{m-3} = f_{m-2} = f_{x-3} = f_{x-2} = 4$. So, we have $6n - 18 - 2n - 10 \le 5x - 40$ and $x \ge \left\lceil \frac{4n+12}{5} \right\rceil = \left\lceil \frac{4(n+1)}{5} \right\rceil + 1$. In this case, $|X| = x \ge \left\lceil \frac{4n+12}{5} \right\rceil = \left\lceil \frac{4(n+1)}{5} \right\rceil + 1$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{2\})) \ge \left\lceil \frac{4(n+1)}{5} \right\rceil + 1$.

If $n \equiv 3 \pmod 5$, then $\sum_{t_1=1}^{m-2} f_{t_1} + \sum_{t_2=m+1}^{x-2} f_{t_2} \le 5(x-4)$. Thus, we get $\sum_{t_1=1}^{m-2} f_{t_1} + \sum_{t_2=m+1}^{x-2} f_{t_2} = (u_m + u_{m-1} + u_x + u_{x-1}) - (2n+10)$. For $n \equiv 3 \pmod 5$, $u_m = n,\ u_{m-1} = n-1$, and $u_x = 2n,\ u_{x-1} = 2n-1$. So, we have $6n - 2 - 2n - 10 \le 5x - 20$ and $x \ge \left\lceil \frac{4n+8}{5} \right\rceil$. In this case, $|X| = x \ge \left\lceil \frac{4n+8}{5} \right\rceil = \left\lceil \frac{4(n+1)}{5} \right\rceil$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{2\})) \ge \left\lceil \frac{4(n+1)}{5} \right\rceil$.

If $n \equiv 4 \pmod 5$, then $\sum_{t_1=1}^{m-2} f_{x_1} + \sum_{t_2=m+1}^{x-2} f_{t_2} \le 5(x-4)$. Thus, we get $\sum_{t_1=1}^{m-2} f_{t_1} + \sum_{t_2=m+1}^{x-2} f_{t_2} = (u_m + u_{m-1} + u_x + u_{x-1}) - (2n+10)$. For $n \equiv 4 \pmod 5$, $u_m = n-1,\ u_{m-1} = n-2$, $u_x = 2n-1$ and $u_{x-1} = 2n-2$ So, we have $6n - 6 - 2n - 10 \le 5x - 20$ and $x \ge \left\lceil \frac{4n+4}{5} \right\rceil$. In this case, $|X| = x \ge \left\lceil \frac{4n+4}{5} \right\rceil = \left\lceil \frac{4(n+1)}{5} \right\rceil$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{2\})) \ge \left\lceil \frac{4(n+1)}{5} \right\rceil$. Thus, the desired equality is obtained as a result of the lower and upper bounds on $\gamma_{\times 2}(D_{sd}(G,\{2\}))$.

This completes the proof.

■

Theorem 2.6. *If $G \cong C_n$ for $n \ge 11$, then*

$$\gamma_{\times 2}(D_{sd}(G,\{2\})) = \begin{cases} \left\lceil \dfrac{4n}{5} \right\rceil & , n \equiv 0,\ 4 \ (\textit{mod } 5) \\ \left\lceil \dfrac{4n}{5} \right\rceil + 1 & , n \equiv 1, 3 \ (\textit{mod } 5) \\ \left\lceil \dfrac{4n}{5} \right\rceil + 2 & , n \equiv 2 \ (\textit{mod } 5) \end{cases}$$

Proof. Let the vertices of the $D_{sd}(G,\{2\})$ graph be divided into two sets of $V(D_{sd}(G,\{2\})) = V_1 \cup V_2$ where $V_1 = \{1, 2, \ldots, n\}$ and $V_2 = \{n+1, n+2, \ldots, 2n\}$.We first establish upper bounds on $\gamma_{\times 2}(D_{sd}(G,\{2\}))$. Let

$$D_1 = \{1, n\} \cup \left\{ \bigcup_{i=0}^{\lceil \frac{n-6}{5} \rceil - 1} \{(5i+5), (5i+6)\} \right\},$$

$$D_2 = \{(n+1), (2n)\} \cup \left\{ \bigcup_{i=0}^{\lceil \frac{n-6}{5} \rceil - 1} \{(n+5i+5), (n+5i+6)\} \right\} \text{ and } D = D_1 \cup D_2.$$

If $n \equiv 1 \pmod 5$, let $S = D \cup \{(n-1), (2n-1)\}$, in other cases $S = D$. In all cases, the set S is a $DD-set$ of $D_{sd}(G,\{2\})$. Further if $n \equiv 0, 4 \pmod 5$, then $|S| = \left\lceil \frac{4n}{5} \right\rceil$, while if $n \equiv 1, 3 \pmod 5$, then $|S| = \left\lceil \frac{4n}{5} \right\rceil + 1$.

Finally, if $n \equiv 2 \pmod 5$, then $|S| = \left\lceil \frac{4n}{5} \right\rceil + 2$. Hence, $\gamma_{\times 2}(D_{sd}(G,\{2\})) \leq \left\lceil \frac{4n}{5} \right\rceil$ if $n \equiv 0, 4 \pmod 5$, $\gamma_{\times 2}(D_{sd}(G,\{2\})) \leq \left\lceil \frac{4n}{5} \right\rceil + 1$ if $n \equiv 1, 3 \pmod 5$ and $\gamma_{\times 2}(D_{sd}(G,\{2\})) \leq \left\lceil \frac{4n}{5} \right\rceil + 2$ if $n \equiv 2 \pmod 5$.

Now let's prove the lower bounds on $\gamma_{\times 2}(D_{sd}(G,\{2\}))$. Let's assume that the set $X = \{u_1, u_2, \ldots, u_i, \ldots, u_m, u_{m+1}, \ldots, u_j, \ldots, u_x\}$ is a $\gamma_{\times 2} - set$. Here; u_i and u_j are any two positive integers such that $u_1 < u_2 < \cdots < u_i < \cdots < u_m < u_{m+1} < \cdots < u_j < \cdots < u_x$, where $1 \leq u_i \leq n$ $i \in \{1, 2, \ldots, m\}$ and $n+1 \leq u_j \leq 2n$ $j \in \{n+1, \ldots, x\}$. We have $f_t = u_{t+2} - u_t$ for $t \in \{1, 2, \ldots, x-2\}$and $t \neq m-1$. To show the inverse of the inequality, we need to show that $f_t \leq 5$.

Suppose $f_t \geq 6$ for at least one value of t. Without loss of generality, assume that$f_t = 6$. In accordance with this claim; the following sets are obtained.

$$D_1{}' = \{1\ ,\ n\} \cup \left\{ \bigcup_{i=0}^{\left\lceil \frac{n-7}{5} \right\rceil - 1} \{(5i+6), (5i+7)\} \right\} \text{ and}$$

$$D_2{}' = \{(n+1), (n+2), (n+5), (2n)\} \cup \left\{ \bigcup_{i=0}^{\left\lceil \frac{n-7}{5} \right\rceil - 1} \{(n+5i+6), (n+5i+7)\} \right\}.$$

In this case, $X = D_1{}' \cup D_2{}'$ and $|X| = 6 + 4\left\lceil \frac{n-7}{5} \right\rceil$. If $n \equiv 0 \pmod 5$, then $|X| = 6 + 4\left(\frac{n-5}{5}\right) = \frac{4n+10}{5}$. However, this value contradicts the upper value we found earlier as $|S| = \frac{4n}{5}$ for $n \equiv 0 \pmod 5$. A similar situation can easily be seen that the values obtained for $n \equiv i \pmod 5$, $i \in \{1, 2, 3, 4\}$ according to the X set contradict the upper limits we obtained earlier. This contradicts our claim. Thus, it must be$f_x \leq 5$. In this case, we have $\sum\limits_{t_1=1}^{m-2} f_{t_1} + \sum\limits_{t_2=m+1}^{x-2} f_{t_2} \leq 5(x-4)$. Furthermore, for all values of n according to $mod\ 5$, it is easily seen that $u_1 + u_2 + u_{m+1} + u_{m+2} = 2n + 13$ since $u_1 = 1$, $u_2 = 6$, $u_{m+1} = n+1$ and $u_{m+2} = n+5$.

If $n \equiv 0 \pmod 5$, then $\sum\limits_{t_1=1}^{m-3} f_{t_1} + \sum\limits_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} \leq 5(x-6) + 8$. Thus, we get $\sum\limits_{t_1=1}^{m-3} f_{t_1} + \sum\limits_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} = (u_{m-1} + u_{m-2} + u_{x-1} + u_{x-2}) - (2n+13) + f_{m-2} + f_{x-2}$. For $n \equiv 0 \pmod 5$, $u_{m-1} = n-4$, $u_{m-2} = n-5$, $u_{x-1} = 2n-4$, and $u_{x-2} = 2n-5$, $f_{m-2} = f_{x-2} = 4$. So, we have $6n - 18 - 2n - 13 \leq 5(x-6)$ and $x \geq \left\lceil \frac{4n-1}{5} \right\rceil$. In this case, $|X| = x \geq \left\lceil \frac{4n-1}{5} \right\rceil = \left\lceil \frac{4n}{5} \right\rceil$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{2\})) \geq \left\lceil \frac{4n}{5} \right\rceil$.

If $n \equiv 1 \pmod 5$, then $\sum\limits_{t_1=1}^{m-2} f_{t_1} + \sum\limits_{t_2=m+1}^{x-2} f_{t_2} \leq 5(x-4)$. Thus, we get $\sum\limits_{t_1=1}^{m-2} f_{t_1} + \sum\limits_{t_2=m+1}^{x-2} f_{t_2} = (u_m + u_{m-1} + u_x + u_{x-1}) - (2n+13)$. For $n \equiv 1 \pmod 5$, $u_m = n$, $u_{m-1} = n-1$, $u_x = 2n$ and $u_{x-1} = 2n-1$. So, we have $6n - 2 - 2n - 13 \leq 5x - 20$ and $x \geq \left\lceil \frac{4n+5}{5} \right\rceil = \left\lceil \frac{4n}{5} \right\rceil + 1$. In this case, $|X| = x \geq \left\lceil \frac{4n}{5} \right\rceil + 1$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{2\})) \geq \left\lceil \frac{4n}{5} \right\rceil + 1$.

If $n \equiv 2 \pmod 5$, then $\sum\limits_{t_1=1}^{m-3} f_{t_1} + \sum\limits_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} \leq 5(x-6) + 4$. Thus, we get $\sum\limits_{t_1=1}^{m-3} f_{t_1} + \sum\limits_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} = (u_{m-1} + u_{m-2} + u_{x-1} + u_{x-2}) + f_{m-2} + f_{x-2} - (2n+13)$. For $n \equiv 2 \pmod 5$, $u_{m-1} = n-1$, $u_{m-2} = n-2$, $u_{x-1} = 2n-1$, $u_{x-2} = 2n-2$. So, we have $6n - 6 - 2n - 13 \leq 5(x-6)$ and $x \geq \left\lceil \frac{4n+12}{5} \right\rceil = \left\lceil \frac{4n}{5} \right\rceil + 2$. In this case, $|X| = x \geq \left\lceil \frac{4n}{5} \right\rceil + 2$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{2\})) \geq \left\lceil \frac{4n}{5} \right\rceil + 2$.

If $n \equiv 3 \pmod 5$, then $\sum\limits_{t_1=1}^{m-3} f_{t_1} + \sum\limits_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} \leq 5(x-6) + 6$. Thus, we get $\sum\limits_{t_1=1}^{m-3} f_{t_1} + \sum\limits_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} = (u_{m-1} + u_{m-2} + u_{x-1} + u_{x-2}) - (2n+13) + f_{m-2} + f_{x-2}$. For $n \equiv 3 \pmod 5$, $u_{m-1} = n-2$, $u_{m-2} = n-3$, $u_{x-1} = 2n-2$, $u_{x-2} = 2n-3$ and $f_{m-2} = f_{x-2} = 3$. So, we have $6n - 10 - 2n - 13 \leq 5(x-6)$ and $x \geq \left\lceil \frac{4n+7}{5} \right\rceil = \left\lceil \frac{4n}{5} \right\rceil + 1$. In this case, $|X| = x \geq \left\lceil \frac{4n}{5} \right\rceil + 1$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{2\})) \geq \left\lceil \frac{4n}{5} \right\rceil + 1$.

If $n \equiv 4 \pmod 5$, then $\sum_{t_1=1}^{m-3} f_{t_1} + \sum_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} \leq 5(x-6) + 8$. Thus, we get $\sum_{t_1=1}^{m-3} f_{t_1} + \sum_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} = (u_{m-1} + u_{m-2} + u_{x-1} + u_{x-2}) - (2n+13) + f_{m-2} + f_{x-2}$. For $n \equiv 4 \pmod 5$, $u_{m-1} = n-3$, $u_{m-2} = n-4$, $u_{x-1} = 2n-3$, $u_{x-2} = 2n-4$ and $f_{m-2} = f_{x-2} = 4$. So, we have $6n - 14 - 2n - 13 \leq 5(x-6)$ and $x \geq \left\lceil \frac{4n+3}{5} \right\rceil = \left\lceil \frac{4n}{5} \right\rceil$. In this case, $|X| = x \geq \left\lceil \frac{4n}{5} \right\rceil$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{2\})) \geq \left\lceil \frac{4n}{5} \right\rceil$.
Thus, the desired equality is obtained as a result of the lower and upper bounds on $\gamma_{\times 2}(D_{sd}(G,\{2\}))$.

This completes the proof.

■

Theorem 2.7. *For $m \geq 1$ and $n \geq 2$, let $G \cong K_{m,n}$ be a bipartite complete graph with $(m+n)$-vertices. Then, the double dominance number of the graph $(D_{sd}(G,\{2\})$ is $\gamma_{\times 2}(D_{sd}(G,\{2\})) = 4$.*

Proof. Let the vertices of the $D_{sd}(G,\{2\})$ graph be divided into four sets of $V(D_{sd}(G,\{2\})) = V_1 \cup V_2 \cup V_1' \cup V_2'$, where $V_1 = \{v_1, v_2, \ldots, v_m\}$, $V_2 = \{v_1, v_2, \ldots, v_n\}$, $V_1' = \{v_1', v_2', \ldots, v_m'\}$ and $V_2' = \{v_1', v_2', \ldots, v_n'\}$. We first establish upper bounds on $\gamma_{\times 2}(D_{sd}(G,\{2\}))$. If $S = \{v_1, v_1, v_1', v_1'\}$, then the set S is the $DD-set$ of the graph $D_{sd}(G,\{2\})$. Thus, $\gamma_{\times 2}(D_{sd}(G,\{2\})) \leq 4$.
For the lower bound, let the set T be the $\gamma_{\times 2}(D_{sd}(G,\{2\}))-set$. Assume that $|T| = 3$. This requires that every vertex in T has at least one neighbor still in T. Taking into account that $V_1 \cong V_1'$and $V_2 \cong V_2'$, the following cases are obtained.

Case 1. Let $u_i \in V_1$, $v_j \in V_2$, $v_t' \in V_2'$. Assume that $T = \{u_i, v_j, v_t'\}$ $i \in \{1,\ldots,m\}$, $j,t \in \{1,\ldots,n\}$ and $j \neq t$. However, in this case, there will be vertices in the graph $D_{sd}(G,\{2\})$ that are not double dominated.

Case 2. Let $u_i, u_j \in V_1$, $u_t' \in V_1'$. Assume that $T = \{u_i, u_j, u_t'\}$ $i,j,t \in \{1,\ldots,m\}$ and $i \neq j \neq t$. However, in this case, there will be vertices in the graph $D_{sd}(G,\{2\})$ that are not double dominated.

Case 3. Let $v_i, v_j \in V_2$, $v_t' \in V_2'$. Assume that $T = \{v_i, v_j, v_t'\}$ $i,j,t \in \{1,\ldots,n\}$ and $i \neq j \neq t$. However, in this case, there will be vertices in the graph $D_{sd}(G,\{2\})$ that are not double dominated.

Case 4. Let $u_i \in V_1$, u$_j' \in V_1'$, $v_t' \in V_2'$. Assume that $T = \{u_i, u_j', v_t'\}$ $i,j \in \{1,\ldots,m\}$, $t \in \{1,\ldots,n\}$ and $i \neq j$. However, in this case, there will be vertices in the graph $D_{sd}(G,\{2\})$ that are not double dominated.

Case 5. Let $v_j \in V_2$, $v_t' \in V_2'$, u$_i \in V_1'$. Assume that $T = \{v_j, v_t', u_i\}$ $j,t \in \{1,\ldots,n\}$, $i \in \{1,\ldots,m\}$ and $j \neq t$. However, in this case, there will be vertices in the graph $D_{sd}(G,\{2\})$ that are not double dominated.

Case 6. Let $u_i \in V_1$, $v_j \in V_2$, u$_t' \in V_1'$. Assume that $T = \{u_i, v_j, u_t'\}$ $i,t \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$ and $i \neq t$. However, in this case, there will be vertices in the graph $D_{sd}(G,\{2\})$ that are not double dominated.

In all cases, some vertices of the graph cannot be double dominated. Thus, we get $\gamma_{\times 2}(D_{sd}(G,\{2\})) = |T| \geq 4$. Thus, the desired equality is obtained as a result of the lower and upper bounds on $\gamma_{\times 2}(D_{sd}(G,\{2\}))$.

This completes the proof.

■

Corollary 2.8. *Let $G \cong S_{1,n}$ be a star graph with $(n+1)$-vertices. Then, the double dominance number of the graph $(D_{sd}(G,\{2\})$ is $\gamma_{\times 2}(D_{sd}(G,\{2\})) = 4$.*

Proof. If $m = 1$ and $n \geq 2$, then $K_{m,n} \cong K_{1,n}$. Thus, the proof of the result is easily seen from Theorem 2.7. ■

Theorem 2.9. *Let $G \cong W_{1,n}$ be a wheel graph with $(n+1)$-vertices. Then, the double dominance number of the graph $(D_{sd}(G, \{2\})$ is $\gamma_{\times 2}(D_{sd}(G, \{2\})) = 4$.*

Proof. Let the vertices of the $D_{sd}(G, \{2\})$ graph be divided into two sets of $V(D_{sd}(G, \{2\})) = V(G) \cup V(G')$, where $V(G) = \{c_1, u_1, \ldots, u_n\}$ and $V(G') = \{c'_1, u'_1, \ldots, u'_n\}$. Let c_1 be the central vertex of the graph G. We first establish upper bounds on $\gamma_{\times 2}(D_{sd}(G, \{2\}))$. If $S = \{c_1, u_1, c'_1, u'_1\}$, then the set S is the $DD - set$ of the graph $D_{sd}(G, \{2\})$. Thus, $\gamma_{\times 2}(D_{sd}(G, \{2\})) \leq 4$.

To complete the proof, we need to prove the lower bound. Let the set T be the $\gamma_{\times 2}(D_{sd}(G, \{2\})) - set$. Assume that $|T| = 3$. For double dominating of vertices in T, at least one neighbor of each vertices must be in T. Thus, we have the following states.

Case 1. Let every vertex in T be at $V(G)$. Since $\deg(c_1) = n$, one of the vertices must be c_1 (or every vertex in T be at $V(G')$). However, in this case, there will be vertices in the graph $D_{sd}(G, \{2\})$ that are not double dominated.

Case 2. Let two vertices in T be at $V(G)$ and the other at $V(G')$. Since $\deg(c_1) = n$, one of the vertices must be c_1 (or two vertices in T be at $V(G')$and the other at $V(G)$). However, in this case, there will be vertices in the graph $D_{sd}(G, \{2\})$ that are not double dominated.

In all cases, some vertices of the graph cannot be double dominated. Thus, we get $\gamma_{\times 2}(D_{sd}(G, \{2\})) = |T| \geq 4$. The desired bounds are obtained as a result of the upper bounds on $\gamma_{\times 2}(D_{sd}(G, \{2\}))$ that were established earlier.

This completes the proof. ■

Theorem 2.10. *If $G \cong P_n$ for $n \geq 10$, then*

$$\gamma_{\times 2}(D_{sd}(G, \{3\})) = \begin{cases} \left\lceil \dfrac{4n+8}{5} \right\rceil + 1 & , n \equiv 1 \ \textit{(mod 5)} \\ \left\lceil \dfrac{4n+8}{5} \right\rceil & , \textit{otherwise} \end{cases}$$

Proof. We first establish upper bounds on $\gamma_{\times 2}(D_{sd}(G, \{3\}))$. We have $\deg(u_1) = \deg(u_n) = \deg(u_{n+1}) = \deg(u_{2n}) = 2$, $\deg(u_i) = 2,\ i \in \{2,\ 3,\ n-1,\ n-2,\ n+2,\ n+3,\ 2n-1,\ 2n-2\}$ and $\deg(u_j) = 4$, $j \in \{4\ ,\ \ldots\ ,\ n-3\ ,\ n+4\ ,\ 2n-3\}$. Let the set D be a $DD - set$ of the graph $D_{sd}(G, \{3\})$. Therefore, in order to double dominate the vertex u_1, it must have neighbors as well. Similarly, this is valid for the vertex u_{n+1}. So, $\{u_2, u_4, u_{n+2}, u_{n+4}\} \in D$. In order for the vertices in D to be double dominated, u_5 and its duplicate, u_{n+5}, must be added to S. In this case the vertices u_6 , u_7 and similarly the vertices u_{n+6} , u_{n+7} that are copies of these peaks are double dominated by the set D. For double dominating of the vertices u_6 and u_7, the vertices u_{n+9} , u_{n+10} are added to D since D is a $DD - set$. Add the vertices u_9 , u_{10} for u_{n+6} and u_{n+7}. Continuing in this way, upper limits on $\gamma_{\times 2}(D_{sd}(G, \{3\}))$ are obtained. Let

$$D = \bigcup_{i=0}^{\lfloor \frac{n}{5} \rfloor - 1} \{u_{5i+4}\ ,\ u_{5i+5}\ ,\ u_{n+5i+4}\ ,\ u_{n+5i+5}\} \cup \{u_2\ ,\ u_{n+2}\}.$$

If $n \equiv 0$ (mod 5), let $S = D$. If $n \equiv 1, 2, 3$ (mod 5), let $S = D \cup \{u_n, u_{2n}\}$. Otherwise, $n \equiv 4$ (mod 5), $S = D \cup \{u_n, u_{n-1}, u_{2n}, u_{2n-1}\}$. In all cases, the set S is a $DD - set$ of $D_{sd}(G, \{3\})$. Further if $n \equiv 1$ (mod 5), then $|S| = \left\lceil \frac{4n+8}{5} \right\rceil + 1$, while if $n \neq 1$ (mod 5), then $|S| = \left\lceil \frac{4n+8}{5} \right\rceil$. Hence, $\gamma_{\times 2}(D_{sd}(G, \{3\})) \leq \left\lceil \frac{4n+8}{5} \right\rceil + 1$ if $n \equiv 1$ (mod 5) and otherwise$\gamma_{\times 2}(D_{sd}(G, \{3\})) \leq \left\lceil \frac{4n+8}{5} \right\rceil + 1$.

Now let's prove the lower bounds on $\gamma_{\times 2}\left(D_{sd}\left(G,\{3\}\right)\right)$. Let's assume that the set $X = \{u_1, u_2, \ldots, u_i, \ldots, u_m, u_{m+1}, \ldots, u_j, \ldots, u_x\}$ is a $\gamma_{\times 2} - set$. Here; u_i and u_j are any two positive integers such that $u_1 < u_2 < \ldots < u_i < \ldots < u_m < u_{m+1} < \ldots < u_j < \ldots < u_x$, where $1 \leq u_i \leq n$ $i \in \{1, 2, \ldots, m\}$ and $n+1 \leq u_j \leq 2n$ $j \in \{n+1, \ldots, x\}$. We have $f_t = u_{t+2} - u_t$ for $t \in \{1, 2, \ldots, x-2\}$ and $t \neq m-1,\ m$. To show the inverse of the inequality, we need to show that $f_t \leq 5$.
Suppose $f_t \geq 6$ for at least one value of t. Without loss of generality, assume that $f_t = 6$. In accordance with this claim; the following sets are obtained.

$$D' = \{2, 4, 5, (n+2), (n+4), (n+6), (n+9)\} \cup \left\{ \bigcup_{i=0}^{\lceil \frac{n-9}{5} \rceil - 1} \{(5i+10), (5i+11), (n+5i+10), (n+5i+11)\} \right\}$$

In this case, $X = D'$ and $|X| = 8 + 4\left\lceil \frac{n-9}{5} \right\rceil$. If $n \equiv 0 \pmod 5$, then $|X| = 8 + 4\left(\frac{n-5}{5}\right) = \frac{4n+20}{5}$. However, this value contradicts the upper value we found earlier as $|S| = \frac{4n+10}{5}$ for $n \equiv 0 \pmod 5$. A similar situation can easily be seen that the values obtained for $n \equiv i \pmod 5$, $i \in \{1, 2, 3, 4\}$ according to the X set contradict the upper limits we obtained earlier. This contradicts our claim. It must be $f_t \leq 5$. So, we have $\sum_{t_1=1}^{m-2} f_{t_1} + \sum_{t_2=m+1}^{x-2} f_{t_2} \leq 5(x-4)$. For all values of n according to $mod\ 5$, it is easily seen that $u_1 + u_2 + u_m + u_{m+1} = 2n + 12$ since $u_1 = 2$, $u_2 = 4$, $u_m = n+2$, $u_{m+1} = n+4$.
If $n \equiv 0 \pmod 5$, then $\sum_{t_1=1}^{m-2} f_{t_1} + \sum_{t_2=m+1}^{x-2} f_{t_2} = (u_m + u_{m-1} + u_x + u_{x-1}) - (2n+12)$. For $n \equiv 0 \pmod 5$, $u_m = n$, $u_{m-1} = n-1$, $u_x = 2n$ and $u_{x-1} = 2n-1$. So, we have $6n - 2 - 2n - 12 \leq 5(x-4)$ and $x \geq \left\lceil \frac{4n+6}{5} \right\rceil$. In this case, $|X| = x \geq \left\lceil \frac{4n+6}{5} \right\rceil = \left\lceil \frac{4n+8}{5} \right\rceil$. This implies that $\gamma_{\times 2}\left(D_{sd}\left(G,\{3\}\right)\right) \geq \left\lceil \frac{4n+8}{5} \right\rceil$.
If $n \equiv 1, 2, 3, 4 \pmod 5$, then $\sum_{t_1=1}^{m-3} f_{t_1} + \sum_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} \leq 5(x-6) + f_{m-2} + f_{x-2}$. Moreover, $\sum_{t_1=1}^{m-3} f_{t_1} + \sum_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} = (u_{m-2} + u_{m-1} + u_{x-2} + u_{x-1}) - (2n+12) + f_{m-2} + f_{x-2}$.
If $n \equiv 1 \pmod 5$, then we have $4n - 18 \leq 5(x-6)$ and $x \geq \left\lceil \frac{4n+12}{5} \right\rceil$since $u_{m-2} = n-2$, $u_{m-1} = n-1$, $u_{x-2} = 2n-2$, $u_{x-1} = 2n-1$ and $f_{m-2} = f_{x-2} = 2$. In this case, $|X| = x \geq \left\lceil \frac{4n+12}{5} \right\rceil = \left\lceil \frac{4n+8}{5} \right\rceil + 1$. This implies that $\gamma_{\times 2}\left(D_{sd}\left(G,\{3\}\right)\right) \geq \left\lceil \frac{4n+8}{5} \right\rceil + 1$.
If $n \equiv 2 \pmod 5$, then we have $6n - 10 - 2n - 12 \leq 5(x-6)$ and $x \geq \left\lceil \frac{4n+8}{5} \right\rceil$since $u_{m-2} = n-3$, $u_{m-1} = n-2$, $u_{x-2} = 2n-3$, $u_{x-1} = 2n-2$ and $f_{m-2} = f_{x-2} = 3$. In this case, $|X| = x \geq \left\lceil \frac{4n+8}{5} \right\rceil$. This implies that $\gamma_{\times 2}\left(D_{sd}\left(G,\{3\}\right)\right) \geq \left\lceil \frac{4n+8}{5} \right\rceil$.
If $n \equiv 3 \pmod 5$, then we have $6n - 14 - 2n - 12 \leq 5(x-6)$ and $x \geq \left\lceil \frac{4n+4}{5} \right\rceil$since $u_{m-2} = n-4$, $u_{m-1} = n-3$, $u_{x-2} = 2n-4$, $u_{x-1} = 2n-3$ and $f_{m-2} = f_{x-2} = 4$. In this case, $|X| = x \geq \left\lceil \frac{4n+5}{5} \right\rceil = \left\lceil \frac{4n+8}{5} \right\rceil$. This implies that $\gamma_{\times 2}\left(D_{sd}\left(G,\{3\}\right)\right) \geq \left\lceil \frac{4n+8}{5} \right\rceil$.

If $n \equiv 4 \pmod 5$, then

$$\begin{aligned}
&\sum_{t_1=1}^{m-4} f_{t_1} + \sum_{t_2=m+1}^{x-4} f_{t_2} + f_{m-3} + f_{m-2} + f_{x-3} + f_{x-2} \\
&\leq 5(x-8) + f_{m-3} + f_{m-2} + f_{x-3} + f_{x-2}
\end{aligned}$$

Moreover, $\sum_{t_1=1}^{m-4} f_{t_1} + \sum_{t_2=m+1}^{x-4} f_{t_2} = (u_{m-2} + u_{m-3} + u_{x-2} + u_{x-3}) - (2n+12)$. For $n \equiv 4 \pmod 5$, we have $6n - 18 - 2n - 12 \leq 5x - 40$ and $x \geq \left\lceil \frac{4n+10}{5} \right\rceil$ since $u_{m-2} = n-4$, $u_{m-3} = n-5$, $u_{x-2} = 2n-4$,

$u_{x-3} = 2n-5$ and $f_{m-3} = f_{m-2} = f_{x-3} = f_{x-2} = 4$. In this case, $|X| = x \geq \left\lceil \frac{4n+10}{5} \right\rceil = \left\lceil \frac{4n+8}{5} \right\rceil$. This implies that $\gamma_{\times 2}\left(D_{sd}\left(G,\{3\}\right)\right) \geq \left\lceil \frac{4n+8}{5} \right\rceil$.
Thus, the desired equality is obtained as a result of the lower and upper bounds on $\gamma_{\times 2}\left(D_{sd}\left(G,\{3\}\right)\right)$.
This completes the proof. ■

Theorem 2.11. *If $G \cong C_n$ for $n \geq 10$, then*

$$\gamma_{\times 2}\left(D_{sd}\left(G,\{3\}\right)\right) = \begin{cases} \left\lceil \dfrac{4n+10}{5} \right\rceil - 1 & , n \equiv 1 \ (mod\ 5) \\ \left\lceil \dfrac{4n+10}{5} \right\rceil + 1 & , n \equiv 3 \ (mod\ 5) \\ \left\lceil \dfrac{4n+10}{5} \right\rceil & , otherwise \end{cases}$$

Proof. Let the vertices of the $D_{sd}\left(G,\{3\}\right)$ graph be divided into two sets of $V\left(D_{sd}\left(G,\{3\}\right)\right) = V_1 \cup V_2$ where$V_1 = \{1,2,\ldots,n\}$ and $V_2 = \{n+1,n+2,\ldots,2n\}$. We first establish upper bounds on $\gamma_{\times 2}\left(D_{sd}\left(G,\{3\}\right)\right)$. Let

$$D_1 = \{n,(n-1),(n-2)\} \cup \left\{ \bigcup_{i=0}^{\left\lceil \frac{n-7}{5} \right\rceil - 1} \{(5i+4),(5i+5)\} \right\},$$

$$D_2 = \{2n\ ,\ 2n-1\ ,\ 2n-2\} \cup \left\{ \bigcup_{i=0}^{\left\lceil \frac{n-7}{5} \right\rceil - 1} \{(n+5i+4),(n+5i+5)\} \right\} \text{ and } D = D_1 \cup D_2.$$

If $n \equiv 0,1,3,4 \pmod 5$, let $S = D$. If $n \equiv 2 \pmod 5$, let $S = D \cup \{n-3\ ,\ 2n-3\}$. Otherwise, $n \equiv 4 \pmod 5$, $S = D \cup \{u_n, u_{n-1}, u_{2n}, u_{2n-1}\}$. In all cases, the set S is a $DD-set$ of $D_{sd}\left(G,\{3\}\right)$. Further if $n \equiv 1 \pmod 5$, then $|S| = \left\lceil \frac{4n+10}{5} \right\rceil - 1$, while if $n \equiv 3 \pmod 5$, then $|S| = \left\lceil \frac{4n+10}{5} \right\rceil + 1$ and otherwise $|S| = \left\lceil \frac{4n+10}{5} \right\rceil$. Hence, $\gamma_{\times 2}\left(D_{sd}\left(G,\{3\}\right)\right) \leq \left\lceil \frac{4n+10}{5} \right\rceil - 1$ if $n \equiv 1 \pmod 5$, $\gamma_{\times 2}\left(D_{sd}\left(G,\{3\}\right)\right) \leq \left\lceil \frac{4n+10}{5} \right\rceil + 1$ if $n \equiv 3 \pmod 5$ and otherwise $\gamma_{\times 2}\left(D_{sd}\left(G,\{3\}\right)\right) \leq \left\lceil \frac{4n+10}{5} \right\rceil$.
Now let's prove the lower bounds on $\gamma_{\times 2}\left(D_{sd}\left(G,\{3\}\right)\right)$. Let's assume that the set $X = \{u_1,u_2,\ldots,u_i,\ldots,u_m,u_{m+1},\ldots,u_j,\ldots,u_x\}$ is a $\gamma_{\times 2}-set$. Here; u_i and u_j are any two positive integers such that $u_1 < u_2 < \ldots < u_i < \ldots < u_m < u_{m+1} < \ldots < u_j < \ldots < u_x$, where $1 \leq u_i \leq n$ $i \in \{1,2,\ldots,m\}$ and $n+1 \leq u_j \leq 2n$ $j \in \{n+1,\ldots,x\}$. We have $f_t = u_{t+2} - u_t$ for $t \in \{1,2,\ldots,x-2\}$ and $t \neq m\ ,\ m-1\ ,\ m-2$. To show the inverse of the inequality, we need to show that $f_t \leq 5$.
Suppose $f_t \geq 6$ for at least one value of t. Without loss of generality, assume that$f_t = 6$. In accordance with this claim; the following sets are obtained. Let

$$D' = \{n,\ (n-1)\,,\ (n-2)\,,\ 2n,\ 2n-1,\ 2n-2\} \cup \left\{ \bigcup_{i=0}^{\left\lceil \frac{n-7}{6} \right\rceil - 1} \{(6i+4),(6i+5),(n+6i+4),(n+6i+5)\} \right\}$$

However, the vertices $(6i+6)$, $(n+6i+6)$ cannot be double dominated with this set. In this case, some vertices must be added to the set D'. This contradicts the upper bound we found earlier. Hence, it must be $f_t \leq 5$. So, we get

$$\sum_{t_1=1}^{m-5} f_{t_1} + \sum_{t_2=m+1}^{x-5} f_{t_2} + f_{m-4} + f_{m-3} + f_{m-2} + f_{x-4} + f_{x-3} + f_{x-2}$$
$$\leq 5\,(x-10) + f_{m-4} + f_{m-3} + f_{m-2} + f_{x-4} + f_{x-3} + f_{x-2}.$$

Also, the right-hand side of the inequality is equal to $(u_{m-4} + u_{m-3} + u_{x-4} + u_{x-3}) + f_{m-4} + f_{m-3} + f_{m-2} + f_{x-4} + f_{x-3} + f_{x-2}$. For all values of n according to mod 5, it is easily seen that $u_1 + u_2 + u_{m+1} + u_{m+2} = 2n + 18$ since $u_1 = 4$, $u_2 = 5$, $u_{m+1} = n + 4$, $u_{m+2} = n + 5$.
For $n \equiv 0$ (mod 5), we have $6n - 22 - 2n - 18 \leq 5x - 50$ and $x \geq \lceil \frac{4n+10}{5} \rceil$ since $u_{m-4} = n - 6$, $u_{m-3} = n - 5$, $u_{x-4} = 2n - 6$, $u_{x-3} = 2n - 5$ and $f_{m-4} = f_{x-4} = f_{m-3} = f_{x-3} = 4$, $f_{m-2} = f_{x-2} = 2$. In this case, $|X| = x \geq \lceil \frac{4n+10}{5} \rceil$. This implies that $\gamma_{\times 2} (D_{sd} (G, \{3\})) \geq \lceil \frac{4n+10}{5} \rceil$.
For $n \equiv 1$ (mod 5), we have $6n - 26 - 2n - 18 \leq 5(x - 10)$ and $x \geq \lceil \frac{4n+6}{10} \rceil$since $u_{m-4} = n - 7$, $u_{m-3} = n - 6$, $u_{x-4} = 2n - 7$, $u_{x-3} = 2n - 6$ and $f_{m-4} = f_{x-4} = f_{m-3} = f_{x-3} = 3$, $f_{m-2} = f_{x-2} = 2$. In this case, $|X| = x \geq \lceil \frac{4n+6}{10} \rceil = \lceil \frac{4n+10}{6} \rceil - 1$. This implies that $\gamma_{\times 2} (D_{sd} (G, \{3\})) \geq \lceil \frac{4n+10}{5} \rceil + 1$.
If $n \equiv 2$ (mod 5) , then $\sum_{t_1=1}^{m-4} f_{t_1} + \sum_{t_2=m+1}^{x-4} f_{t_2} + f_{m-3} + f_{m-2} + f_{x-3} + f_{x-2} \leq 5(x - 8) + f_{m-3} + f_{m-2} + f_{x-3} + f_{x-2}$. For $n \equiv 2$ (mod 5), $f_{m-3} = f_{m-2} = f_{x-3} = f_{x-2} = 2$. Then, we have $\sum_{t_1=1}^{m-4} f_{t_1} + \sum_{t_2=m+1}^{x-4} f_{t_2} + 28 = (u_{m-3} + u_{m-2} + u_{x-3} + u_{x-2}) - (2n + 18) + 8$. Furthermore, we get $6n - 10 - 2n - 18 \leq 5x - 40$ and $x \geq \lceil \frac{4n+12}{5} \rceil$ since $u_{m-3} = n - 3$, $u_{m-2} = n - 2, u_{x-3} = 2n - 3$ $u_{x-2} = 2n - 2$. In this case, $|X| = x \geq \lceil \frac{4n+12}{5} \rceil = \lceil \frac{4n+10}{5} \rceil$. This implies that $\gamma_{\times 2} (D_{sd} (G, \{3\})) \geq \lceil \frac{4n+10}{5} \rceil$.
If $n \equiv 3$ (mod 5), then the formula in $n \equiv 0, 1$ (mod 5) is valid. For $n \equiv 3$ (mod 5), we have $6n-14-2n-18 \leq 5(x - 10)$ and $x \geq \lceil \frac{4n+18}{5} \rceil$since $u_{m-4} = n - 4$, $u_{m-3} = n - 3$, $u_{x-4} = 2n - 4$, $u_{x-3} = 2n - 3$ and $f_{m-4} = f_{m-3} = f_{m-2} = f_{x-4} = f_{x-3} = f_{x-2} = 2$. In this case, $|X| = x \geq \lceil \frac{4n+18}{5} \rceil = \lceil \frac{4n+10}{5} \rceil + 1$. This implies that $\gamma_{\times 2} (D_{sd} (G, \{3\})) \geq \lceil \frac{4n+10}{5} \rceil + 1$.
If $n \equiv 4$ (mod 5), then the formula in $n \equiv 0, 1, 3$ (mod 5) is valid. For $n \equiv 4$ (mod 5), we have $6n - 18 - 2n - 18 \leq 5(x - 10)$ and $x \geq \lceil \frac{4n+14}{5} \rceil = \lceil \frac{4n+10}{5} \rceil$since $u_{m-4} = n - 5$, $u_{m-3} = n - 4, u_{x-4} = 2n - 5, u_{x-3} = 2n - 4$ and $f_{m-4} = f_{m-3} = f_{x-4} = f_{x-3} = 3$ and $f_{m-2} = f_{x-2} = 2$. In this case, $|X| = x \geq \lceil \frac{4n+10}{5} \rceil$. This implies that $\gamma_{\times 2} (D_{sd} (G, \{3\})) \geq \lceil \frac{4n+10}{5} \rceil$.
Thus, the desired equality is obtained as a result of the lower and upper bounds on $\gamma_{\times 2} (D_{sd} (G, \{3\}))$.
This completes the proof. ■

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